On Bunched Predicate Logic

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Abstract

We present the logic of bunched implications, **BI**, in which a multiplicative (or linear) and an additive (or intuitionistic) implication live side-by-side. The propositional version of **BI** arises from an analysis of the proof-theoretic relationship between conjunction and implication, and may be viewed as a merging of intuitionistic logic and multiplicative, intuitionistic linear logic. The predicate version of **BI** includes, in addition to usual additive quantifiers, multiplicative (or intensional) quantifiers $\forall_{\mathbf{new}}$ and $\exists_{\mathbf{new}}$, which arise from observing restrictions on structural rules on the level of terms as well as propositions. Moreover, these restrictions naturally allow the distinction between additive predication and multiplicative predication for each propositional connective. We provide a natural deduction system, a sequent calculus, a Kripke semantics and a BHK semantics for **BI**. We mention computational interpretations, based on locality and sharing, at both the propositional and predicate levels. We explain **BI**'s relationship with intuitionistic logic, linear logic and other relevant logics.

1 Introduction

One of the most important outcomes of the study of linear logic, much more than the formal system itself, is its revealing of the computational significance of the structural rules of Weakening and Contraction [11]. Logically, their absence leads to the decomposition of conjunction into *additive* (&) and *multiplicative* ($\otimes$) forms, which can be given a sequential natural deduction presentation as follows:

$$\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \& \psi} \&I \qquad \frac{\Gamma \vdash \phi_1 \& \phi_2}{\Gamma \vdash \phi_i} \ (i = 1,2) \quad \&E$$

$$\frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \otimes \psi} \otimes I \qquad \frac{\Gamma, \phi, \psi \vdash \chi \quad \Delta \vdash \phi \otimes \psi}{\Gamma, \Delta \vdash \chi} \otimes E$$

If we have the rules of Weakening and Contraction

$$\frac{\Gamma \vdash \psi}{\Gamma, \phi \vdash \psi} W \qquad \frac{\Gamma, \phi, \phi \vdash \psi}{\Gamma, \phi \vdash \psi} C,$$

then these rules for $\otimes$ and & define the same connective, but without them the connectives are distinct. A similar decomposition obtains for disjunction, the multiplicative version arising in the classical (as opposed to intuitionistic) setting.

The decomposition of the connectives has a long history in the relevant logic tradition but the possibilities revealed by restricting structural rules were given a new perspective by the appealing "resource-interpretations" of linear logic. The leading example is perhaps the *number-of-uses* reading in which a proof of a linear implication $\phi \multimap \psi$ determines a function that *uses* its argument exactly once. Like $\otimes$, the linear implication is "multiplicative", which is to say that it uses separate contexts in its elimination rule.

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \multimap \psi} \multimap I \qquad \frac{\Gamma \vdash \phi \multimap \psi \quad \Delta \vdash \phi}{\Gamma, \Delta \vdash \psi} \multimap E.$$

However, an important message of linear logic is that, in order to obtain an expressive system, one cannot stay in a pretty-but-weak system: it seems crucial to allow access to structurals in some manner. In linear logic, this is done via the "!" modality, which admits a recovery of conventional intuitionistic (or additive) implication $\phi \rightarrow \psi$ as $!\phi \multimap \psi$. The number-of-uses reading of implication is extended to $!\phi$ as "as many ϕs as required".

Access to the structurals can be recovered in another, rather different way, not involving a modality. Just as we decomposed conjunction directly into multiplicative and additive parts, so we can decompose *implication* directly. The technical cost of this conceptual symmetry is that we must work with a more richly structured notion of sequent, entailing a more delicate analysis of the proof-theoretic relationship between implication and conjunction.

Just as the propositional assumptions in a sequent can be seen as resources, supporting the distinction between additive and multiplicative connectives, so the the collection of (first-order) variables over which such a sequent is formed can also be seen as resources, supporting similar distinctions. By controlling access to the structural rules in the formation of predicates, in the same way as for propositions, each connective can be formulated either with additive maintenance of variables or with multiplicative maintenance of variables. So, just as restriction of the structural rules at the *propositional* level decomposes conjunction and implication into additive and multiplicative parts, so restriction of the structural rules in the formation of terms further decomposes each *predicate* connective into additive and multiplicative parts.

Our purpose herein is to describe such an alternative route to combining multiplicative and additive implication and to explain its consequences at the propositional and predicate levels. We begin by considering the proof-theoretic background to **BI** and proceed by introducing propositional **BI**, the logic of *bunched implications*, giving a natural deduction system, a BHK-style categorical semantics of proofs and a Kripke-style resource semantics; predicate **BI** is introduced in § 8 and the quantifiers are considered in § 9. The general point, however, is that structural restrictions make just as much sense on level of terms in predicate logic as they do on the level of propositions. The quantifiers reflect this, giving rise to a form of "resource-sensitivity" over individuals or term denotations in a predicate logic. The details of propositional **BI** can be found in [24] and the details of predicate **BI** can be found in [25]. Computational interpretations of **BI**, at both the propositional predicate levels, are briefly discussed in § 10; more details can be found in [20, 19, 23, 22]. We compare **BI** with intuitionistic logic, linear logic and other relevant logics, also in § 10. A similar discussion is in [20] (joint work with Peter O'Hearn).

