Petri Nets

Computational Models for Complex Systems

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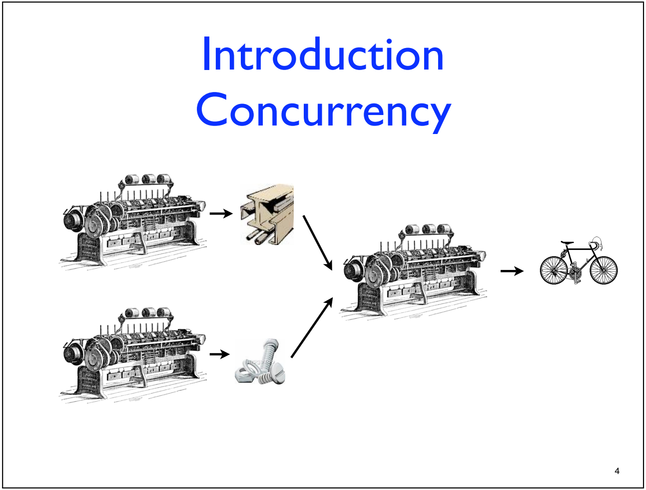
Introduction

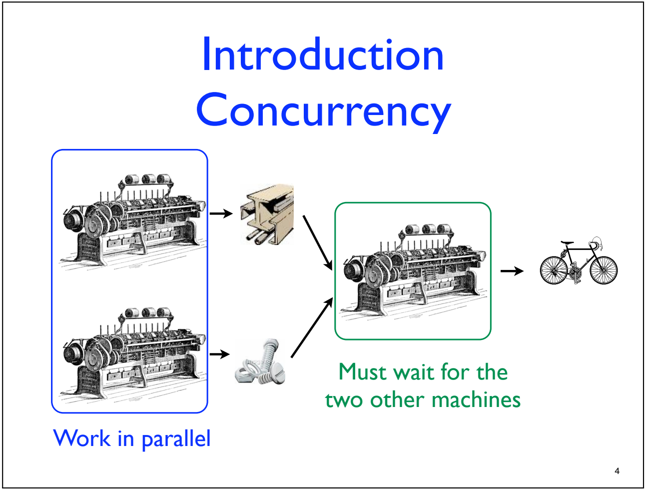
- Petri nets have been proposed to model concurrent systems, with applications mainly in manufactury and concurrent programming
- Recent applications are in the context of business processes
  - Have a look at Roberto Bruni's course: `http://didawiki.di.unipi.it/doku.php/magistraleinformaticaeconomia/mpb/start`
- Many books have been written on Petri nets. One of the most famous is
  - Wolfgang Reisig, "Petri Nets. An Introduction". Springer-Verlag.
- This lesson is based on the tutorial by G. Geeraerts available here: `http://di.ulb.ac.be/verif/ggeeraer/Tutorial-Perti-Nets-Geeraerts.pdf`

Introduction

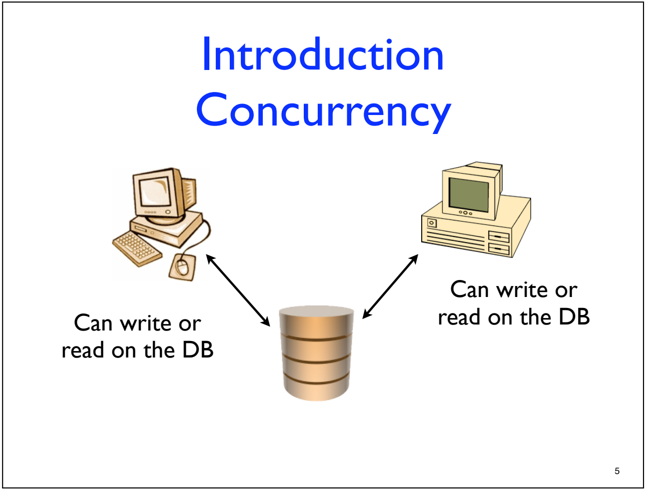


ⓒG. Geeraerts

Introduction



Introduction



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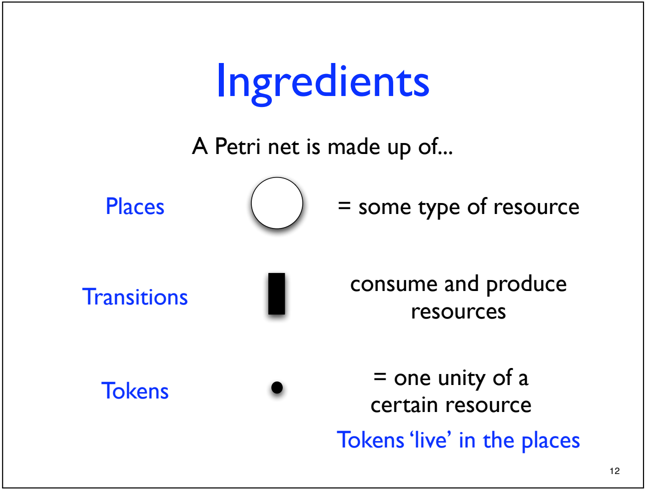
Petri nets



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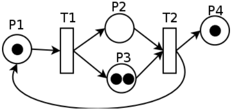
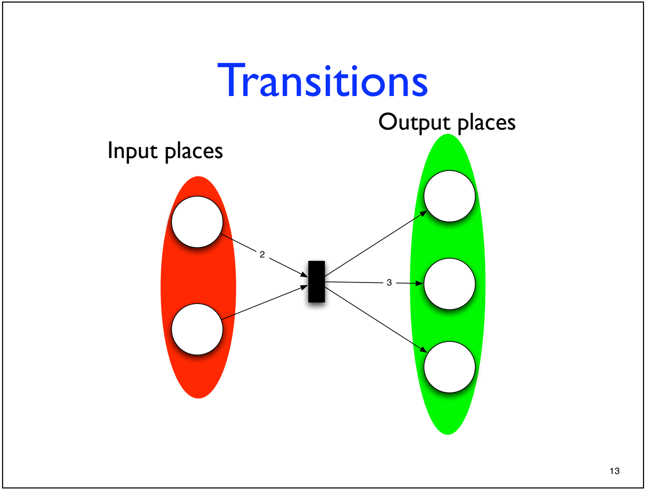
Places and Transitions are edges of a bipartite graph (the Petri net)

Tokens are inside places



Petri nets



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A Petri net trasition corresponds exactly to a MultiSet Rewriting rule (or a chemical reaction)

MSR:

$$a^2b \mapsto cd^3e$$

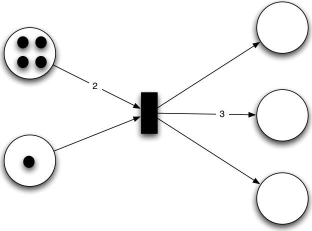
Chem:

$$2A + B \rightarrow C + 3D + E$$

Petri nets

Firing a transition

Transitions consume tokens from the input places
and produce tokens in the output places



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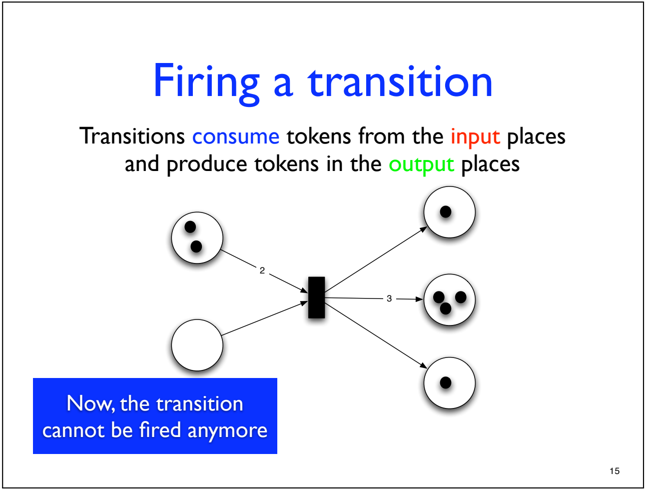
Petri nets


Firing a transition
Transitions consume tokens from the input places
and produce tokens in the output places
2
3
Now, the transition
cannot be fired anymore
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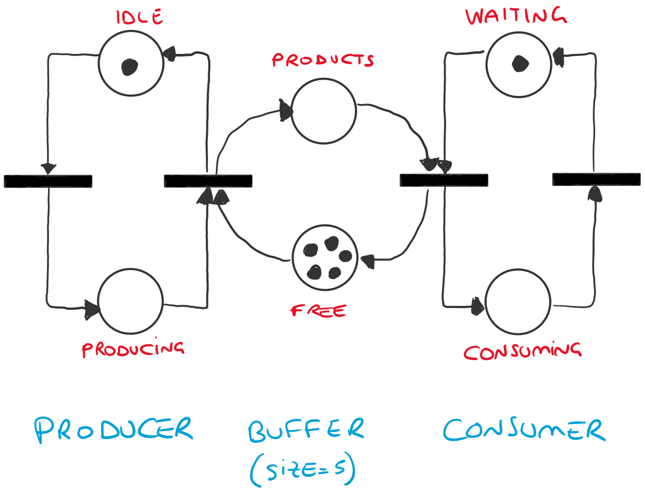
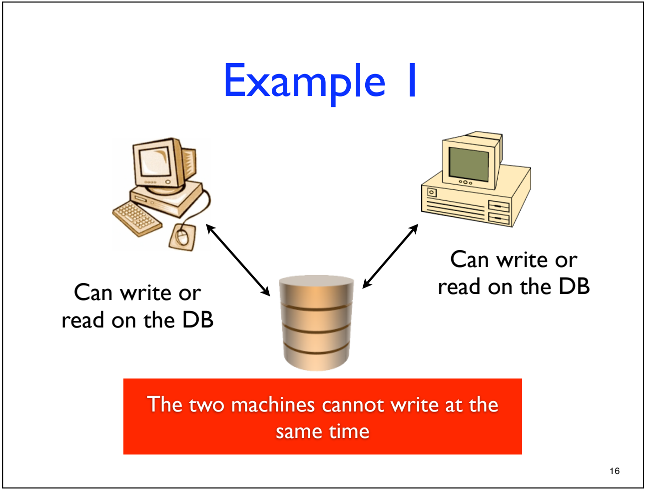
Petri nets

Example: producer/consumer with bounded buffer

22:29



Petri nets



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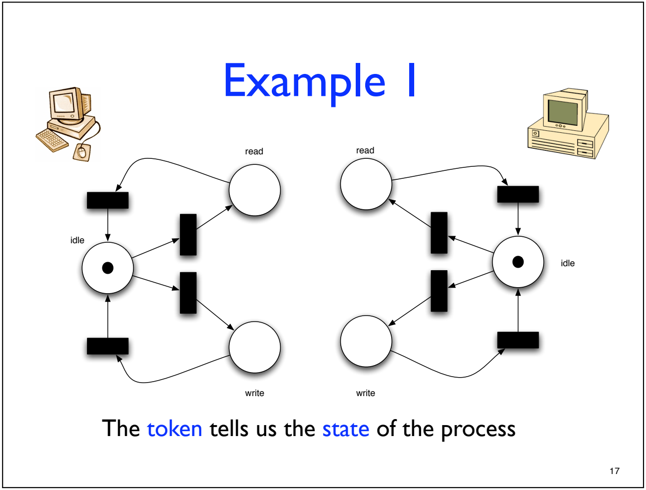
Petri nets



