Noncooperative game theory

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Game theory

Game theory deals with the analysis of conflictual situations among different decision makers (players) which have different interests.

The decision (strategy) of each player can produce different results depending on the strategies chosen by the other players.

Game theory studies the possibility to forecast the strategies that will be chosen by each player which is assumed to be "rational".

Definition

A noncooperative game in normal form is given by a set of N players, each player i has a set Ω_i of strategies and a cost function $f_i : \Omega_1 \times \cdots \times \Omega_N \to \mathbb{R}$.
The aim of each player i consists in solving the optimization problem

$$\begin{cases} \min \ f_i(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_N) \\ x_i \in \Omega_i \end{cases}$$

Nash equilibrium

From now on, we will consider noncooperative games with 2 players:

Player 1: $\begin{cases} \min \ f_1(x, y) \\ x \in X \end{cases}$ Player 2: $\begin{cases} \min \ f_2(x, y) \\ y \in Y \end{cases}$

How to define an *equilibrium* notion?

Definition

In a two players noncooperative game, a pair of strategies $(\bar{x}, \bar{y})$ is a Nash equilibrium if no player can decrease his/her cost by unilateral deviation, i.e.,

$$f_1(\bar{x}, \bar{y}) = \min_{x \in X} \ f_1(x, \bar{y}), \qquad f_2(\bar{x}, \bar{y}) = \min_{y \in Y} \ f_2(\bar{x}, y).$$

Equivalent definition: $\bar{x}$ is the best response of player 1 to strategy $\bar{y}$ of player 2 and $\bar{y}$ is the best response of player 2 to strategy $\bar{x}$ of player 1.

Matrix games

A **matrix game** is a two players noncooperative game where:

- $f_2 = -f_1$ (zero-sum game),
- X and Y are finite sets: $X = \{1, \dots, m\}$, $Y = \{1, \dots, n\}$.

It can be represented by a $m \times n$ matrix C, where c_{ij} is the amount of money player 1 pays to player 2 if player 1 chooses strategy i and player 2 chooses strategy j.

Example 1.

		Player 2		
		1	2	3
Player 1	1	1	−1	0
	2	3	−2	−1
	3	2	3	−2

Are there Nash equilibria?

Example

Example 1.

Player 2 (columns), Player 1 (rows)

	1	2	3
1	1	-1	0
2	3	-2	-1
3	2	3	-2

For player 2, strategy 3 is worse than strategy 1 because his/her profit is less than the one obtained playing strategy 1 for any strategy of player 1. Hence, player 2 will never choose strategy 3, which can be deleted from the game. The game is equivalent to

Player 2 (columns), Player 1 (rows)

	1	2
1	1	-1
2	3	-2
3	2	3

Now, for player 1 strategy 3 is worse than strategy 1.

Example

The reduced game is

Player 2

		1	2
	1	1	-1
Player 1	2	3	-2

For player 2, strategy 2 is worse than strategy 1. Thus, player 2 will always choose strategy 1. The reduced game is

Player 2

		1
	1	1
Player 1	2	3

Finally, for player 1, strategy 2 is worse than strategy 1. Therefore, player 1 will always choose strategy 1.
Hence $(1, 1)$ is a Nash equilibrium.

Strictly dominated strategies

Definition

Given a 2 players noncooperative game, a strategy $x \in X$ is strictly dominated by $\widetilde{x} \in X$ if

$$f_1(x, y) > f_1(\widetilde{x}, y) \qquad \forall\ y \in Y.$$

Similarly, a strategy $y \in Y$ is strictly dominated by $\widetilde{y} \in Y$ if

$$f_2(x, y) > f_2(x, \widetilde{y}) \qquad \forall\ x \in X.$$

Strictly dominated strategies can be deleted from the game.

Exercises

Exercise 1. Find all the Nash equilibria of the following matrix game:

Player 2

Player 1		1	2	3	4	5
	1	1	−1	1	−2	−3
	2	2	−2	3	4	0
	3	1	0	1	−3	−4
	4	4	−3	2	−1	−1
	5	5	−2	4	−3	2

Exercise 2. Prove that if (i, j) and (p, q) are Nash equilibria of a matrix game, then $c_{ij} = c_{pq}$. Furthermore, (i, q) and (p, j) are Nash equilibria as well.