2 A Proof-theoretic introduction

Implication is inextricably bound up with conjunction, or with antecedent-forming operations used to formulate sequents. This connection goes so far that it is sometimes said that an introduction rule

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi}$$

can be regarded, proof-theoretically, as defining the *meaning* of $\rightarrow$. However far we may wish to take this point, it is clear that the character of the implication in a logic is, in a sense, determined by, that of the comma or conjunction.

If, as is the case in **BI**, we have two forms of implication then we are faced with the question of which of them to use in the introduction rule.

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \, ? \, \psi}$$

That is, should the conclusion $-\,?\,-$ in this rule be a multiplicative or additive implication ?

The connection between introduction rules and implications suggests a solution: If an antecedent-forming operation determines the behaviour of an implication, and we have two implications, then we should have two antecedent-forming operations. So, we postulate a new context-forming operation, ";", and stipulate that Contraction and Weakening are possible for ";" but not for ",". The introduction rules then become

$$\frac{\Gamma \, , \, \phi \vdash \psi}{\Gamma \vdash \phi \mathrel{-\!\!*} \psi} \qquad \text{and} \qquad \frac{\Gamma \, ; \, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi}$$

The antecedents are no longer sequences; rather, they are trees with propositions as leaves and internal nodes labelled with "," or ";", or in short, *bunches* [10, 6, 27].

It is all very well to postulate proof rules in this way, but what meaning or significance, if any, does the resulting logic have ? We argue herein that **BI** possesses two very natural semantics. The first, a BHK-style semantics of the proof theory, arises from *doubly closed categories* (DCCs), in which a single category admits two closed structures or function spaces. It shows clearly the difference with linear logic, where two closed categories are usually used. We shall return to this point in §§4, 10.

The second semantics is a Kripke-style semantics of formulæ, which combines Kripke's semantics of intuitionistic logic and Urquhart's semantics of multiplicative, intuitionistic linear logic. This second semantics gives **BI** more of a genuine status as a logic: It gives us a way to read the formulæ as propositions that are true (or not) relative to a given world.

Moving on to predicate **BI**, we extend our treatment of additivity and multiplicativity to the formation of predicates by allowing the collection of variables over which a sequent is formed to be structured not merely as a finite sequence (or set) but as a bunch. With this formulation, each propositional connective gives rise to both an additive *and* a multiplicative predicate connective.

We also consider the vexed question of multiplicative quantifiers. In **BI**, the quantifiers come in two varieties:

Additive quantifiers	$\forall$	$\exists$
Multiplicative quantifiers	$\forall_{\mathbf{new}}$	$\exists_{\mathbf{new}}$

The additive ones are exactly the quantifiers found in classical, intuitionistic and linear logic. The multiplicative ones are new, being *intensional* quantifiers of the kind long-sought in relevant logic.

3 Propositional BI

In this section, we give a presentation of **BI** in sequential natural deduction form, *i.e.*, a sequent presentation based on introduction and elimination rules. A sequent calculus presentation can be found in [24].

CONNECTIVES

Additive connectives	$1 \;\wedge\; \rightarrow \;\; 0 \;\vee$
Multiplicative connectives	$I \;\ast\; \mathrel{-\!\!*}$

The additive connectives correspond to those of intuitionistic logic (IL), whereas the multiplicatives correspond to those of multiplicative, intuitionistic linear logic (MILL).

BUNCHES

$$\begin{array}{rcll} \Gamma & ::= & \phi & \text{propositional assumption} \\ & \mid & \emptyset_m & \text{multiplicative unit} \\ & \mid & \Gamma, \Gamma & \text{multiplicative combination} \\ & \mid & \emptyset_a & \text{additive unit} \\ & \mid & \Gamma; \Gamma & \text{additive combination} \end{array}$$

We write $\Gamma(\Delta)$ to indicate a bunch in which the bunch Δ appears as a subtree. We write $\Gamma(\Delta')$ for the similar tree where Δ' replaces Δ. (So $\Gamma(-)$ is a kind of positional notation indicating an incomplete tree that can be completed by placing various trees in its hole.) We don't present a more compact representation of bunches using, say, sets or sequences instead of binary operators; the real point of bunches is to let us get the implications right. We use an equivalence on trees instead of worrying about representation.

COHERENT EQUIVALENCE: $\Gamma \equiv \Gamma'$

1. Commutative monoid equations for $\emptyset_a$ and ;
2. Commutative monoid equations for $\emptyset_m$ and ,
3. Congruence: if $\Delta \equiv \Delta'$ then $\Gamma(\Delta) \equiv \Gamma(\Delta')$

Note that ";" and "," do not distribute over one another.