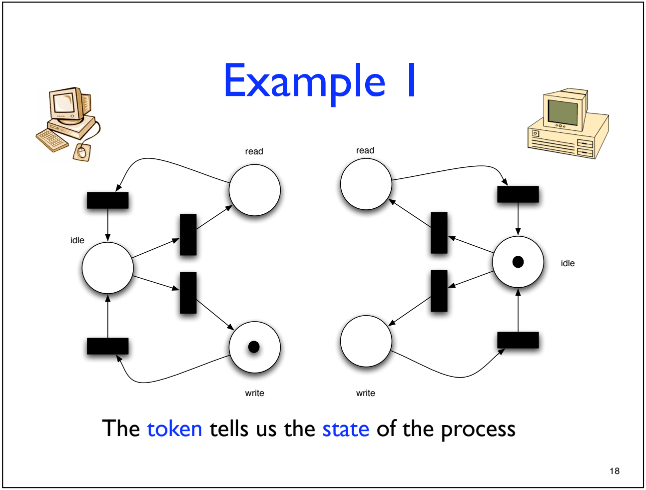
With MSR rules:

$$
\begin{aligned}
idle_1 &\mapsto read_1\\
idle_1 &\mapsto write_1\\
read_1 &\mapsto idle_1\\
write_1 &\mapsto idle_1\\
idle_2 &\mapsto read_2\\
idle_2 &\mapsto write_2\\
read_2 &\mapsto idle_2\\
write_2 &\mapsto idle_2
\end{aligned}
$$

state (multiset):

$$idle1\ idle2$$

Petri nets



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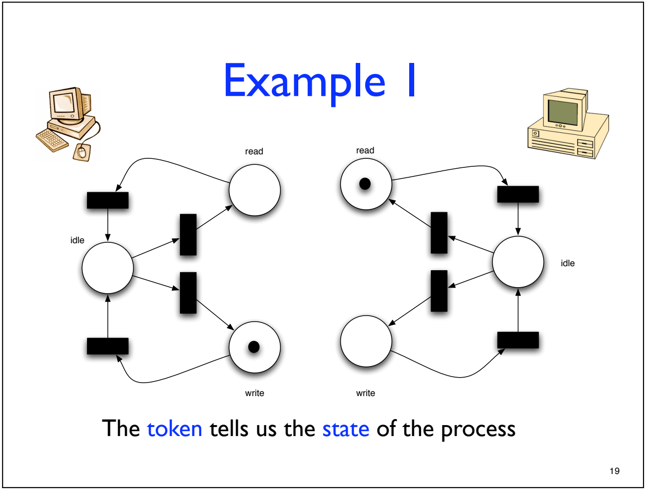
With MSR rules:

$$idle_1 \mapsto read_1$$
$$idle_1 \mapsto write_1$$
$$read_1 \mapsto idle_1$$
$$write_1 \mapsto idle_1$$
$$idle_2 \mapsto read_2$$
$$idle_2 \mapsto write_2$$
$$read_2 \mapsto idle_2$$
$$write_2 \mapsto idle_2$$

state (multiset):

$$\color{red}{write1}\ idle2$$

Petri nets



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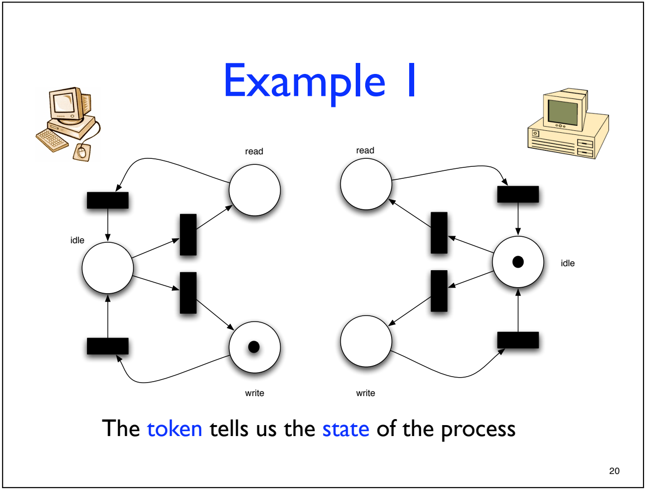
With MSR rules:

$$idle_1 \mapsto read_1$$
$$idle_1 \mapsto write_1$$
$$read_1 \mapsto idle_1$$
$$write_1 \mapsto idle_1$$
$$idle_2 \mapsto read_2$$
$$idle_2 \mapsto write_2$$
$$read_2 \mapsto idle_2$$
$$write_2 \mapsto idle_2$$

state (multiset):

$$write1 \; read2$$

Petri nets



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With MSR rules:

$$idle_1 \mapsto read_1$$
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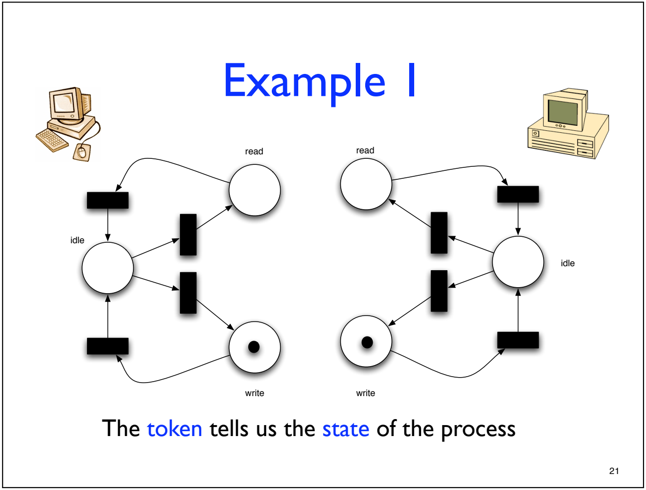
state (multiset):

$$write1\ idle2$$

Petri nets



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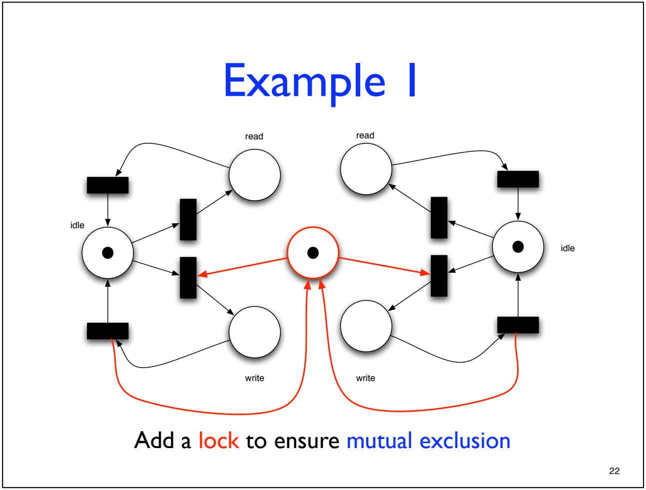
With MSR rules:

$$idle_1 \mapsto read_1$$
$$idle_1 \mapsto write_1$$
$$read_1 \mapsto idle_1$$
$$write_1 \mapsto idle_1$$
$$idle_2 \mapsto read_2$$
$$idle_2 \mapsto write_2$$
$$read_2 \mapsto idle_2$$
$$write_2 \mapsto idle_2$$

state (multiset):

$write1$ $write2$!!!

Petri nets



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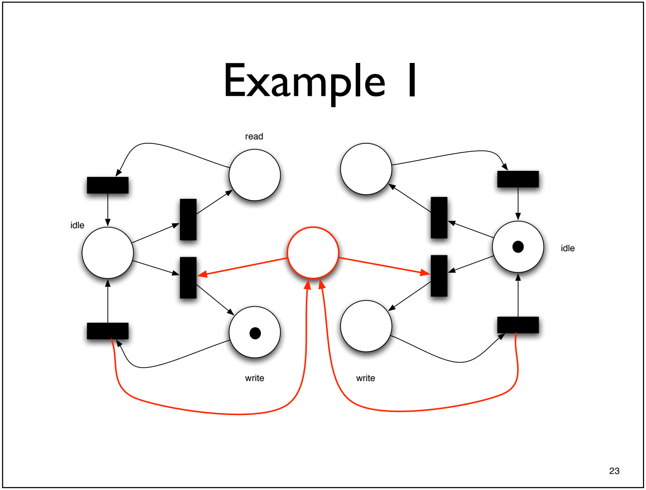
With MSR rules:

$$idle_1 \mapsto read_1$$
$$idle_1 \ M \mapsto write_1$$
$$read_1 \mapsto idle_1$$
$$write_1 \mapsto idle_1 \ M$$
$$idle_2 \mapsto read_2$$
$$idle_2 \ M \mapsto write_2$$
$$read_2 \mapsto idle_2$$
$$write_2 \mapsto idle_2 \ M$$

state (multiset):

$$idle1 \ idle2 \ M$$

Petri nets



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With MSR rules:

$$idle_1 \mapsto read_1$$
$$idle_1\ \color{red}{M} \mapsto write_1$$
$$read_1 \mapsto idle_1$$
$$write_1 \mapsto idle_1\ \color{red}{M}$$
$$idle_2 \mapsto read_2$$
$$idle_2\ \color{red}{M} \mapsto write_2$$
$$read_2 \mapsto idle_2$$
$$write_2 \mapsto idle_2\ \color{red}{M}$$

state (multiset):

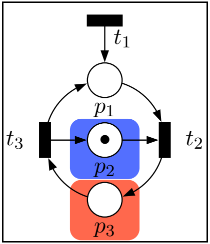
$\color{red}{write1}\ idle2$

Petri nets

Example 2

```
mutex M ;

Process P {
   repeat {
      take M ;
      critical ;
      release M ;
   }
}
```



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t_1 represents the start of a new (concurrent) instance of process P

p_1: processes
p_2: free mutex
p_3: critical session

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Formal definition

- A Petri net is a tuple ⟨P,T⟩ where:
  - P is the (finite) set of places
  - T is the (finite) set of transitions. Each transition t is a tuple ⟨I, O⟩ where:
    - I: is a function s.t. t consumes I(p) tokens in each place p
    - O is a function s.t. t produces O(p) tokens in each place p

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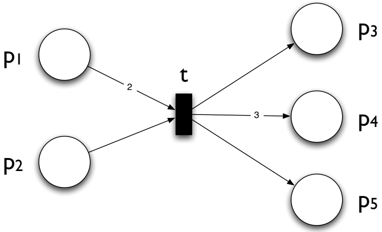
Example

I(p1)=2   I(p2)=1   I(p3)=0   I(p4)=0   I(p5)=0

O(p1)=0   O(p2)=0   O(p3)=1   O(p4)=3   O(p5)=1



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Dynamics of Petri nets (semantics)

Markings

- The distribution of the tokens in the places is formalised by the notion of marking, which can be seen:
  - either as a function m, s.t. $m(p)$ is the number of tokens in place p
  - or as a vector $m = \langle m_1, m_2, \ldots m_n \rangle$ where m_i is the number of tokens in place p_i

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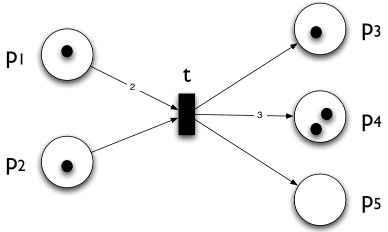
Dynamics of Petri nets (semantics)

Example

$m = \langle 1,1,1,2,0 \rangle$

$m = \langle p_1, p_2, p_3, 2p_4 \rangle$

$m(p_1)=1, m(p_2)=1, m(p_3)=1, m(p_4)=2, m(p_5)=0$



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Markings are multisets!

Dynamics of Petri nets (semantics)

Firing a transition

- A transition t = $\langle I, O \rangle$ can be fired from m iff for any place p:

$$m(p) \geq I(p)$$

- The firing transforms the marking m into a marking m' s.t. for any place p:

$$m'(p) = m(p) - I(p) + O(p)$$

- Notation: $m \rightarrow m'$
- Notation: $Post(m) = \{m' \mid m \rightarrow m'\}$

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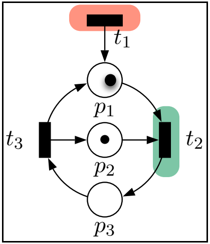
Post(*m*) is the set of markings that can be obtained by firing transitions from *m*

- *Post* corresponds to the transition relation in Transition Systems terminology

Dynamics of Petri nets (semantics)

Example

Post(⟨1, 1, 0⟩)=
{ ⟨2, 1, 0⟩ , ⟨0, 0, 1⟩ }



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Dynamics of Petri nets (semantics)

Example

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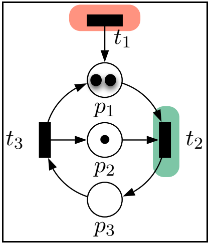

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Dynamics of Petri nets (semantics)

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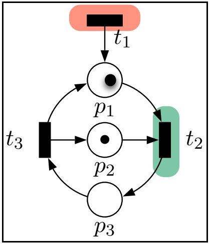

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Dynamics of Petri nets (semantics)

Example

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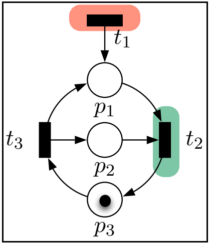

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Dynamics of Petri nets (semantics)

Initial marking
Reachable markings

- All PN are equipped with an initial marking m_0
- If two markings m and m' are s.t.:

$$m \rightarrow m_1 \rightarrow m_2 \rightarrow \cdots \rightarrow m'$$

Then m' is reachable from m

- Let N be a PN with initial marking m_0:

$$Reach(N) = \{m \text{ reachable from } m_0\}$$

is the set of reachable markings of N.