Mixed strategies

Example 3. (Odds and evens)

		Player 2	
		1	2
Player 1	1	1	−1
	2	−1	1

Are there strictly dominated strategies? Are there Nash equilibria?

Definition

If C is a $m \times n$ matrix game, then a mixed strategy for player 1 is a m-vector of probabilities and we consider $X = \{x \in \mathbb{R}^m : \ x \geq 0, \quad \sum_{i=1}^{m} x_i = 1\}$ the set of mixed strategies of player 1. The vertices of X, i.e., vectors $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ are called pure strategies of player 1.
Similarly, $Y = \{y \in \mathbb{R}^n : \ y \geq 0, \quad \sum_{j=1}^{n} y_j = 1\}$ is the set of mixed strategies of player 2.
The expected cost for player 1 is $f_1(x, y) = x^{\mathsf{T}} C y$
and for player 2 is $f_2(x, y) = -x^{\mathsf{T}} C y$.

Mixed strategies Nash equilibria

Definition

If C is a $m \times n$ matrix game, then $(\bar{x}, \bar{y}) \in X \times Y$ is a mixed strategies Nash equilibrium if

$$\max_{y \in Y} \bar{x}^T C y = \bar{x}^T C \bar{y} = \min_{x \in X} x^T C \bar{y},$$

i.e., $(\bar{x}, \bar{y})$ is a saddle point of the function $x^T C y$.

Theorem

$(\bar{x}, \bar{y})$ is a mixed strategies Nash equilibrium if and only if

$$\begin{cases} \bar{x} \text{ is an optimal solution of } \min\limits_{x \in X} \max\limits_{y \in Y} x^T C y \\ \bar{y} \text{ is an optimal solution of } \max\limits_{y \in Y} \min\limits_{x \in X} x^T C y \end{cases}$$

Mixed strategies Nash equilibria

Theorem

1. The problem $\min_{x \in X} \max_{y \in Y} x^{\mathsf{T}} C y$ is equivalent to the linear programming problem

$$\begin{cases} \min \ v \\ v \geq \sum_{i=1}^{m} c_{ij} x_i \quad \forall \ j = 1, \dots, n \\ x \geq 0, \quad \sum_{i=1}^{m} x_i = 1 \end{cases} \qquad (P_1)$$

2. The problem $\max_{y \in Y} \min_{x \in X} x^{\mathsf{T}} C y$ is equivalent to the linear programming problem

$$\begin{cases} \max \ w \\ w \leq \sum_{j=1}^{n} c_{ij} y_j \quad \forall \ i = 1, \dots, m \\ y \geq 0, \quad \sum_{j=1}^{n} y_j = 1 \end{cases} \qquad (P_2)$$

3. (P_2) is the dual of (P_1).

Corollary. Any matrix game has at least a mixed strategies Nash equilibrium.

Mixed strategies Nash equilibria

Example 3. (Odds and evens)

		Player 2	
		1	2
Player 1	1	1	−1
	2	−1	1

$$(P_1)\begin{cases} \min\ v \\ v \geq x_1 - x_2 \\ v \geq -x_1 + x_2 \\ x \geq 0 \\ x_1 + x_2 = 1 \end{cases} \quad \text{is equivalent to} \quad \begin{cases} \min\ v \\ v \geq 2x_1 - 1 \\ v \geq 1 - 2x_1 \\ 0 \leq x_1 \leq 1 \end{cases} \quad \Rightarrow \bar{x} = (1/2, 1/2)$$

$$(P_2)\begin{cases} \max\ w \\ w \leq y_1 - y_2 \\ w \leq -y_1 + y_2 \\ y \geq 0 \\ y_1 + y_2 = 1 \end{cases} \quad \text{is equivalent to} \quad \begin{cases} \max\ w \\ w \leq 2y_1 - 1 \\ w \leq 1 - 2y_1 \\ 0 \leq y_1 \leq 1 \end{cases} \quad \Rightarrow \bar{y} = (1/2, 1/2)$$

Mixed strategies Nash equilibria

Exercise 3. Consider the following matrix game:

		Player 2			
		1	2	3	4
Player 1	1	-4	4	-1	2
	2	4	-4	2	-3

- Are there strictly dominated strategies?
- Are there pure strategies Nash equilibria?
- Find all mixed strategies Nash equilibria.