IDENTITY AND STRUCTURE

$$\frac{}{\phi \vdash \phi}\ Id \qquad \frac{\Gamma \vdash \phi}{\Delta \vdash \phi} \equiv \ (\text{where } \Delta \ \equiv \ \Gamma)$$

$$\frac{\Gamma(\Delta) \vdash \phi}{\Gamma(\Delta; \Delta') \vdash \phi}\ W \qquad \frac{\Gamma(\Delta; \Delta) \vdash \phi}{\Gamma(\Delta) \vdash \phi}\ C$$

MULTIPLICATIVES

$$\frac{}{\emptyset_m \vdash I}\ I\,I \qquad \frac{\Gamma(\emptyset_m) \vdash \chi \quad \Delta \vdash I}{\Gamma(\Delta) \vdash \chi}\ I\,E$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \mathrel{-\!\!*} \psi}\ {-\!\!*}\,I \qquad \frac{\Gamma \vdash \phi \mathrel{-\!\!*} \psi \quad \Delta \vdash \phi}{\Gamma, \Delta \vdash \psi}\ {-\!\!*}\,E$$

$$\frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi * \psi}\ {*I} \qquad \frac{\Gamma(\phi, \psi) \vdash \chi \quad \Delta \vdash \phi * \psi}{\Gamma(\Delta) \vdash \chi}\ {*E}$$

ADDITIVES

$$\frac{}{\emptyset_a \vdash 1}\ 1\,I \qquad \frac{\Gamma(\emptyset_a) \vdash \chi \quad \Delta \vdash 1}{\Gamma(\Delta) \vdash \chi}\ 1\,E$$

$$\frac{\Gamma; \phi \vdash \psi}{\Gamma \vdash \phi \to \psi} \to I \qquad \frac{\Gamma \vdash \phi \to \psi \quad \Delta \vdash \phi}{\Gamma; \Delta \vdash \psi} \to E$$

$$\frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma; \Delta \vdash \phi \wedge \psi}\ \wedge I \qquad \frac{\Gamma(\phi; \psi) \vdash \chi \quad \Delta \vdash \phi \wedge \psi}{\Gamma(\Delta) \vdash \chi}\ \wedge E$$

$$\frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2}\ (i = 1,\, 2) \quad \vee I$$

$$\frac{\Gamma \vdash \phi \vee \psi \quad \Delta(\phi) \vdash \chi \quad \Delta(\psi) \vdash \chi}{\Delta(\Gamma) \vdash \chi}\ \vee E$$

Notice that the introduction and elimination rules for additive and multiplicative implications, conjunctions and units are identical in form. The difference between them is the antecedent-combining operations they use. Because ";" admits Weakening and Contraction, rules where additive maintenance is explicit are admissible:

$$\frac{\Gamma \vdash \phi \to \psi \quad \Gamma \vdash \phi}{\Gamma \vdash \psi} \qquad \frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}$$

The question of whether these rules can replace the more explicit ones above is delicate [19, 24].

Proposition 1 (conservativity) BI *is conservative over both* **IL** *and* **MILL***:*

$$\phi_1 \,;\, \ldots \,;\, \phi_n \vdash_{\mathbf{BI}} \phi \textit{ iff } \phi_1 \,,\, \ldots \,,\, \phi_n \vdash_{\mathbf{IL}} \phi,$$

where each ϕ_i and ϕ is a formula built up using additive connectives only, and $\emptyset_a$ plays the role of the empty context in **IL***. Similarly,*

$$\phi_1 \,,\, \ldots \,,\, \phi_n \vdash_{\mathbf{BI}} \phi \textit{ iff } \phi_1 \,,\, \ldots \,,\, \phi_n \vdash_{\mathbf{MILL}} \phi,$$

where each ϕ_i and ϕ is a formula built up using multiplicative connectives only, and $\emptyset_m$ plays the role of the empty context in **MILL***.*

These conservativity properties can be seen straightforwardly from a semantics of **BI**'s proof theory.

4 Doubly closed categories

In order to explain the structure of a proof theory, it is usually useful to give a categorical description of it in which the connectives enjoy universal properties and satisfy naturality conditions.

In order to do this for **BI**, note that the semantics of proofs for **IL** is given using cartesian closed categories and that the semantics for **MILL** using symmetric monoidal closed categories. In each case, the introduction rule for implication corresponds to an adjunction where the internal hom is a right adjoint: to a cartesian product, for **IL**, and a tensor product, for **MILL**.

To model **BI** we ask for a category that has all the structure necessary to model *both* **MILL** and **IL**.

A *doubly closed category*, or DCC for short, is a category equipped with two monoidal closed structures. A DCC is called *cartesian* if one of the closed structures is cartesian and the other symmetric monoidal and *bicartesian* if, in addition, it has finite co-products.