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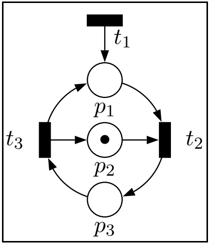
Analogous to initial state and reachable states in Transition Systems terminology

Dynamics of Petri nets (semantics)

Example

$\text{Reach}(\mathcal{N}) =$
$\{\langle i, 1, 0\rangle \mid i \in \mathbb{N}\}$
$\cup$
$\{\langle i, 0, 1\rangle \mid i \in \mathbb{N}\}$

This set allows us to prove that the mutual exclusion is indeed enforced



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Ordering on markings

- Markings can be compared thanks to $\preccurlyeq$:

  $m \preccurlyeq m'$ iff for any place p: $m(p) \leqslant m'(p)$

  $m \prec m'$ iff $m \preccurlyeq m'$ and $m \neq m'$

- Examples:
  - $\langle 1,0,0\rangle \prec \langle 1,1,0\rangle \preccurlyeq \langle 1,1,0\rangle \preccurlyeq \langle 5,7,2\rangle$
  - $\langle 1,0,0\rangle$ is not comparable to $\langle 0,1,0\rangle$

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Structural/Dynamical properties

Questions on PN

- Meaningful questions about PN include:
  - Boundedness: is the number of reachable markings bounded ?
  - Place boundedness: is there a bound on the maximal number of tokens that can be created in a given place ?
  - Semi-liveness: is there a reachable marking from which a given transition can fire ?
  - Coverability

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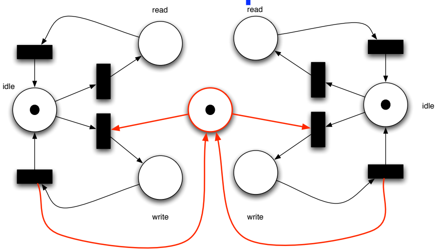
Structural/Dynamical properties

Example



Bounded PN — All the places are bounded

All the transitions are semi-live

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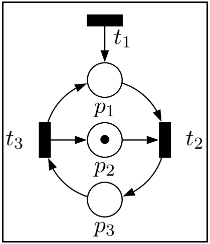
Structural/Dynamical properties

Example

- Unbounded PN
- p_2 and p_3 are bounded
- p_1 is unbounded
- All the transitions are semi-live



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Reachability graph

Reachability graph

- Idea: build a node for each reachable marking and add an edge from m to m' if some transition transforms m into m'
  - remark: now, if we meet the same marking twice, we do not create a new node, but re-use the previously created node.

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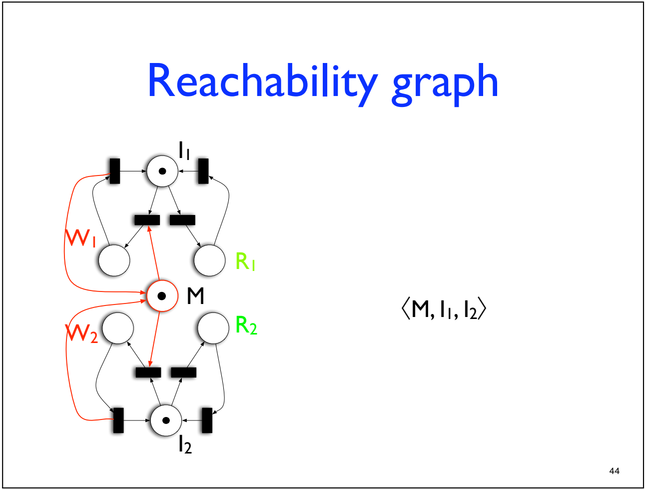
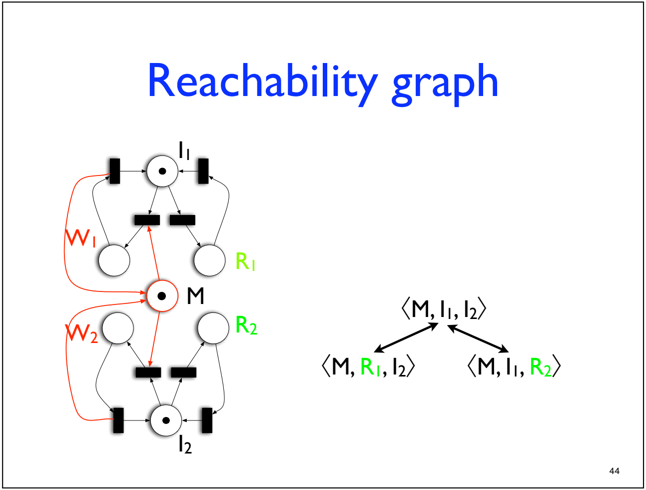
It is a Transition System

- Exactly the same Transition System that would be obtained from the MultiSet Rewriting representation of the Petri net

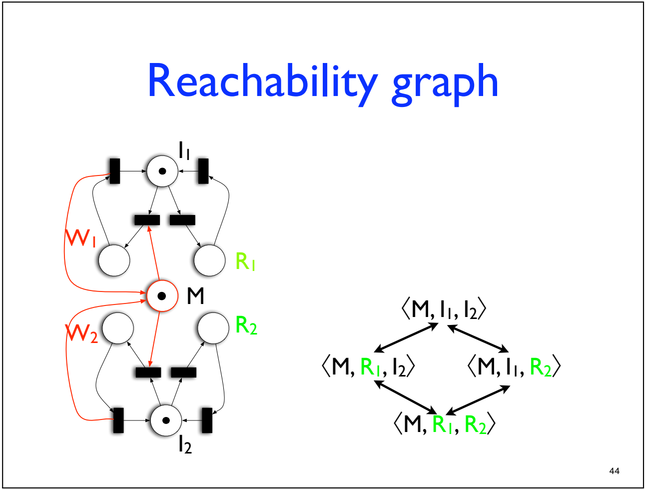
Reachability graph



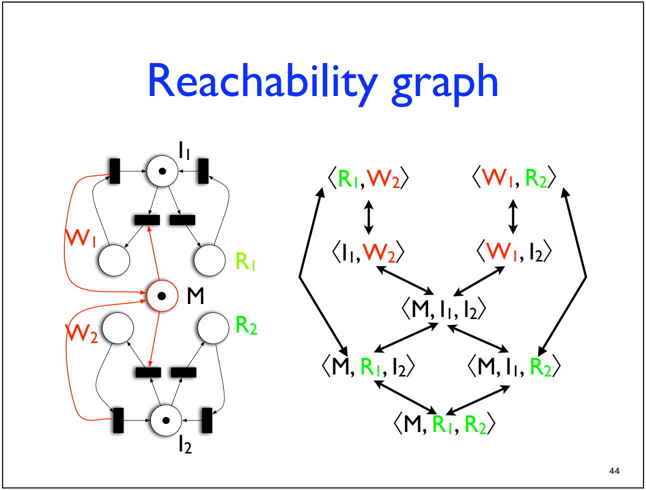
Reachability graph



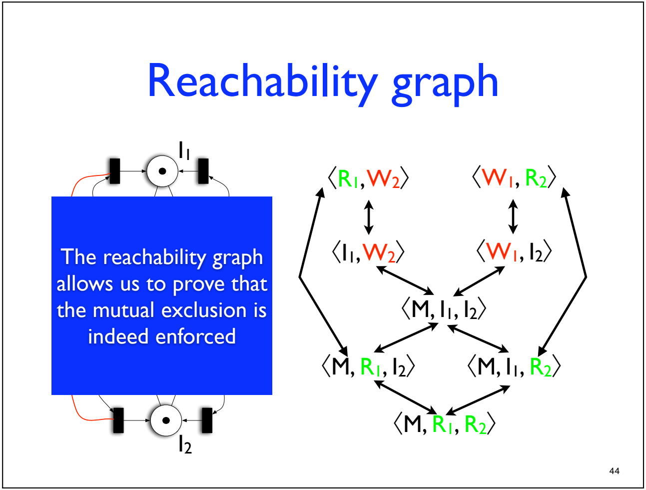
Reachability graph



Reachability graph



Reachability graph



Reachability graph

Reachability graph

- The reachability graph of a PN contains all the necessary information to decide:
  - boundedness
  - place boundedness
  - semi-liveness
  - ...

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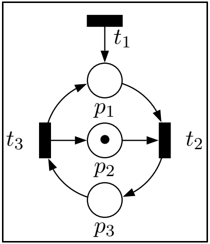
Reachability graph

Reachability graph

- Unfortunately...

  ⟨p2⟩



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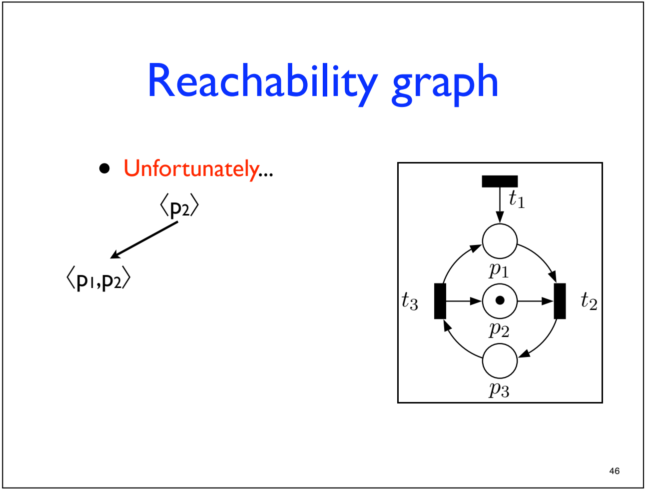
Reachability graph



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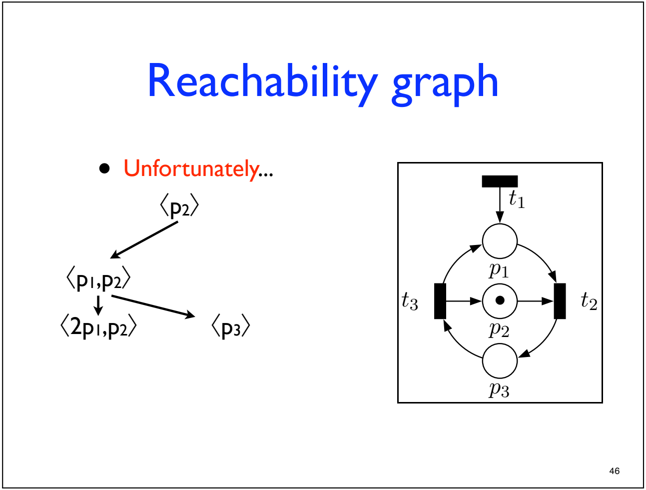
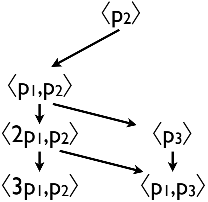
Reachability graph

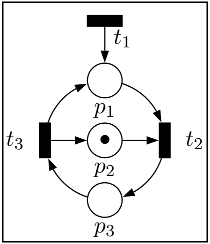


Reachability graph

Reachability graph

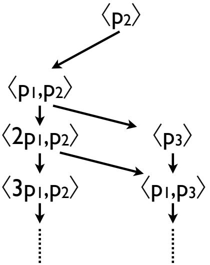
- Unfortunately...

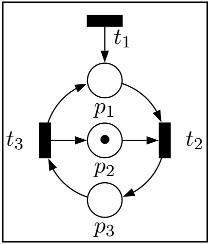




Reachability graph

- Unfortunately...





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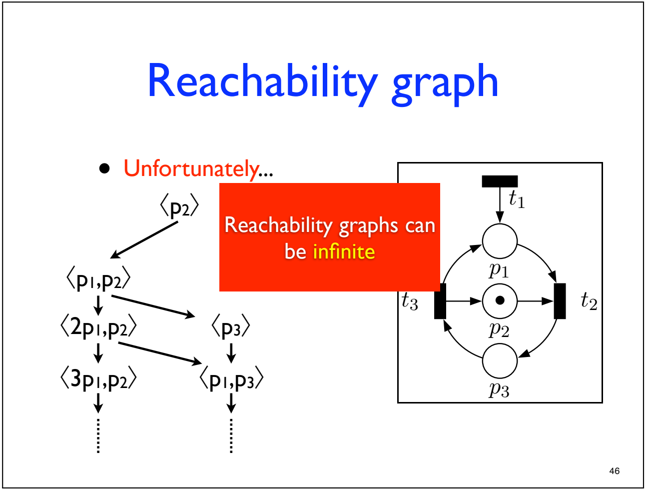
Reachability graph



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Infinite, but decidable!