Mixed strategies Nash equilibria

Exercise 4. Consider the following matrix game:

		Player 2			
		1	2	3	4
Player 1	1	-1	5	-2	0
	2	-3	5	0	5
	3	0	-4	1	-1
	4	-1	3	5	3

- Are there strictly dominated strategies?
- Are there pure strategies Nash equilibria?
- Find a mixed strategies Nash equilibrium.

Bimatrix games

A **bimatrix game** is a two players noncooperative game where:

- the sets of mixed strategies are $X = \{x \in \mathbb{R}^m : \ x \geq 0, \quad \sum_{i=1}^m x_i = 1\}$ and $Y = \{y \in \mathbb{R}^n : \ y \geq 0, \quad \sum_{j=1}^n y_j = 1\}$
- the cost functions are $f_1(x,y) = x^\mathsf{T} C_1 y$ and $f_2(x,y) = x^\mathsf{T} C_2 y$, where C_1 and C_2 are $m \times n$ matrices.

Theorem

Any bimatrix game has at least a mixed strategies Nash equilibrium.

Example 4. (Prisoner's dilemma)

$$C_1 = \begin{pmatrix} 5 & 1 \\ 10 & 2 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 5 & 10 \\ 1 & 2 \end{pmatrix}$$

Are there strictly dominated strategies?

Best response mappings

Example 5. (Battle of the buddies)

$$C_1 = \begin{pmatrix} -5 & 0 \\ 0 & -1 \end{pmatrix} \qquad C_2 = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$$

Are there strictly dominated strategies?
Are there pure strategies Nash equilibria?
Are there mixed strategies Nash equilibria? How to compute them?

Theorem

If we define the best response mappings $B_1 : Y \to X$ and $B_2 : X \to Y$ as

$$B_1(y) = \{\text{optimal solutions of } \min_{x \in X} \; x^{\mathsf{T}} C_1 y\}$$

$$B_2(x) = \{\text{optimal solutions of } \min_{y \in Y} \; x^{\mathsf{T}} C_2 y\}$$

then $(\bar{x}, \bar{y})$ is a Nash equilibrium if and only if $\bar{x} \in B_1(\bar{y})$ and $\bar{y} \in B_2(\bar{x})$.

Best response mappings

Example 5 (continued).

$$C_1 = \begin{pmatrix} -5 & 0 \\ 0 & -1 \end{pmatrix} \qquad C_2 = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$$

Given $y \in Y$ we have to solve the problem

$$\begin{cases} \min \ x^{\mathsf{T}} C_1 y = -5x_1 y_1 - x_2 y_2 \\ x \in X \end{cases} \equiv \begin{cases} \min \ (1 - 6y_1)x_1 + y_1 - 1 \\ 0 \le x_1 \le 1 \end{cases}$$

hence the optimal solution is

$$B_1(y_1) = \begin{cases} 0 & \text{if } y_1 \in [0, 1/6] \\ [0,1] & \text{if } y_1 = 1/6 \\ 1 & \text{if } y_1 \in [1/6, 1] \end{cases}$$

Similarly, given $x \in X$ we have to solve the problem

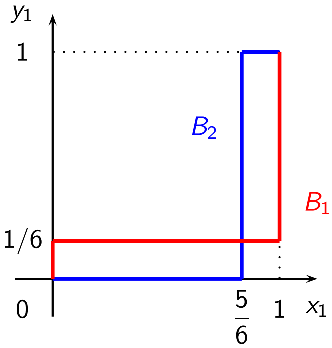
$$\begin{cases} \min \ x^{\mathsf{T}} C_2 y = -x_1 y_1 - 5x_2 y_2 \\ y \in Y \end{cases} \equiv \begin{cases} \min \ (5 - 6x_1)y_1 + 5x_1 - 5 \\ 0 \le y_1 \le 1 \end{cases}$$

hence the optimal solution is

$$B_2(x_1) = \begin{cases} 0 & \text{if } x_1 \in [0, 5/6] \\ [0,1] & \text{if } x_1 = 5/6 \\ 1 & \text{if } x_1 \in [5/6, 1] \end{cases}$$

Best response mappings

Nash equilibria are given by the <u>intersections</u> of the graphs of the best response mappings B_1 and B_2:



There are 3 Nash equilibria:

- $\bar{x} = (0, 1)$, $\bar{y} = (0, 1)$ (pure strategies)
- $\bar{x} = (5/6, 1/6)$, $\bar{y} = (1/6, 5/6)$ (mixed strategies)
- $\bar{x} = (1, 0)$, $\bar{y} = (1, 0)$ (pure strategies)

KKT conditions for Nash equilibria

Theorem

$(\bar{x}, \bar{y})$ is a Nash equilibrium if and only if there are $\mu_1, \mu_2 \in \mathbb{R}$ such that

$$\begin{cases} C_1 \bar{y} + \mu_1 e \geq 0 \\ \bar{x} \geq 0, \quad \sum_{i=1}^{m} \bar{x}_i = 1 \\ \bar{x}_i (C_1 \bar{y} + \mu_1 e)_i = 0 \quad \forall\ i = 1, \ldots, m \\ C_2^{\mathsf{T}} \bar{x} + \mu_2 e \geq 0 \\ \bar{y} \geq 0, \quad \sum_{j=1}^{n} \bar{y}_j = 1 \\ \bar{y}_j (C_2^{\mathsf{T}} \bar{x} + \mu_2 e)_j = 0 \quad \forall\ j = 1, \ldots, n \end{cases}$$

where $e = (1, 1, \ldots, 1)$.

Exercise 5. Find the Nash equilibria of the following bimatrix game by exploiting the KKT conditions:

$$C_1 = \begin{pmatrix} 3 & 3 \\ 4 & 1 \\ 6 & 0 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 3 & 4 \\ 4 & 0 \\ 3 & 5 \end{pmatrix}$$

Characterization of Nash equilibria

Theorem

Assume that $C_1 < 0$ and $C_2 < 0$.

- If $(\bar{x}, \bar{y})$ is a Nash equilibrium then there are $u > 0$, $v > 0$ such that $\widetilde{x} = \bar{x}/u$ and $\widetilde{y} = \bar{y}/v$ solve the following system:

$$\begin{cases} \widetilde{x} \geq 0, \quad C_1 \widetilde{y} + e \geq 0, \quad \widetilde{x}_i (C_1 \widetilde{y} + e)_i = 0 \quad \forall\ i = 1, \dots, m \\ \widetilde{y} \geq 0, \quad C_2^{\mathsf{T}} \widetilde{x} + e \geq 0, \quad \widetilde{y}_j (C_2^{\mathsf{T}} \widetilde{x} + e)_j = 0 \quad \forall\ j = 1, \dots, n \end{cases} \tag{S}$$

- If $(\widetilde{x}, \widetilde{y})$ solves system (S) with $\widetilde{x} \neq 0$ and $\widetilde{y} \neq 0$, then $\left(\frac{\widetilde{x}}{\sum_{i=1}^m \widetilde{x}_i}, \frac{\widetilde{y}}{\sum_{j=1}^n \widetilde{y}_j} \right)$ is a Nash equilibrium.

Characterization of Nash equilibria

Define the polyhedra

$$P = \left\{ x \in \mathbb{R}^m : \begin{array}{rl} x_i \geq 0 & \forall\ i = 1, \dots, m \\ (C_2^{\mathsf{T}} x + e)_j \geq 0 & \forall\ j = m+1, \dots, m+n \end{array} \right\}$$

$$Q = \left\{ y \in \mathbb{R}^n : \begin{array}{rl} (C_1 y + e)_i \geq 0 & \forall\ i = 1, \dots, m \\ y_j \geq 0 & \forall\ j = m+1, \dots, m+n \end{array} \right\}$$

Theorem

- $(\widetilde{x}, \widetilde{y})$ solves system (S) if and only if $\widetilde{x} \in P$, $\widetilde{y} \in Q$ and for any $k \in \{1, \dots, m+n\}$ either the k-th constraint of P is active in $\widetilde{x}$ or the k-th constraint of Q is active in $\widetilde{y}$.
- If the vertices of P and Q are non-degenerate and $(\widetilde{x}, \widetilde{y})$ solves system (S), then $\widetilde{x}$ is a vertex of P and $\widetilde{y}$ is a vertex of Q.