Briefly, models of proofs for **BI** are given using bicartesian DCCs, $\mathcal{D}$. Roughly, we intend that $E = [\![\phi]\!]_{\mathcal{D}}$, $F = [\![\psi]\!]_{\mathcal{D}}$ and $G = [\![\chi]\!]_{\mathcal{D}}$. To see how closely DCC structure corresponds to bunches, consider the two adjunctions

$$[G * E, F] \;\cong\; [G, E \mathrel{-\!\!*} F] \qquad [G \wedge E, F] \;\cong\; [G, E \to F]$$

where $*$ is a symmetric monoidal product and $\wedge$ a cartesian (or categorical) product. To mimic these adjunctions in the proof theory one would like to have a context extension corresponding to $*$ and another to $\wedge$. This leads directly to

$$\frac{\Gamma\,,\,\phi \vdash \psi}{\Gamma \vdash \phi \mathrel{-\!\!*} \psi} \quad \text{and} \quad \frac{\Gamma\,;\,\phi \vdash \psi}{\Gamma \vdash \phi \to \psi}$$

and to the tree-like structure of antecedents.

Propositional **BI**'s proofs can be represented in the usual way as the terms of a *$\beta\eta\zeta$-normalizing* simply-typed λ-calculus, $\alpha\lambda$, with two forms of abstraction, one additive and one multiplicative; see [24] for details. We will not give a detailed interpretation of **BI**'s proofs in a bicartesian DCC here. For now, the main point is *soundness* [24].

Proposition 2 (soundness) *Any* **BI** *proof Φ of a judgement $\Gamma \vdash \phi$ determines a morphism $[\![\Phi]\!]_{\mathcal{D}} : [\![\Gamma]\!]_{\mathcal{D}} \to [\![\phi]\!]_{\mathcal{D}}$ in a DCC, $\mathcal{D}$, where the interpretation of formulæ and bunches uses the corresponding DCC structure. Moreover, if $\Phi = \Phi'$ as $\alpha\lambda$-terms, then $[\![\Phi]\!]_{\mathcal{D}} = [\![\Phi']\!]_{\mathcal{D}}$.*

The assigning of morphisms to derivations should be evident, except for one point that deserves mention. To interpret the elimination rule for $\vee$ we need to use distributivity of $\vee$ over both $*$ and $\wedge$. To see why we get the needed distributivities in bicartesian DCCs note first that, since $\phi * (-)$ and $\phi \wedge (-)$ are both left adjoints, they both preserve all colimits. Second, $\vee$ is a coproduct. Then, putting these points together, we obtain the isomorphisms

$$\begin{array}{c} E \wedge (F \vee G) \;\cong\; (E \wedge F) \vee (E \wedge G) \\ E * (F \vee G) \;\cong\; (E * F) \vee (E * G) \end{array}$$

We have *completeness* [24], via model existence.

Proposition 3 (model existence) *There is a DCC, $\mathcal{T}$, such that if $m : E \to F$ is a morphism in $\mathcal{T}$, then there exist Γ and ψ and Ψ such that $[\![\Gamma]\!]_{\mathcal{T}} = E$, $[\![\psi]\!]_{\mathcal{T}} = F$, $[\![\Psi]\!]_{\mathcal{T}} = m$ and $\Gamma \vdash \Psi : \psi$ is provable in $\alpha\lambda$. Moreover, if $m' : E \to F$, $m = m'$ and $[\![\Psi']\!]_{\mathcal{T}} = m'$, then $\Gamma \vdash \Psi = \Psi' : \psi$ in $\alpha\lambda$.*

The conservativity of **BI** over **IL** can be seen immediately in terms of DCCs. For suppose $\mathcal{C}$ is a categorical model of **IL**, *i.e.*, a bicartesian closed category. Then we can regard it as a bicartesian DCC, in which the two closed structures are the same. When we restrict to the additive fragment of **BI**, this shows that the denotations of **BI** proofs are exactly the same as those in the model of **IL**.

The DCC semantics suggests some, at first sight, unusual properties. In particular, a morphism from E to F can variously be viewed as a map $I \longrightarrow (E \mathrel{-\!\!*} F)$ or $1 \longrightarrow (E \to F)$ using the adjunctions and, indeed, we have the following isomorphisms of hom sets:

$$[1, E \to F] \;\cong\; [E, F] \;\cong\; [I, E \mathrel{-\!\!*} F]$$

In **BI**, these isomorphisms are realized by the following deductions:

$$\dfrac{\dfrac{\phi \vdash \psi}{\emptyset_a ; \phi \vdash \psi}}{\emptyset_a \vdash \phi \to \psi} \qquad \dfrac{\dfrac{\phi \vdash \psi}{\emptyset_m , \phi \vdash \psi}}{\emptyset_m \vdash \phi \mathrel{-\!\!*} \psi}$$

$$\dfrac{\dfrac{\emptyset_a \vdash \phi \to \psi \quad \overline{\phi \vdash \phi}}{\emptyset_a ; \phi \vdash \psi}}{\phi \vdash \psi} \qquad \dfrac{\dfrac{\emptyset_m \vdash \phi \mathrel{-\!\!*} \psi \quad \overline{\phi \vdash \phi}}{\emptyset_m , \phi \vdash \psi}}{\phi \vdash \psi}$$