The reachability graph of a Petri net (aka the Transition System of a MultiSet Rewriting system) can be infinite

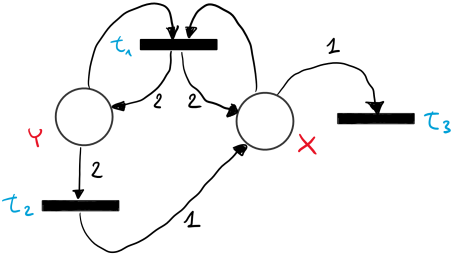
But, the reachability property (i.e. is a given state/marking reachable?) is DECIDABLE

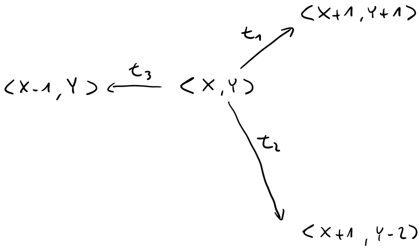
- Its computation has been proven to require EXPONENTIAL time

The reason for decidability is that the reachability graph has a regular structure!

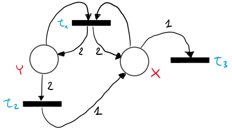
Infinite, but decidable!

For example:

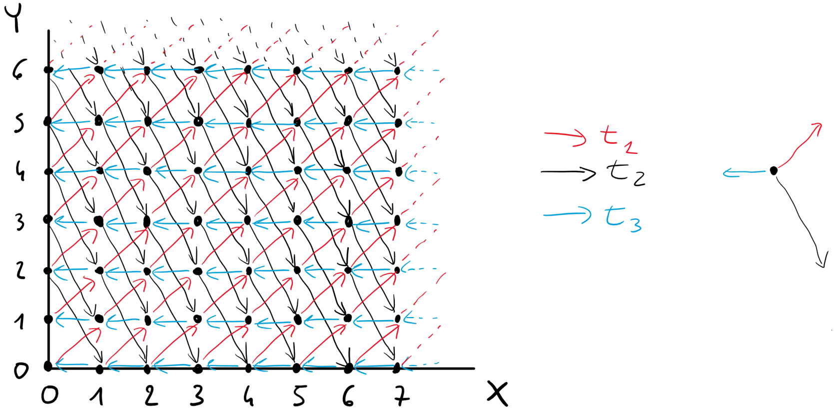




Infinite, but decidable!



The reachability graph, plotted on the cartesian plane:



Infinite, but very regular!

Decidable, but exponential!

Reachability of a marking M is decidable, but exponential...

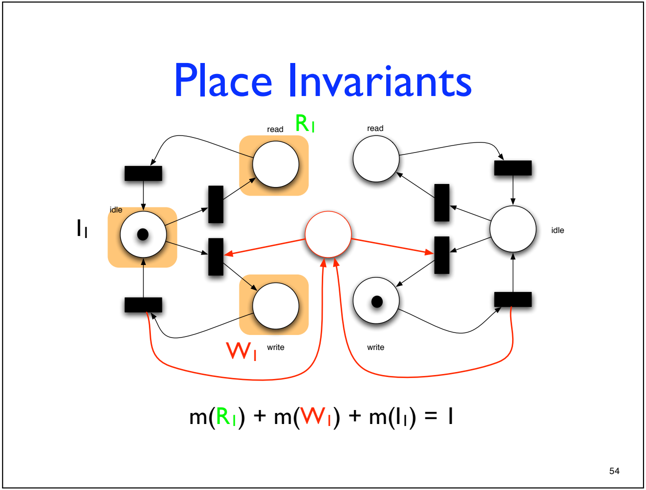
SOLUTION 1: consider overapproximations of the set of reachable states

1. based on Place Invariants
2. based on Karp and Miller tree

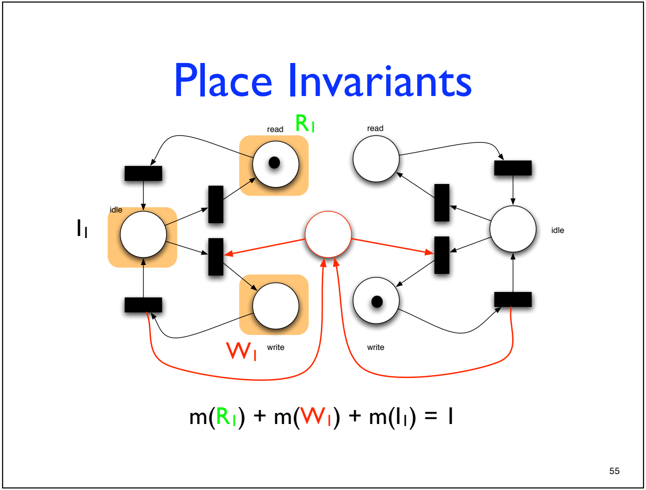
If M does not belong to the overapproximation it is not reachable (this is a sufficient condition: if M belongs, nothing can be said...)

SOLUTION 2: consider coverability instead of reachability (weaker than reachability, but meaningful in the context of Petri Nets and easy to compute)

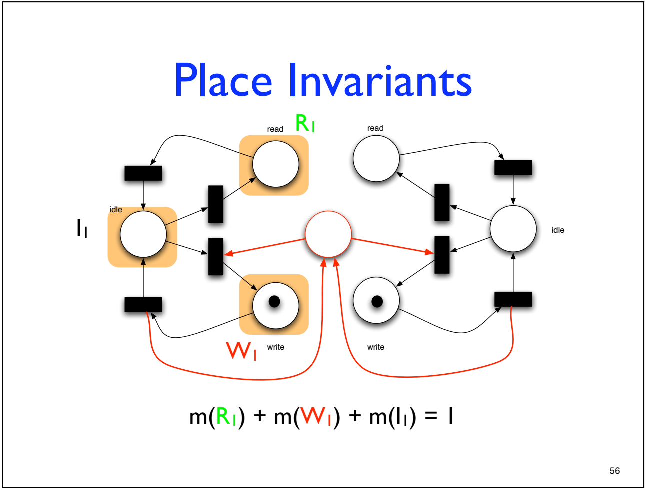
Place Invariants



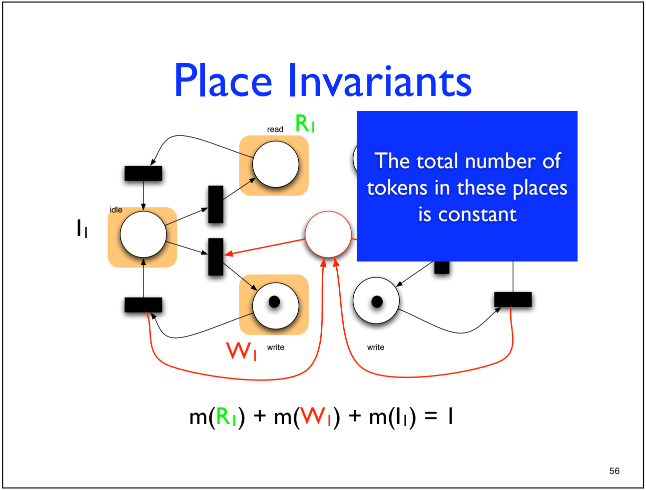
The idea is to identify groups of places whose overall number tokens is (more or less) constant


Place Invariants
read
R_I
read
idle
I_I
idle
W_I
write
write
$m(R_I) + m(W_I) + m(I_I) = 1$
55

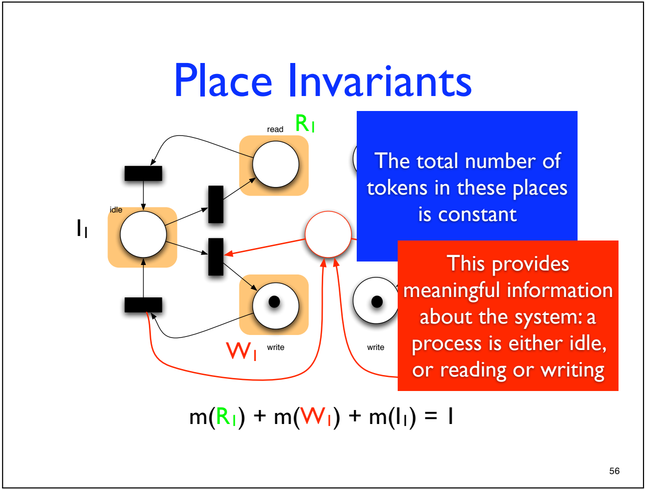
Place Invariants



Place Invariants



Place Invariants



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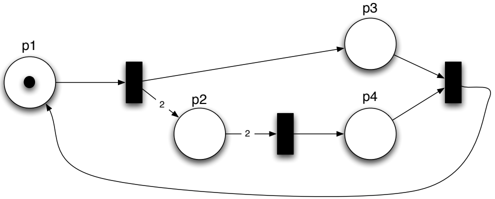
Place Invariants

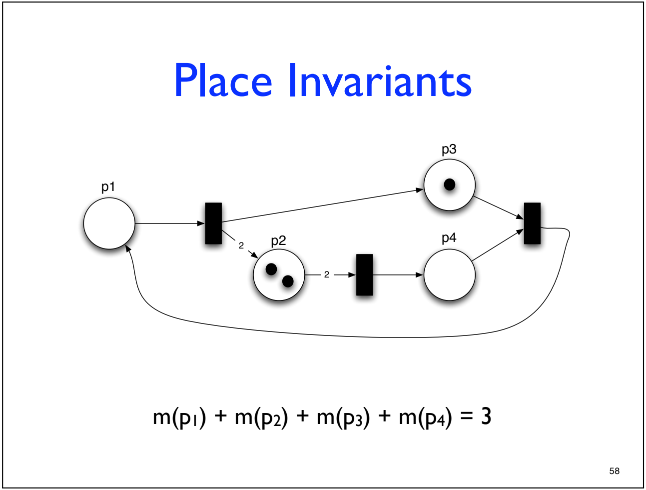
Place Invariants



$$m(p_1) + m(p_2) + m(p_3) + m(p_4) = 1$$

57

Place Invariants



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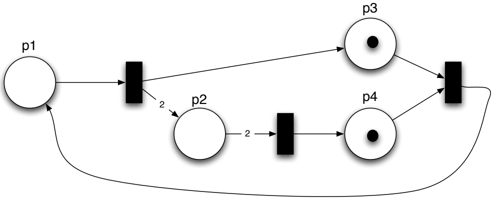
Place Invariants

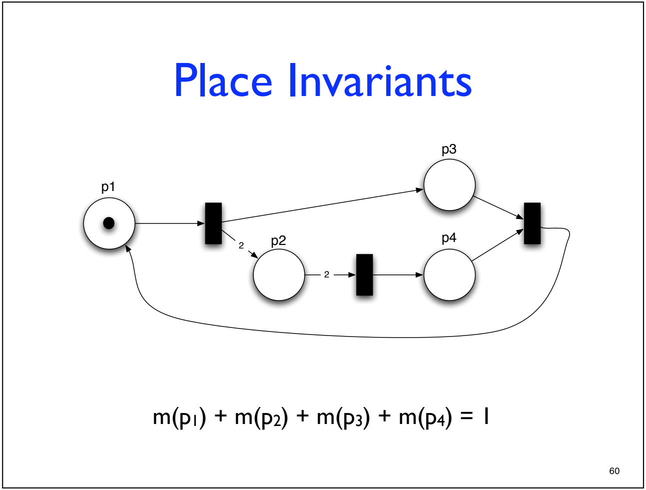
Place Invariants



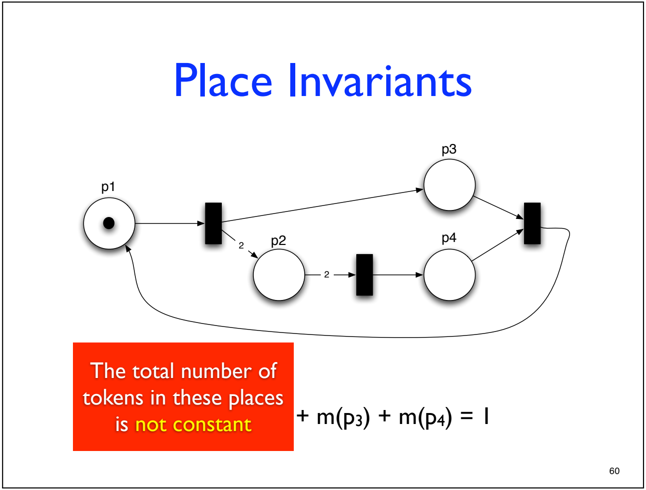
$$m(p_1) + m(p_2) + m(p_3) + m(p_4) = 2$$