Therefore, if $C_1 < 0$, $C_2 < 0$ and vertices of P and Q are non-degenerate, then <u>we can find all the Nash equilibria</u> analyzing all the pairs (x, y) of vertices of P and Q, checking if each constraint $k = 1, \dots, m+n$ is active either in x or in y.

Characterization of Nash equilibria

Example 6. (Battle of the buddies)

$$C_1 = \begin{pmatrix} -5 & 0 \\ 0 & -1 \end{pmatrix} \qquad C_2 = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$$

Since the elements of C_1 and C_2 are not all negative, we can reformulate the game setting $(C_1)_{ij} = (C_1)_{ij} - 1$ and $(C_2)_{ij} = (C_2)_{ij} - 1$:

$$C_1 = \begin{pmatrix} -6 & -1 \\ -1 & -2 \end{pmatrix} \qquad C_2 = \begin{pmatrix} -2 & -1 \\ -1 & -6 \end{pmatrix}$$

$P = \{x \in \mathbb{R}^2 : \ x_1 \geq 0, \ x_2 \geq 0, \ -2x_1 - x_2 + 1 \geq 0, \ -x_1 - 6x_2 + 1 \geq 0\}.$
$Q = \{y \in \mathbb{R}^2 : \ -6y_3 - y_4 + 1 \geq 0, \ -y_3 - 2y_4 + 1 \geq 0, \ y_3 \geq 0, \ y_4 \geq 0\}.$

Solutions of system (S)	Nash equilibria
$x = (0, 1/6)$ $y = (0, 1/2)$	$x = (0, 1)$ $y = (0, 1)$
$x = (5/11, 1/11)$ $y = (1/11, 5/11)$	$x = (5/6, 1/6)$ $y = (1/6, 5/6)$
$x = (1/2, 0)$ $y = (1/6, 0)$	$x = (1, 0)$ $y = (1, 0)$

Convex games

Now, we consider a two players noncooperative game

$$\text{Player 1: } \begin{cases} \min_x \ f_1(x,y) \\ g_i^1(x) \le 0 \quad \forall\, i = 1,\ldots,p \end{cases} \qquad \text{Player 2: } \begin{cases} \min_y \ f_2(x,y) \\ g_j^2(y) \le 0 \quad \forall\, j = 1,\ldots,q \end{cases}$$

where f_1, g^1, f_2 and g^2 are continuously differentiable.
The game is said convex if the optimization problem of each player is convex.

Theorem

If the feasible regions $X = \{x \in \mathbb{R}^m : \ g_i^1(x) \le 0 \quad i = 1,\ldots,p\}$ and $Y = \{y \in \mathbb{R}^n : \ g_j^2(y) \le 0 \quad j = 1,\ldots,q\}$ are closed, convex and bounded, the cost function $f_1(\cdot, y)$ is quasiconvex for any $y \in Y$ and $f_2(x, \cdot)$ is quasiconvex for any $x \in X$, then there exists at least a Nash equilibrium.

The quasiconvexity of the cost functions is crucial.
Example. The game defined as $X = Y = [0,1]$, $f_1(x,y) = -x^2 + 2xy$, $f_2(x,y) = y(1-2x)$ has no Nash equilibria.

KKT conditions

Theorem

- If $(\bar{x}, \bar{y})$ is a Nash equilibrium and the Abadie constraints qualification holds both in $\bar{x}$ and $\bar{y}$, then there are $\lambda^1 \in \mathbb{R}^p$, $\lambda^2 \in \mathbb{R}^q$ such that

$$\begin{cases} \nabla_x f_1(\bar{x}, \bar{y}) + \sum_{i=1}^{p} \lambda_i^1 \nabla g_i^1(\bar{x}) = 0 \\ \lambda^1 \geq 0, \qquad g^1(\bar{x}) \leq 0 \\ \lambda_i^1 g_i^1(\bar{x}) = 0, \quad i = 1, \dots, p \\ \nabla_y f_2(\bar{x}, \bar{y}) + \sum_{j=1}^{q} \lambda_j^2 \nabla g_j^2(\bar{y}) = 0 \\ \lambda^2 \geq 0, \qquad g^2(\bar{y}) \leq 0 \\ \lambda_j^2 g_j^2(\bar{y}) = 0, \quad j = 1, \dots, q \end{cases}$$

- If $(\bar{x}, \bar{y}, \lambda^1, \lambda^2)$ solves the above system and the game is convex, then $(\bar{x}, \bar{y})$ is a Nash equilibrium.