However, the two implications are not interconvertible:

$$\phi \mathrel{-\!\!*} \psi \not\vdash \phi \to \psi \qquad \text{and} \qquad \phi \to \psi \not\vdash \phi \mathrel{-\!\!*} \psi.$$

Example 4 (O'Hearn and Pym [20]) **Set** $\times$ **Set** *is bicartesian closed, with coproducts and cartesian closed structure defined pointwise from their corresponding versions in* **Set**. *A symmetric monoidal closed structure is given by*

$$\begin{array}{rcl} I & = & (1, 0) \\ (E_0, E_1) * (F_0, F_1) & = & ((E_0 \times F_0) + (E_1 \times F_1))\,, \\ & & (E_0 \times F_1) + (E_1 \times F_0)) \\ (E_0, E_1) \mathrel{-\!\!*} (F_0, F_1) & = & ((E_0 \to F_0) \times (E_1 \to F_1)\,, \\ & & (E_0 \to F_1) \times (E_1 \to F_0)) \end{array}$$

This model can be used to show that, in general, there is no functor $!$: **Set** $\times$ **Set** $\to$ **Set** $\times$ **Set** *admitting an isomorphism $!E \mathrel{-\!\!*} F \;\cong\; E \to F$, thus indicating that a DCC is not simply a model of linear logic in disguise: Consider that $(1, 0) \to (2, 2) = (2, 1)$ but that, no matter what C is, $C \mathrel{-\!\!*} (2, 2) = (X, Y)$ for sets X and Y of the same cardinality. Therefore, for any "!" we try to pick, $!E \mathrel{-\!\!*} (2, 2)$ cannot be $(2, 1)$.*

Example 4 shows that **BI** and **MILL**, with !, treat intuitionistic implication in essentially different ways.

5 Day's construction

We can generate a rich class of models of **BI** using a general construction due to Day [9]. He shows that any monoidal (not necessarily closed) category $(\mathcal{C}, *, I)$ induces a monoidal closed structure on the functor category $\mathbf{Set}^{\mathcal{C}^{op}}$, and that if $(\mathcal{C}, *, I)$ is symmetric monoidal, then so is $\mathbf{Set}^{\mathcal{C}^{op}}$. (In fact, Day describes his results much more generally, in the context of enriched categories.) This, combined with the standard fact that $\mathbf{Set}^{\mathcal{C}^{op}}$ is bicartesian closed, yields a host of bicartesian DCCs.

The construction is as follows: The unit I of the monoidal structure is $\mathcal{C}[-, I]$. Given functors E and F, the formula for the tensor product is written using co-ends:

$$(E * F)Z \;=\; \int^{X,Y} EX \times FY \times \mathcal{C}[Z, X * Y]$$

The formula for $-\!\!*$ uses an end:

$$\begin{array}{rcl}(E -\!\!* F)Z & = & \int_X \mathbf{Set}^{\mathcal{C}^{op}}[EX, F(Z * X)] \\ & \cong & \mathbf{Set}^{\mathcal{C}^{op}}[E, F(Z * -)]\end{array}$$

The formulæ for $(E * F)Z$ and $(E -\!\!* F)Z$ are both contravariant in Z, giving the morphism parts of the functors.

Day's construction gives us a way of embedding any symmetric monoidal closed category into a bicartesian DCC: we apply the Yoneda embedding $\mathcal{C} \longrightarrow \mathbf{Set}^{\mathcal{C}^{op}}$, which sends X to $\mathcal{C}[-, X]$. It is standard that Yoneda is full and faithful, and it is not difficult to show that it preserves monoidal closed structure [19]. From this we may conclude that **BI** is conservative over **MILL**, again not only on the level of provability but also on the semantics of proofs.

Two observations are useful for working with the tensor product. The first is that we have a form of pairing operation: given $a \in EX$ and $b \in BY$ we can form an element $[a, b] \in (E * F)(X * Y)$. To see how this element is defined, consider that the co-end $(E * F)Z$ can be described as a quotient of quintuples $(X, Y, f : Z \to_{\mathcal{C}} X * Y, a \in EX, b \in FY)$. The pair $[a, b]$ is then the equivalence class of (X, Y, Id_{X*Y}, a, b).

The second is a representation result which characterizes maps out of tensor products: natural transformations $E * F \longrightarrow G$ are in bijection with families of functions

$$EX \times FY \longrightarrow G(X * Y)$$

natural in X and Y. To see why this is true, consider the definition of $-\!\!*$, and the isomorphism $[E * F, G] \cong [E, F -\!\!* G]$: the multi-map characterization is essentially forced by $-\!\!*$. These observations can be put together to describe the functorial action of $*$. Given natural transformations $\eta : E \to E'$ and $\mu : F \to F'$, we obtain a natural family of maps $EX \times FY \to (E' * F')(X * Y)$ by composing $\eta_X \times \mu_Y$ with pairing. Then we obtain a natural transformation $E * F \to E * F$ by the representation of maps out of a tensor in terms of multi-maps.