59

Place Invariants



Place Invariants



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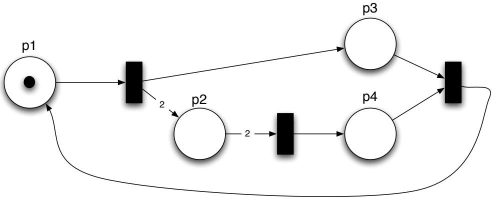
Place Invariants

Place Invariants



The total number of tokens in these places is not constant

+ m(p

In some sense, tokens in p_1 are heavier than those in p_2

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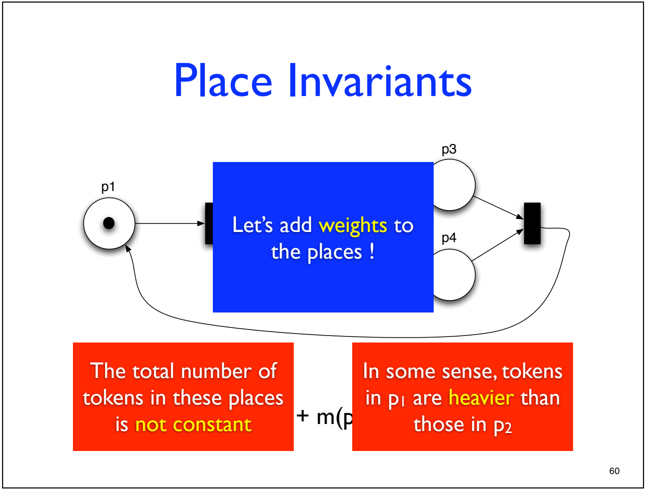
Place Invariants



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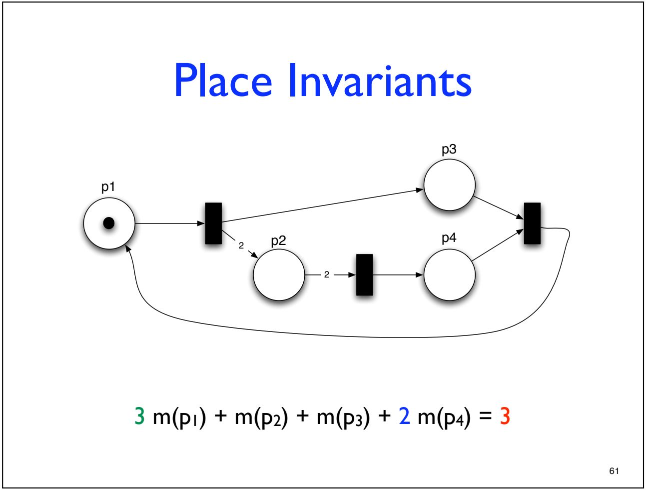
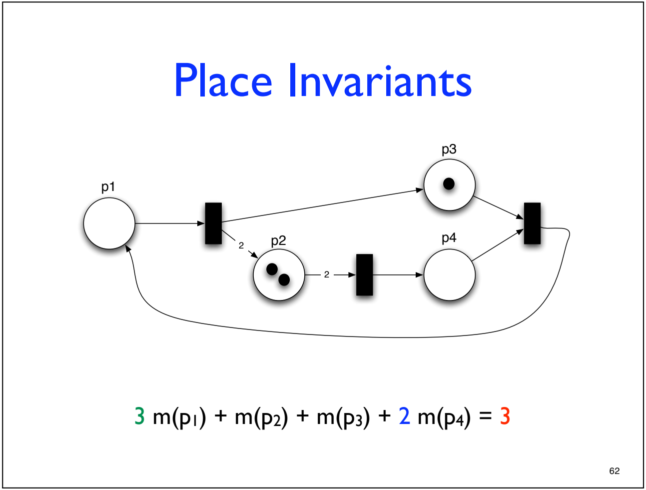
Place Invariants

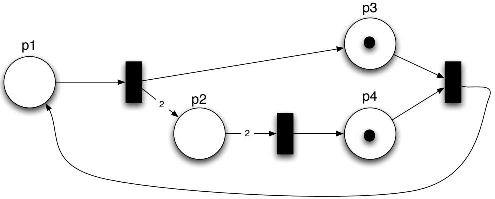


Place Invariants



Place Invariants

Place Invariants



$$3\, m(p_1) + m(p_2) + m(p_3) + 2\, m(p_4) = 3$$

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Place invariant: Definition

- Definition: a place-invariant (or p-semiflow) is a vector i of natural numbers s.t. for any reachable marking m:

$$\sum_{p \in P} i(p) \times m(p) = \sum_{p \in P} i(p) \times m_0(p)$$

remark: there exists a trivial invariant i = ⟨0, 0, .., 0⟩

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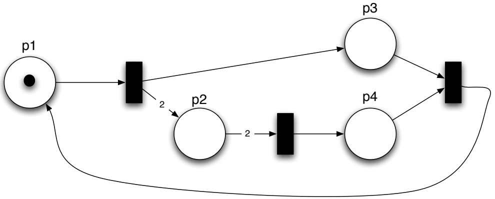
Corresponds to the notion of mass conservation in (bio)chemistry

"Matter can never be created, nor destroyed..."

Place Invariants

Example: other invariants



$$m(p_1) + m(p_3) = 1$$

$$2\, m(p_1) + m(p_2) + 2\, m(p_4) = 2$$

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Invariants as over-approximations

- A place-invariant expresses a constraint on the reachable markings.
  - If m is reachable and i is an invariant, then:

$$\sum_{p \in P} i(p) \times m(p) = \sum_{p \in P} i(p) \times m_0(p)$$

- The reverse is not true !

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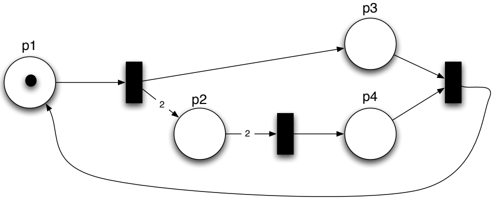
Place Invariants

Example



$m(p_1) + m(p_3) = 1$

is an invariant

but $\langle 1, 25, 0, 234 \rangle$ is not reachable

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Place Invariants

Invariants as over-approximations

- Theorem: For any Petri net N:

$$\text{Reach}(N) \subseteq \{m \mid m \text{ respects some invariant of } N\}$$

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So, every marking that does not respect the invariant is not reachable!

We do not need to explore the reachability graph!

Place invariant and boundedness

- Theorem: If there exists a place invariant i and a place p s.t. i(p)>0 then p is bounded.
- Remark: the reverse is not true.
  - One can find a bounded net that doesn't have a place invariant i with i(p)>0 for each place.

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Place Invariants

Place invariant

- Question: how do we compute them ?

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Matrix characterisation

- The negative effect (consumption) of all the transitions on all the places can be summarised in one matrix:

$$W^- = \begin{pmatrix} I_1(p_1) & I_2(p_1) & \cdots & I_k(p_1) \\ I_1(p_2) & I_2(p_2) & \cdots & I_k(p_2) \\ \vdots & \vdots & \ddots & \vdots \\ I_1(p_n) & I_2(p_n) & \cdots & I_k(p_n) \end{pmatrix}$$

neg. eff. on p_1

neg. eff. on p_2

where, for any i: $t_i = \langle I_i, O_i \rangle$

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Matrix characterisation

- The same can be done with the positive effects:

$$W^+ = \begin{pmatrix} O_1(p_1) & O_2(p_1) & \cdots & O_k(p_1) \\ O_1(p_2) & O_2(p_2) & \cdots & O_k(p_2) \\ \vdots & \vdots & \ddots & \vdots \\ O_1(p_n) & O_2(p_n) & \cdots & O_k(p_n) \end{pmatrix}$$

pos. eff. on p_1

pos. eff. on p_2

where, for any i: $t_i = \langle I_i, O_i \rangle$

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Matrix representation of Petri nets

Incidence Matrix

- The global effect of every transition can be summarised as a single matrix:

$$W = W^{+} - W^{-}$$

W is called the incidence matrix of the net

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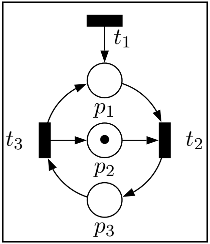
Matrix representation of Petri nets

Example

$$W^+ = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad W^- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$W = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$



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If you think the Petri net as a graph, the incidence matrix is a standard representation...

... more or less, since rows and columns should be defined for all graph nodes (6x6 matrix, in the example)...

... but a Petri net is a bipartite graph

Matrix representation of Petri nets

Example

$$W^+ = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad W^- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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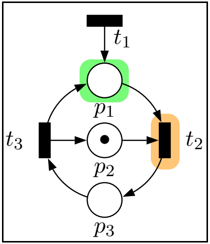

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Place invariants

Computing place invariants

- Intuitively, if i is a place invariant it should assign weights to the places such that the positive and negative effects of every transition are balanced
- Thus, for any transition t = $\langle I, O \rangle$ we should have:

$$\sum_{p \in P} I(p) \times i(p) = \sum_{p \in P} O(p) \times i(p)$$

75

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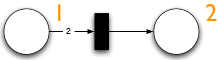

75

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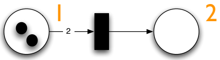

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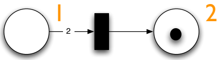

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Place invariants

Computing place invariants

$$\sum_{p \in P} I(p) \times i(p) = \sum_{p \in P} O(p) \times i(p)$$

means

$$\sum_{p \in P} \big(O(p) - I(p)\big) \times i(p) = 0$$

76

Place invariants

Computing place invariants

$$\sum_{p \in P} I(p) \times i(p) = \sum_{p \in P} O(p) \times i(p)$$

means

$$\sum_{p \in P} \big(O(p) - I(p)\big) \times i(p) = 0$$

t = $\langle$I, O$\rangle$

$$W = \begin{pmatrix} \cdots & O(p_1) - I(p_1) & \cdots \\ \cdots & O(p_2) - I(p_2) & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & O(p_n) - I(p_n) & \cdots \end{pmatrix}$$

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Place invariants

Computing place invariants

$$\sum_{p \in P} \left(O(p) - I(p) \right) \times i(p) = 0$$

is thus the scalar product of i and the column of W that corresponds to transition t

77

Place invariants

Computing place invariants

$$\sum_{p \in P} \big(O(p) - I(p)\big) \times i(p) = 0$$

is thus the scalar product of i and the column of W that corresponds to transition t

Since this must hold for any t, we obtain:

Theorem: any solution i to the following system of equations is a place-invariant:

$$i \times W = 0$$

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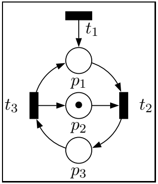
Place invariants

Example



$$W = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

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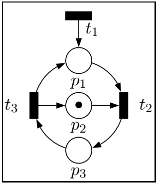
Place invariants

Example



$$W = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\langle i_1, i_2, i_3 \rangle \times W = 0$$

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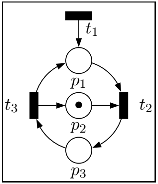
Place invariants

Example



$$W = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\langle i_1, i_2, i_3 \rangle \times W = 0$$

$$\begin{cases} i_1 = 0 \\ -i_1 - i_2 + i_3 = 0 \\ i_1 + i_2 - i_3 = 0 \end{cases}$$

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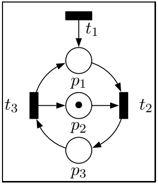
Place invariants

Example



$$W = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

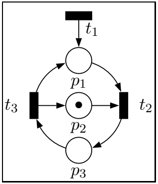
$$\langle i_1, i_2, i_3 \rangle \times W = 0$$

$$\begin{cases} i_1 = 0 \\ -i_1 - i_2 + i_3 = 0 \\ i_1 + i_2 - i_3 = 0 \end{cases} \qquad \begin{cases} i_1 = 0 \\ -i_2 + i_3 = 0 \\ +i_2 - i_3 = 0 \end{cases}$$

78

Place invariants

Example



$$W = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ & & -1 \end{pmatrix}$$

Any vector of the form
⟨0, i, i⟩
is a place invariant

$$\langle i_1, i_2, i_3\rangle \times W = 0$$

$$\begin{cases} i_1 = 0 \\ -i_1 - i_2 + i_3 = 0 \\ i_1 + i_2 - i_3 = 0 \end{cases} \qquad \begin{cases} i_1 = 0 \\ -i_2 + i_3 = 0 \\ +i_2 - i_3 = 0 \end{cases}$$

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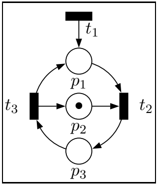
Place invariants

Proving properties



Let us choose $\langle 0, 1, 1 \rangle$ as place-invariant

This means that p_2 and p_3 are bounded !