Exercise 6. Find the Nash equilibria of the following convex game:

Player 1: $\begin{cases} \min_x \; x^2 - x(4y+6) \\ -4 \leq x \leq 6 \end{cases}$ Player 2: $\begin{cases} \min_y \; (x+3)(4-y) \\ -5 \leq y \leq 5 \end{cases}$

Merit functions

Merit functions allow reformulating the Nash equilibrium problem into an equivalent optimization problem.

Assume that the game is convex. Consider the Nikaido-Isoda function

$$f(x, y, u, v) = f_1(u, y) - f_1(x, y) + f_2(x, v) - f_2(x, y),$$

where $x, u \in \mathbb{R}^m$ and $y, v \in \mathbb{R}^n$. Define the gap function as

$$\psi(x, y) = \max_{u \in X, v \in Y} [-f(x, y, u, v)].$$

Then:

- The problem defining ψ is convex
- $\psi(x, y) \geq 0$ for any $(x, y) \in X \times Y$
- $(\bar{x}, \bar{y})$ is a Nash equilibrium if and only if $(\bar{x}, \bar{y}) \in X \times Y$ and $\psi(\bar{x}, \bar{y}) = 0$

Therefore, finding Nash equilibria is equivalent to solve the constrained optimization problem

$$\begin{cases} \min \ \psi(x, y) \\ (x, y) \in X \times Y \end{cases}$$

Merit functions

In general ψ is not differentiable, but it is possible to regularize it.

Given a parameter $\alpha > 0$, the regularized gap function is defined as

$$\psi_\alpha(x, y) = \max_{u \in X, v \in Y} \left[-f(x, y, u, v) - \frac{\alpha}{2} \|(x, y) - (u, v)\|^2 \right].$$

Then:

- The problem defining ψ_α is convex and has a unique optimal solution
- ψ_α is continuously differentiable
- $\psi_\alpha(x, y) \geq 0$ for any $(x, y) \in X \times Y$
- $(\bar{x}, \bar{y})$ is a Nash equilibrium if and only if $(\bar{x}, \bar{y}) \in X \times Y$ and $\psi_\alpha(\bar{x}, \bar{y}) = 0.$

Therefore, finding Nash equilibria is equivalent to solve the smooth constrained optimization problem

$$\begin{cases} \min \ \psi_\alpha(x, y) \\ (x, y) \in X \times Y \end{cases}$$

Merit functions

It is possible to reformulate the problem of finding Nash equilibria as an unconstrained optimization problem.

Given two parameters $\beta > \alpha > 0$, the D-gap function is defined as

$$\psi_{\alpha,\beta}(x, y) = \psi_\alpha(x, y) - \psi_\beta(x, y).$$

Then:

- $\psi_{\alpha,\beta}$ is continuously differentiable
- $\psi_{\alpha,\beta}(x, y) \geq 0$ for any $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$
- $(\bar{x}, \bar{y})$ is a Nash equilibrium if and only if $\psi_{\alpha,\beta}(\bar{x}, \bar{y}) = 0$.

Therefore, finding Nash equilibria is equivalent to solve the smooth, unconstrained optimization problem

$$\begin{cases} \min \ \psi_{\alpha,\beta}(x, y) \\ (x, y) \in \mathbb{R}^m \times \mathbb{R}^n \end{cases}$$

Merit functions

Exercise 7. Consider the following bimatrix game:

$$C_1 = \begin{pmatrix} 3 & 3 \\ 4 & 1 \\ 6 & 0 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 3 & 4 \\ 4 & 0 \\ 3 & 5 \end{pmatrix}$$

- Implement in MATLAB the gap function, the regularized gap function and the D-gap function.
- Exploit the gap function ψ to check if the point $w = (x, y)$, where $x = (1/3, 1/3, 1/3)$ and $y = (1/2, 1/2)$, is a Nash equilibrium.
- Find a local minimum of the regularized gap function ψ_α with $\alpha = 1$ starting from w.
- Try to find a global minimum of the regularized gap function ψ_α with a multistart approach.
- Try to find a global minimum of the D-gap function $\psi_{\alpha,\beta}$, with $\alpha = 1$ and $\beta = 10$, with a multistart approach.