6 Kripke resource semantics

Thus far we have considered **BI** from proof- or type-theoretic perspectives, with a semantic focus on the interpretation of proofs. However, **BI** can also be understood from a model-theoretic point of view, in which we consider a notion of truth for formulæ, which are thus seen to denote propositions in the established sense.

Recall that Kripke gave a semantics of **IL** based on possible worlds, which can be understood as using functor categories $\mathbf{Set}^{\mathcal{P}}$, where $\mathcal{P}$ is a preorder [16]. Similarly, Urquhart's semantics of **MILL** uses $\mathbf{Set}^{\mathcal{M}}$, where $\mathcal{M}$ is a commutative monoid [29]. We obtain a semantics of **BI**, which combines these two semantics directly, by working in categories $\mathbf{Set}^{\mathcal{C}^{op}}$, where $\mathcal{C}$ is a symmetric monoidal category. For simplicity, we describe this for the special case that $\mathcal{C}^{op} = \mathcal{M}$ is a preordered commutative monoid.

Given such a monoid $\mathcal{M} = (M, \cdot, e, \sqsubseteq)$, with $\sqsubseteq$ monotone with respect to $\cdot$, the semantics of **BI** is defined via a forcing relation $m \models F$, for a world $m \in M$ and formula F. All propositions will satisfy the familiar Kripke Monotonicity property from intuitionistic logic:

$$n \models \psi \text{ and } m \sqsubseteq n \text{ implies } m \models \psi$$

In more detail, an interpretation must specify the forcing relation $m \models p$ for propositional variables in a way compatible with Kripke Monotonicity. The clauses for the additive connectives are then

$$\begin{array}{lll}
m \models 1 & & \text{always} \\
m \models \phi \wedge \psi & \text{iff} & m \models \phi \text{ and } m \models \psi \\
m \models 0 & & \text{never} \\
m \models \phi \vee \psi & \text{iff} & m \models \phi \text{ or } m \models \psi \\
m \models \phi \to \psi & \text{iff} & \forall n \sqsubseteq m\ (n \models \phi \text{ implies } n \models \psi)
\end{array}$$

The multiplicatives follow Urquhart's semantics of **MILL**.

$$\begin{array}{lll}
m \models I & \text{iff} & m \sqsubseteq e \\
m \models \phi * \psi & \text{iff} & \exists n, n' \in M (m \sqsubseteq n \cdot n' \text{ and} \\
& & n \models \phi \text{ and } n' \models \psi) \\
m \models \phi -\!\!* \psi & \text{iff} & \forall n \in M\ (n \models \phi \text{ implies } n \cdot m \models \psi)
\end{array}$$

If the order on the monoid $\mathcal{M}$ is discrete then the clauses for $*$ and I reduce to Urquhart's semantics for **MILL**.

The connection with Day's construction can be made precise by observing that any proposition determines a subobject of 1 in $\mathbf{Set}^{\mathcal{M}}$. Then the forcing $n \models \phi * \psi$ holds according to the semantics here iff the evaluation $(E * F)n$

of the tensor product of functors is non-empty. A similar story can be told for all the connectives [24, 25]. Given this definition, we can proceed to establish soundness and completeness theorems, with a model-existence result being established via the existence of normal natural deductions. Again, the details can be found in [24, 25].

We can read the clauses informally in terms of resources. For example, the formula for $*$ says, roughly, that a world forces a multiplicative conjunction if and only if there is a *partition of that world* into components which force the constituents of the formula. This narrative about partitioning suggests a sense in which multiplicative (or intensional, in the relevant terminology) connectives depend on the *internal structure* of worlds. So, one can understand the formula for $-\!\!*$ as follows: if m is the cost of the function and if n is the cost of (any) argument, then the cost of obtaining the result of applying the function to the argument is $m \cdot n$.

The clauses for additives should be understood in terms of conservation, rather than splitting, of information. For example, to establish an additive conjunct one can make use of the same information when showing each conjunct.

The reader may feel that the semantics is merely a straightforward combination of the semantics of Kripke and Urquhart. We would agree but would add that that is in fact the point; the semantics underlines **BI**'s direct route to combining **IL** and **MILL**.

7 BI as a sequent calculus

BI can be formulated as a sequent calculus, in which the elimination rules are replaced by rules which introduce connectives *on the left* [24]. The principal left rules can be stated as follows:

$$\frac{\Gamma(\phi\,,\,\psi) \vdash \chi}{\Gamma(\phi * \psi) \vdash \chi} * L \qquad \frac{\Delta \vdash \phi \quad \Gamma(\psi) \vdash \chi}{\Gamma(\Delta\,,\,\phi -\!\!* \psi) \vdash \chi} -\!\!* L$$

$$\frac{\Gamma(\phi\,;\,\psi) \vdash \chi}{\Gamma(\phi \wedge \psi) \vdash \chi} \wedge L \qquad \frac{\Delta \vdash \phi \quad \Gamma(\psi) \vdash \chi}{\Gamma(\Delta\,;\,\phi \to \psi) \vdash \chi} \to L.$$

The sequent calculus includes the following Cut rule:

$$\frac{\Gamma(\phi) \vdash \psi \quad \Delta \vdash \phi}{\Gamma(\Delta) \vdash \psi}$$

Note that this rule covers both the ";" and "," cases in the construction of bunches. The *Cut-elimination theorem* holds for this calculus [24, 25].