For any reachable marking m:

$$0\, m(p_1) + 1\, m(p_2) + 1\, m(p_3) = 0\, m_0(p_1) + 1\, m_0(p_2) + 1\, m_0(p_3)$$

$$m(p_2) + m(p_3) = 1$$

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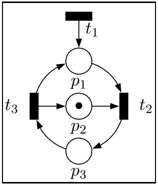
Place invariants

Proving properties



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This means that p_2 and p_3 are bounded !

For any reachable marking m:

$$0\, m(p_1) + 1\, m(p_2) + 1\, m(p_3) = 0\, m_0(p_1) + 1\, m_0(p_2) + 1\, m_0(p_3)$$

$$m(p_2) + m(p_3) = 1$$

Hence, mutual exclusion is enforced !

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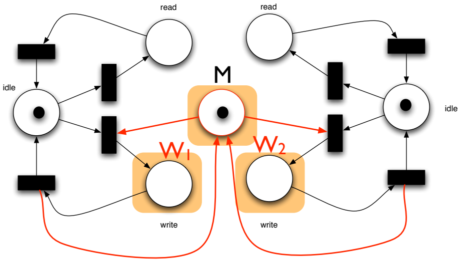
The marking

$$m = \langle p_2, p_3 \rangle$$

is not reachable

Place invariants

Proving properties



i(M) = i(W1) = i(W2) = 1 and i(p) = 0 otherwise
is a place invariant

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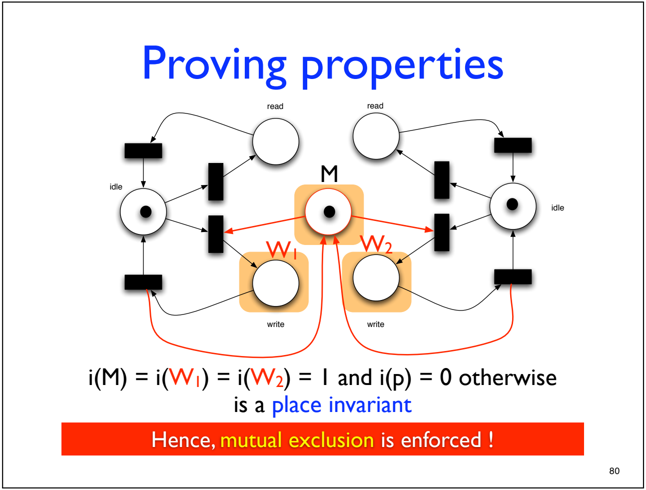
Place invariants



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The marking

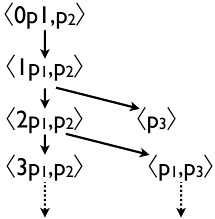
$$m = \langle W_1, W_2 \rangle$$

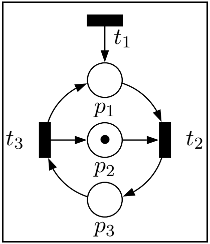
is not reachable

Karp and Miller tree

The reachability tree revisited

- Reminder: reachability trees can be infinite





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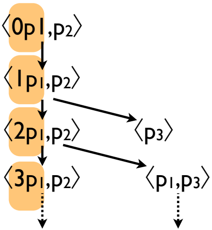
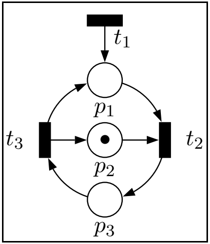
Karp and Miller tree

The reachability tree revisited

- Reminder: reachability trees can be infinite





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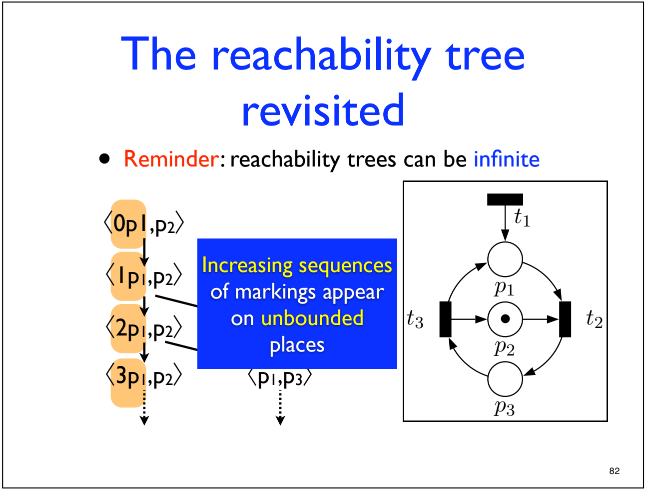
Karp and Miller tree



Karp and Miller tree

The reachability tree revisited

- Let us summarise this infinite sequence

$\langle 0p_1, p_2 \rangle$
↓
$\langle 1p_1, p_2 \rangle$
↓
$\langle 2p_1, p_2 \rangle$
↓
$\langle 3p_1, p_2 \rangle$
⋮

limit → $\langle \omega p_1, p_2 \rangle$

ω must be regarded as: "any number of tokens"

Main idea of the Karp and Miller algorithm

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Karp and Miller tree

Karp & Miller

- Propose in 1969 a solution to detect unbounded places of a Petri net

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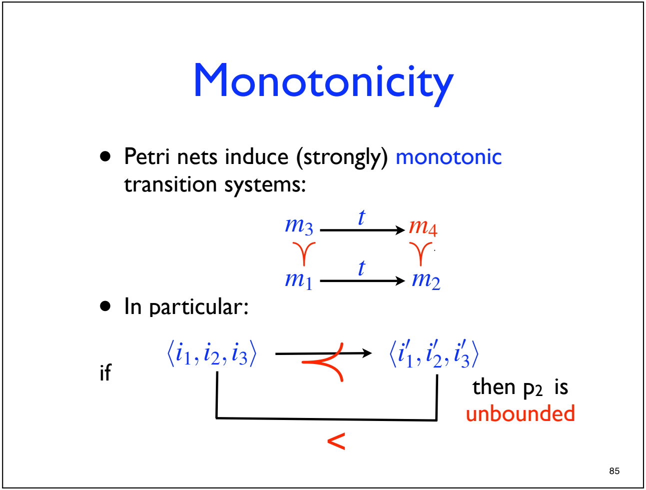
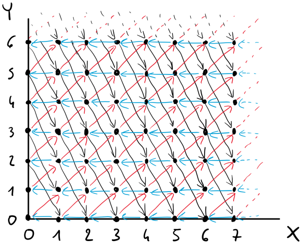
Karp and Miller tree



This is clearer on the cartesian plane representation:



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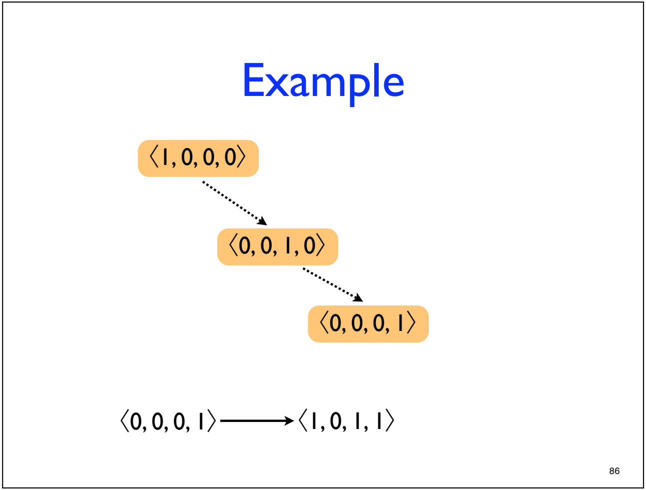
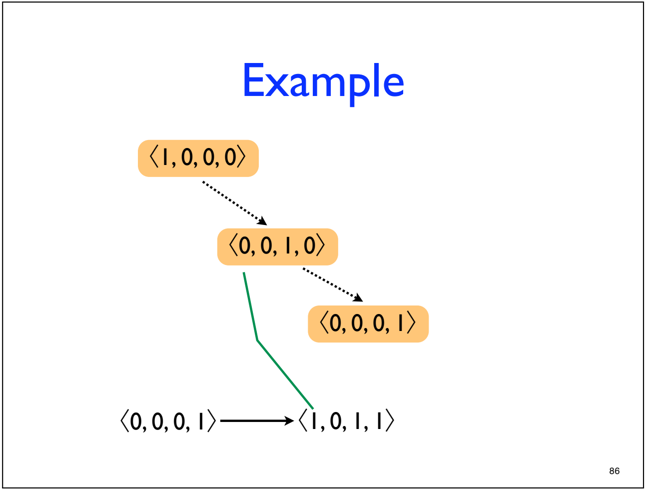
Karp and Miller tree



Karp and Miller tree



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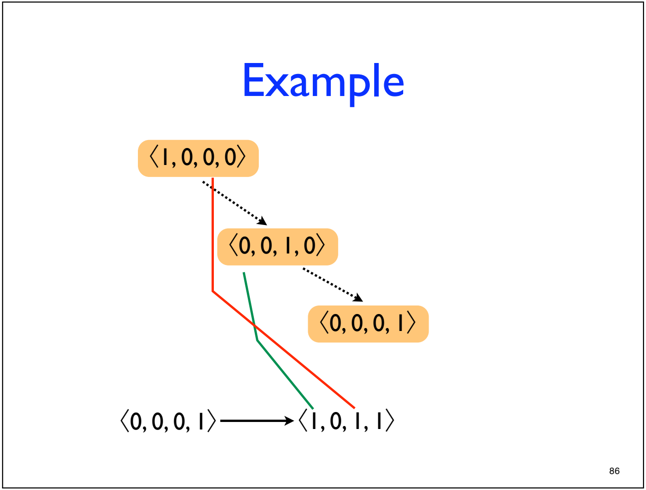
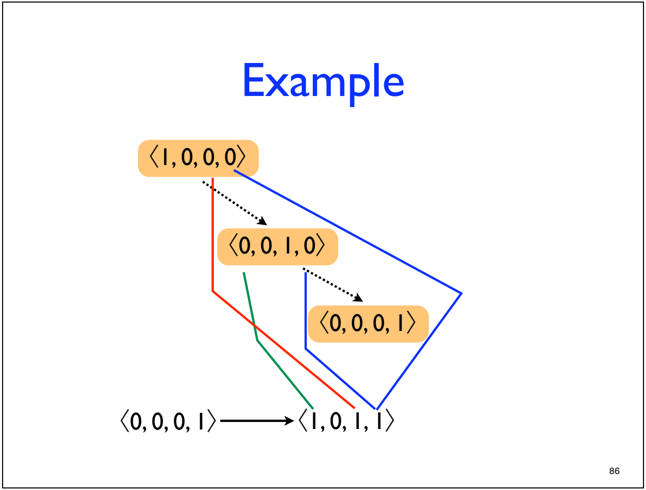
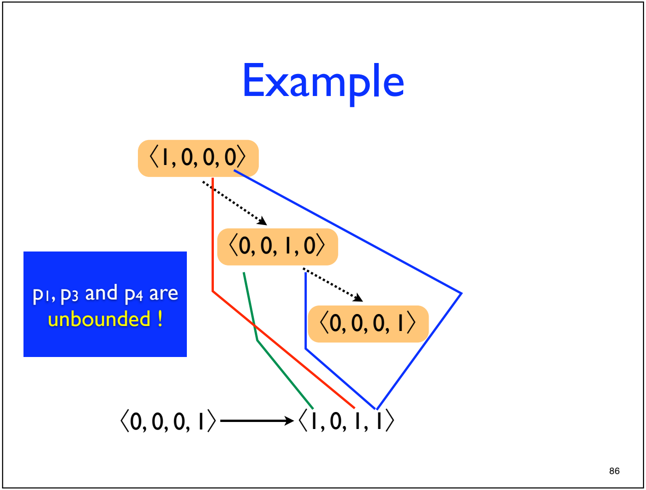
Karp and Miller tree