8 Predicates

Just as the propositional assumptions in a sequent can be structured as bunches, supporting the distinction between additive and multiplicative connectives, so the collection of (first-order) variables over which such a sequent is formed can be structured as *bunches of variables*. With variables structured as bunches, each connective can be formulated either with additive maintenance of variables or with multiplicative maintenance of variables. So, just as explicit control of the structural rules at the *propositional* level decomposes conjunction and implication into additive and multiplicative parts, so explicit control of the structural rules in the formation of terms further decomposes each *predicate* connective into additive and multiplicative parts. We describe this idea here using a single-sorted version of predicate **BI**. We suppress the details of function symbols and the formation of terms.

BUNCHES OF VARIABLES
x ranges over variables, and X over bunches of variables:

$$X \ ::= \ x \mid X; X \mid X, X \mid \emptyset_a \mid \emptyset_m$$

Bunches of variables are subject to the linearity restriction: any variable appears at most once in a bunch. We let $X \cong X'$ denote isomorphism of labelled trees.

JUDGEMENTS
We consider terms- and propositions-in-context, with a syntax of the form

$$X \vdash t : \textbf{Term} \quad \text{and} \quad X \vdash \phi : \textbf{Prop}$$

asserting that a term or predicate is well-formed in context X [17]. Constants and predicate letters can be considered to be given by schematic judgements and, as in the bunched logic itself, Contraction and Weakening are allowed for ";" but not for ",". We omit a formal definition and move on to consider the quantifiers.

Logical judgements have the form

$$(X)\Gamma \vdash \phi$$

asserting that ϕ is a consequence of Γ, where the terms and atomic predicates in the sequent are well-formed in X.

Given this formulation of predication, we can see that for each propositional connective we must choose whether to adopt additive or multiplicative maintenance of variables. For example, the binary rule for introducing $*$ can be formulated either as

$$\frac{(X)\Gamma \vdash \phi \quad (Y)\Delta \vdash \psi}{(X, Y)\Gamma, \Delta \vdash \phi * \psi} \quad *I(m),$$

in which the two premisses use distinct bunches of variables, combined multiplicatively in the conclusion, or as

$$\frac{(X)\Gamma \vdash \phi \quad (Y)\Delta \vdash \psi}{(X; Y)\Gamma, \Delta \vdash \phi * \psi} \quad *I(a),$$

using ";" to combine bunches of variables. It follows, via Contraction

$$\frac{Y(X;X')\Gamma \vdash \phi}{Y(X)\Gamma[X/X'] \vdash \phi[X/X']} \quad (X \cong X') \quad \text{Contraction}$$

that the following, familiar, form of $*I(a)$ is sound:

$$\frac{(X)\Gamma \vdash \phi \quad (X)\Gamma \vdash \psi}{(X)\Gamma, \Delta \vdash \phi * \psi} \quad *I(a).$$

This latter form is the one taken in linear logic [11].

We now extend our Kripke Resource Semantics to predicates. To do this, we must first define a notion of *environment*, which specifies a binding of variables to individuals. For this, we suppose that we are given a functor D in $\mathbf{Set}^{\mathcal{M}}$; this functor is the "domain of individuals". As in the possible worlds semantics of intuitionistic logic, the use of a functor instead of a constant set allows different collections of individuals to exist at different worlds.

Example 5 *Consider the monoid $\mathcal{M} = (N, 0, +, \leq)$ of natural numbers, under addition. Then D can be taken to be the functor sending n to the set $\{0, \ldots, n-1\}$, the morphism part being inclusion.*

Continuing with the definitions, given such a functor D we may define the *environment functor* $[\![X]\!]$, for X a bunch of variables, as follows:

$$\begin{array}{lcllcl}
[\![x]\!] & = & D & & & \\
[\![\emptyset_m]\!] & = & I & [\![X,Y]\!] & = & [\![X]\!] * [\![Y]\!] \\
[\![\emptyset_a]\!] & = & 1 & [\![X;Y]\!] & = & [\![X]\!] \wedge [\![Y]\!]
\end{array}$$

Here, the $*$ is Day's tensor product and $\wedge$ is cartesian product of functors. As a result, each $[\![X]\!]$ is a functor, so that when $n \sqsubseteq m$ and $a \in [\![X]\!]m$ we obtain an element $[\![X]\!](n \sqsubseteq m)a \in [\![X]\!]n$.

The forcing relation for predicate **BI** is of the form

$$(X)u \mid m \models E$$

where $m \in M$ is a world as before, E is a formula, possibly involving quantifiers and predicate letters, and $u \in [\![X]\!]m$ is an *environment* appropriate to a bunch of variables X, at world m. (We emphasise here that an environment is an element of a set, and not a map in $\mathbf{Set}^{\mathcal{M}}$.)