Karp and Miller tree



Karp and Miller tree



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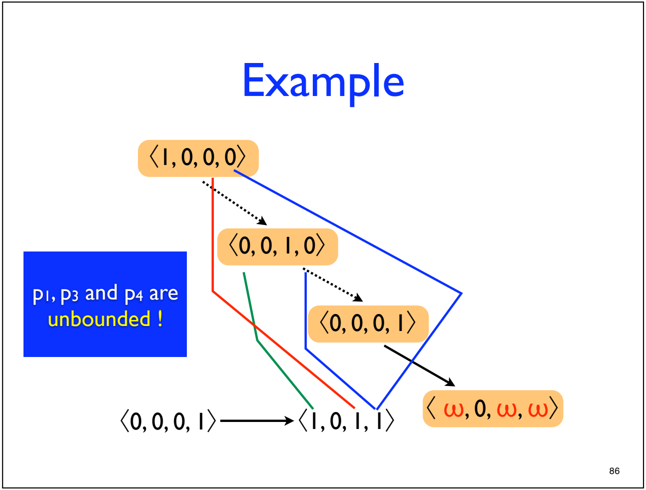
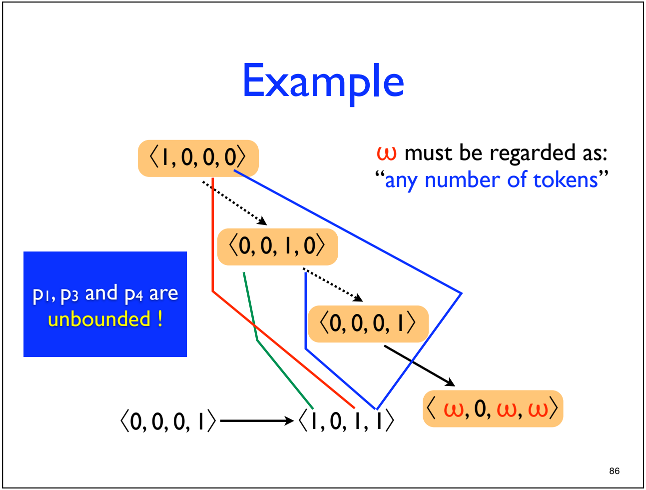
Karp and Miller tree



Karp and Miller tree

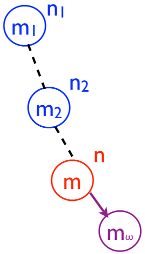


Karp and Miller tree

Karp & Miller Acceleration

This is how we compute the successors of a node n:

foreach *Successor* m' *of* m **do**
  $m_\omega \leftarrow m'$;
  foreach *ancestor* n_i *s.t.* $m_i \prec m'$ **do**
    foreach *place* p *s.t.* $m_i(p) < m'(p)$ **do**
      $m_\omega(p) \leftarrow \omega$;
  Add m_ω as child of n;



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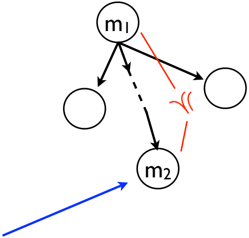
Karp and Miller tree

Karp & Miller
Stopping a branch



This node doesn't have to be developed

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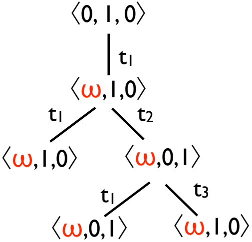
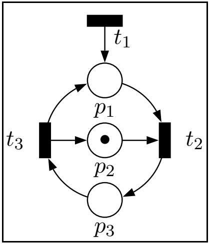
Karp and Miller tree

Example of K&M tree





$$(0,1,0) \xrightarrow{t_1} (1,1,0) \succ (0,1,0)$$

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Properties

- Theorem: the K&M tree is always finite.
  - Idea of the proof:
    - if the net is not bounded, it is because of some infinite increasing sequence of markings.
    - such sequences are detected in a finite amount of time by adding ω in the unbounded places.

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Karp and Miller tree

Properties

- Theorem: a net is bounded iff there is no node containing an ω in its K&M tree.
- Theorem: place p is unbounded iff there exists a node labeled by m in the K&M tree s.t. $m(p) = \omega$.
- Theorem: transition t is semi-live iff there exists a node labeled by m in the K&M tree s.t. t can fire in m.

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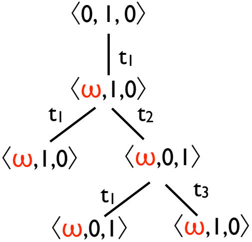
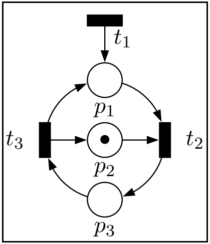
Karp and Miller tree

Example





t_2 is semi-live

p_2 and p_3 are bounded

p_1 is unbounded

The net is unbounded

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Coverability set

Coverability set

- Question: what is the relationship between:
  - the set of reachable markings and
  - the set of labels of the nodes of the K&M tree ?

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Coverability set

Coverability set

- Question: what is the relationship between:
  - the set of reachable markings and
  - the set of labels of the nodes of the K&M tree ?

might be infinite

always finite

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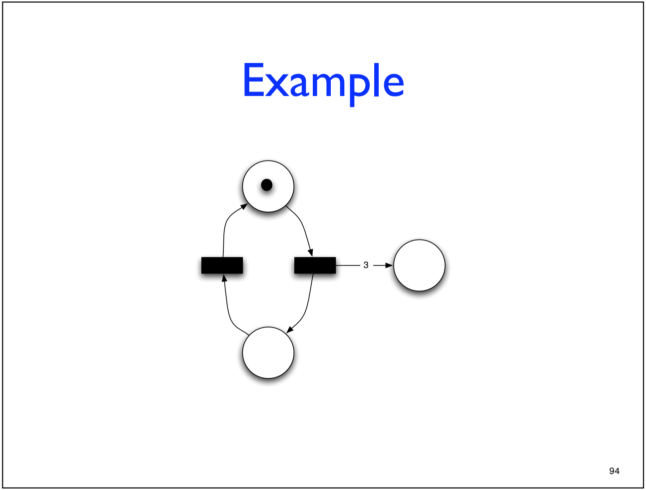
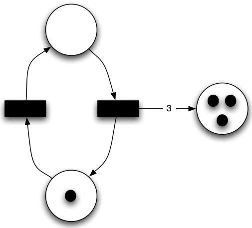
Coverability set



Coverability set

Example



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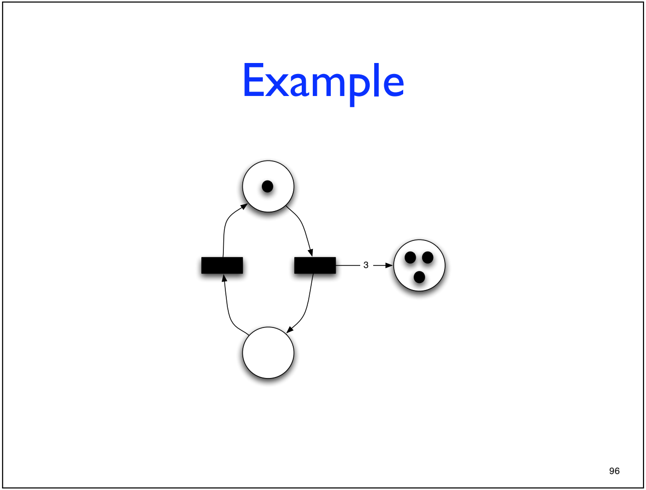
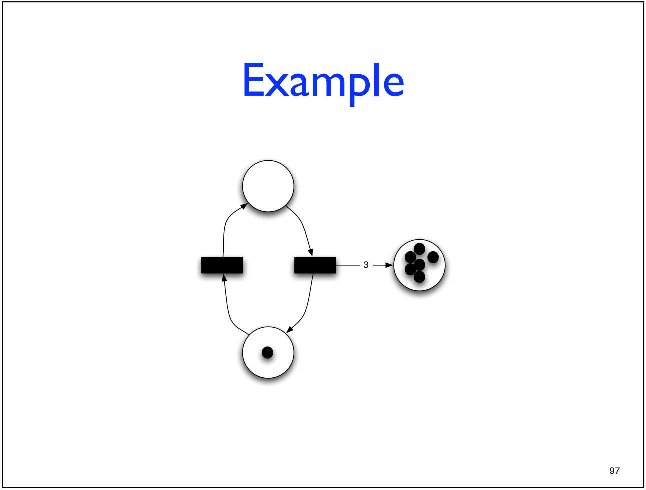
Coverability set



Coverability set



Coverability set

Example

- Set of reachable markings:

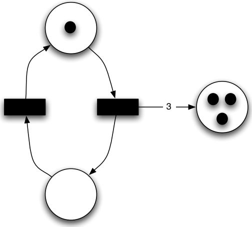
$\{ \langle 1, 0, 3.i \rangle \; , \; \langle 0, 1, 3.i \rangle \;\; | \; i \geqslant 0 \}$

- Set of nodes of the K&M tree:

$\{ \langle 1, 0, 0 \rangle \;\; \langle 1, 0, \omega \rangle \; , \; \langle 0, 1, \omega \rangle \; \}$

- This set "represents":

$\{ \langle 1, 0, i \rangle \; , \; \langle 0, 1, i \rangle \;\; | \; i \geqslant 0 \}$



Clearly: [orange box] ≠ [green box]

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Coverability set

Example

Reach: $\{ \langle 1, 0, 3.i \rangle , \langle 0, 1, 3.i \rangle \mid i \geqslant 0 \}$ vs K&M: $\{ \langle 1, 0, i \rangle , \langle 0, 1, i \rangle \mid i \geqslant 0 \}$

- Clearly, the K&M set contains more markings than the set of reachable markings:

$$\subseteq$$

- However, for every marking m in the K&M set, there exists a reachable marking m' s.t.:

$$m' \succcurlyeq m$$

= + {m| there is m' in with m' $\succcurlyeq$ m}

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Coverability set

Downward-closure

- Let us assume that any natural number i is s.t.

$$i < \omega$$

- Let m be a marking (possibly with ω), then its downward-closure is the set:

$$\downarrow m = \{m' \mid m' \preccurlyeq m\}$$

- Let $S=\{m_1, m_2, \ldots m_k\}$ be a set of markings, then:

$$\downarrow S = \downarrow m_1 \cup \downarrow m_2 \cup \ldots \cup \downarrow m_k$$

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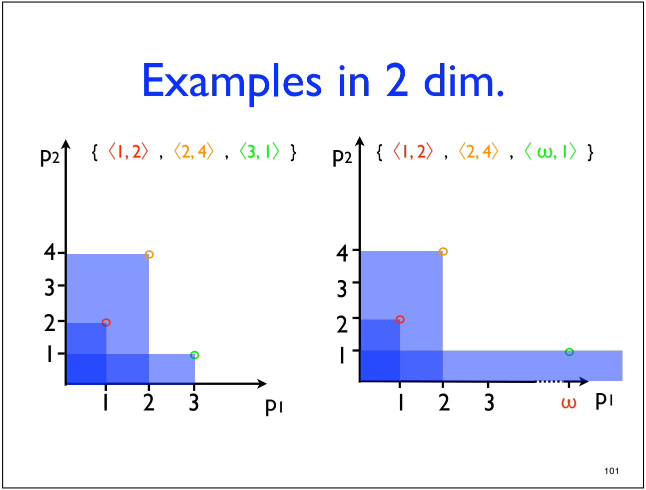
Coverability set



Coverability set

Properties of the K&M tree

- The set of all the markings that appear in a K&M tree is called a coverability set of the net.
  - Notation: Cover(N)
- Theorem: $\downarrow$ Cover(N) = $\downarrow$ Reach(N)
- Theorem: Reach(N) $\subseteq$ $\downarrow$ Cover(N)
- Hence, $\downarrow$ Cover(N) is a finite over-approximation of Reach(N)

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$\downarrow Cover(N)$ is another overapproximation of the set of reachable markings

If a marking is not in $\downarrow Cover(N)$, it is not reachable!

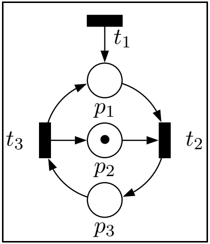
- again: sufficient condition!

Coverability set

Example

Reach(N)
=
{ ⟨i, 1, 0⟩ , ⟨i, 0, 1⟩ | i ≥ 0 }

Cover(N)
=
↓{ ⟨ω, 1, 0⟩ , ⟨ω, 0, 1⟩ }
=
Reach(N) ∪ { ⟨0, 0, 0⟩ }



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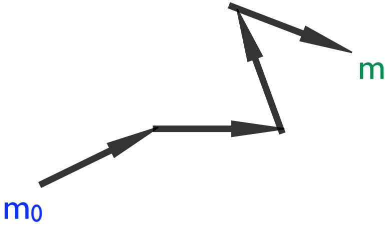
An alternative to reachability

Reachability: a natural question

- The reachability problem: given a marking m is it reachable from m_0 ?



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An alternative to reachability

Reachability: a natural question ??

- In the case of Petri nets, asking whether a given marking is reachable does not always make sense...
- ... because Petri nets are monotonic

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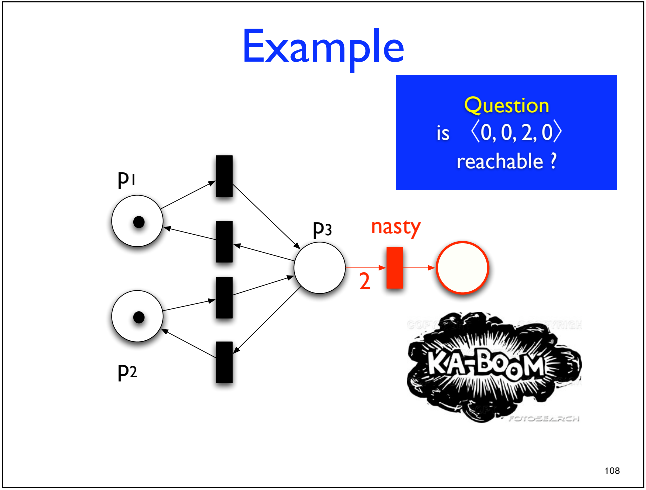
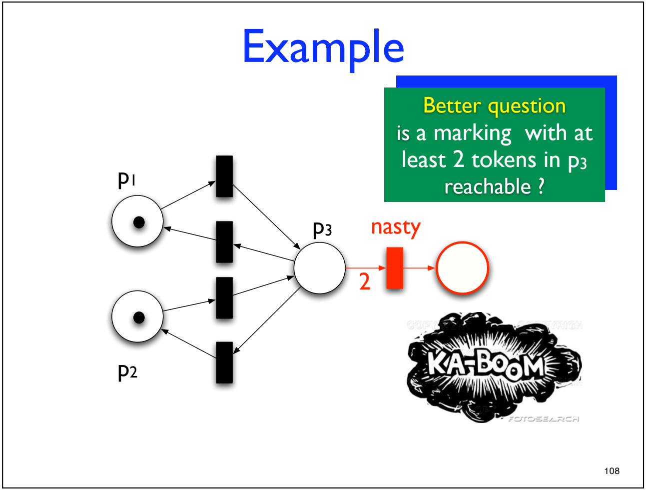
An alternative to reachability



An alternative to reachability



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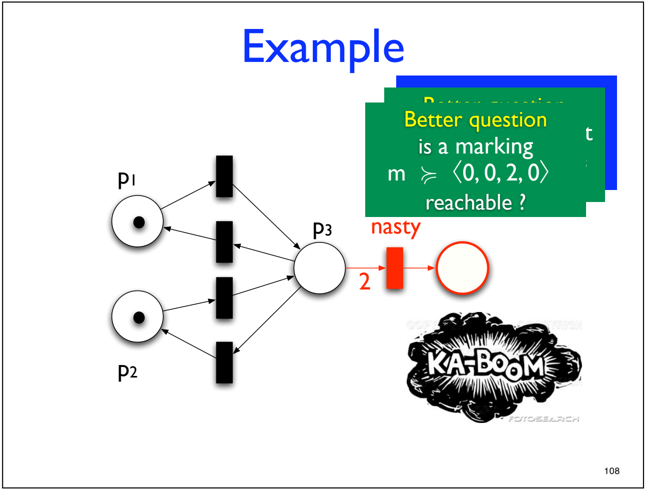
An alternative to reachability



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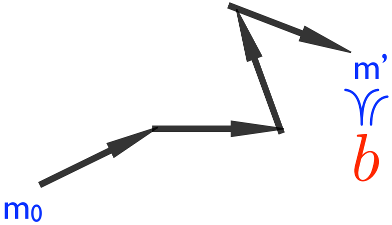
The coverability problem

The coverability problem

Does there exist a reachable marking which is larger than some marking b ?



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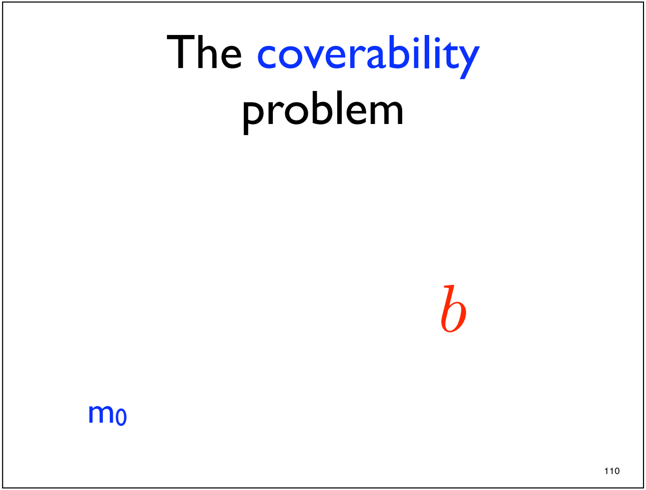
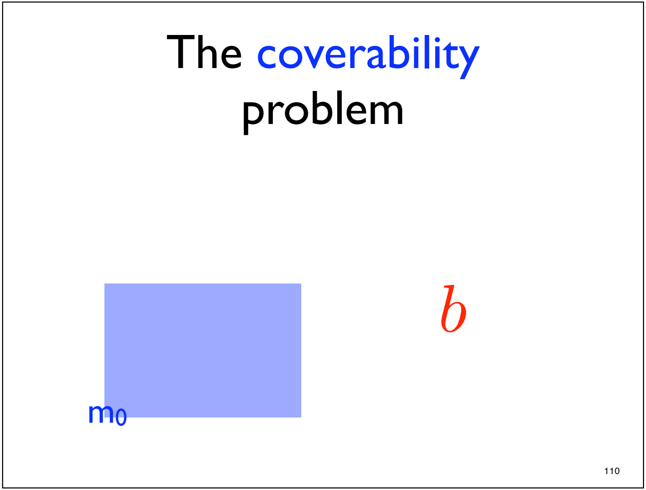
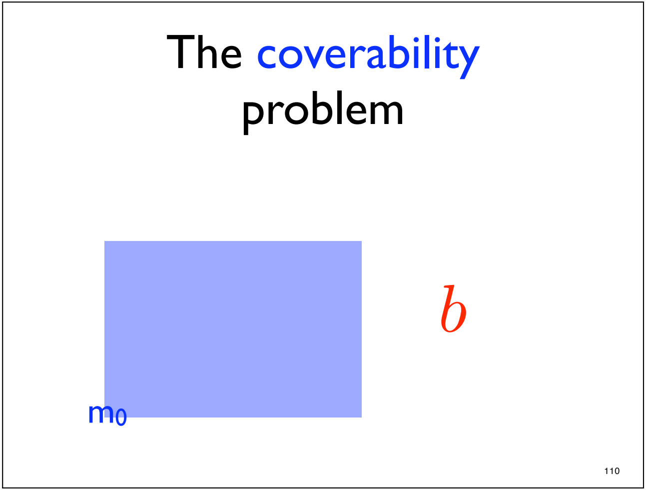
The coverability problem




The coverability problem
b
m0
110


The coverability
problem
b
m0
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The coverability problem

The coverability problem

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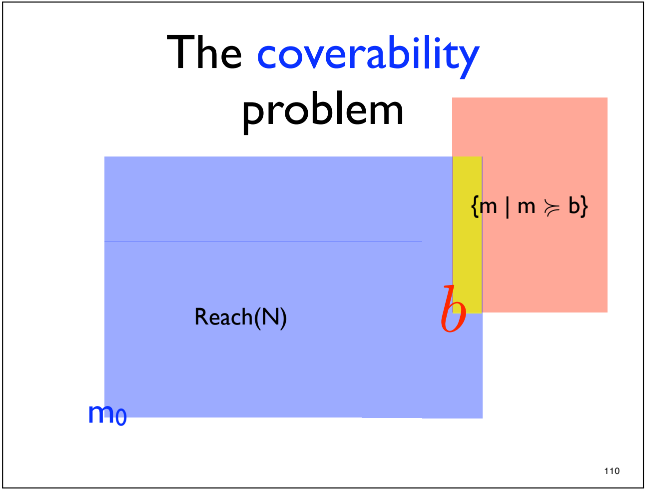
The coverability problem



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The coverability problem

The coverability problem

- Two alternative definitions:
  - Is there a reachable marking m s.t. $m \succcurlyeq b$?
  - Does $Reach(N) \cap \{m \mid m \succcurlyeq b\} \neq \emptyset$?

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The coverability problem

Coverability: a natural question (indeed)

- Coverability might be regarded as the most natural reachability question in the framework of Petri nets
- Besides, coverability is much more easily solved than reachability

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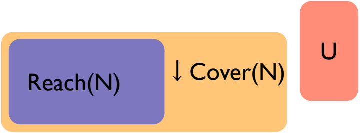
The coverability problem

First idea

- Use the coverability set !
- Remember: the coverability set over-approximates the reachable states:

$$Reach(N) \subseteq \downarrow Cover(N)$$



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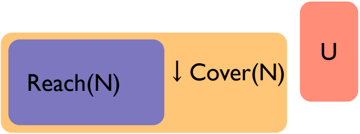
The coverability set can be computed (easily) by using the Karp and Miller algorithm

The coverability problem

First idea



$\downarrow \text{Cover}(N) \cap U = \emptyset$

implies

$\text{Reach}(N) \cap U = \emptyset$

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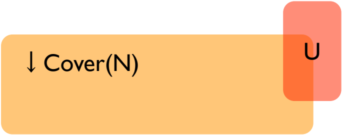
The coverability problem

What if ?



- There is m in ↓ Cover(N) ∩ U
- Hence, there is m' $\succcurlyeq$ m which is in Reach(N)
- However, any m' $\succcurlyeq$ m is also in U
- Thus, there is m' both in Reach(N) and U

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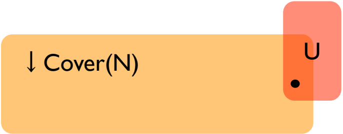
The coverability problem

What if ?



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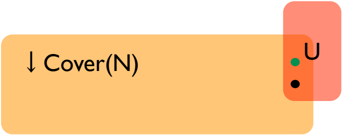
The coverability problem

What if ?



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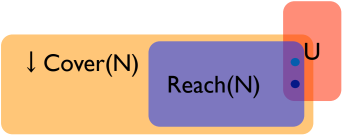
The coverability problem

What if ?



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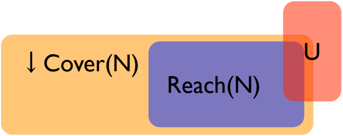
NOTE: $m' > m$ since $\downarrow Cover = \downarrow Reach$

The coverability problem

What if ?



Reach(N) ∩ U = ∅

implies

↓ Cover(N) ∩ U = ∅

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The coverability problem

Coverability set and coverability problem

- Theorem:

  Reach(N) ∩ U = ϕ iff ↓ Cover(N) ∩ U = ϕ

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Summing up:
In order to check whether a marking in U is reachable, we can use the Karp and Miller approach to compute the $\downarrow Cover(N)$ set, and check whether it has a non-empty interesection with U!