The functorial action of environments enables us to formulate an extension of Kripke Monotonicity:

$(X)u \mid n \models \phi$ and $m \sqsubseteq n$ implies $([\![X]\!](m \sqsubseteq n))u \mid m \models \phi$

With these definitions, we can give the forcing relation for multiplicative implication with multiplicative variable maintenance:

$$\begin{array}{ll}
(X,Y)\ [u_X, u_Y] \mid m_X \cdot m_Y \models \phi \mathrel{-\!\!*} \psi & \text{iff} \\
\forall n\, \forall Z\, \forall v \in [\![Z]\!]n\, ((Z,Y)[v, u_Y] \mid n \cdot m_Y \models \phi & \text{implies} \\
(X,Z)\ [u_X, v] \mid m_X \cdot n \models \psi), &
\end{array}$$

where $Y \vdash \phi : \mathbf{Prop}$ and $X \vdash \psi : \mathbf{Prop}$. Notice that we are using the pairing operation of Day's tensor product. The arbitrary nature of the environment v, used in the clause for ϕ, reflects the fact that ϕ does not occur in the conclusion of the $\mathrel{-\!\!*} E$ rule.

Turning, for example, to multiplicative implication with additive variable maintenance, as in first-order linear logic [11], we require a clause such as

$$\begin{array}{ll}
(X;Y)\ (u_X, u_Y) \mid m_X \cdot m_Y \models \phi \mathrel{-\!\!*} \psi & \text{iff} \\
\forall n\, \forall Z\, \forall v \in [\![Z]\!]n\, ((Z;Y)(v, u_Y) \mid n \cdot m_Y \models \phi & \text{implies} \\
(X;Z)\ (u_X, v) \mid m_X \cdot n \models \psi). &
\end{array}$$

Notice that, corresponding to the combination of bunches using ";", we have used the cartesian pairing operation, $(-,-)$, instead of Day's pairing operation, $[-,-]$. Consequently, we must require that (u_X, u_Y) be defined at $m_X \cdot m_Y$, that (v, u_Y) be defined at $n \cdot m_Y$ and that (u_X, v) be defined at $m_X \cdot n$. One way to enforce these conditions would be to require that the domain functor D be constant, so that we should have a *Beth model* [30].

The pairing of the additive propositional implication, $\rightarrow$, with multiplicative variable maintenance is more problematic; its discussion is deferred to another occasion.

9 Quantifiers

The predicate version of **BI** has the familiar intuitionistic quantifiers $\forall$ and $\exists$. It also has multiplicative, or intensional, quantifiers, obtained by observing structural restrictions on the level of terms as well as propositions. The universal introduction rules have the usual restriction that x must not occur in Γ; similarly, the existential elimination rules have the restriction that x must not occur in Δ or ψ. However, in **BI**, these restrictions must be extended to those variables which are constrained by the formation of axioms to be equal to such an x. $X \vdash \Gamma : \mathbf{Prop}$ has its evident meaning.

MULTIPLICATIVES

$$\frac{(X,Y)\ \Gamma \vdash \phi[t/x] \quad Y \vdash t : \mathbf{Term} \quad X \vdash \Gamma : \mathbf{Prop}}{(X)\ \Gamma \vdash \exists_{\mathbf{new}} x.\phi}\ \exists_{\mathbf{new}} I$$

$$\frac{(X)\ \Gamma \vdash \exists_{\mathbf{new}} x.\phi \quad (Y,x)\ \Delta, \phi \vdash \psi}{(X,Y)\ \Gamma, \Delta \vdash \psi}\ \exists_{\mathbf{new}} E$$

$$\frac{(X,x)\Gamma \vdash \phi}{(X)\Gamma \vdash \forall_{\mathbf{new}} x \,.\, \phi}\ \forall_{\mathbf{new}} I$$

$$\frac{(X)\ \Gamma \vdash \forall_{\mathbf{new}} x \,.\, \phi \quad Y \vdash t : \mathbf{Term}}{(X,Y)\ \Gamma \vdash \phi[t/x]}\ \forall_{\mathbf{new}} E$$

ADDITIVES

$$\frac{(X;Y)\ \Gamma \vdash \phi[t/x] \quad Y \vdash t : \mathbf{Term}}{(X)\ \Gamma \vdash \exists x.\phi}\ \exists I$$

$$\frac{(X)\ \Gamma \vdash \exists x.\phi \quad (X;x)\ \Delta, \phi \vdash \psi}{(X)\ \Gamma; \Delta \vdash \psi}\ \exists E$$

$$\frac{(X;x)\ \Gamma \vdash \phi}{(X)\ \Gamma \vdash \forall x\,.\,\phi}\ \forall I \quad \frac{(X)\ \Gamma \vdash \forall x\,.\,\phi \quad Y \vdash t : \mathbf{Term}}{(X;Y)\ \Gamma \vdash \phi[t/x]}\ \forall E$$

The idea of the introduction rule for $\forall_{\mathbf{new}}$ is that we may infer $\forall_{\mathbf{new}} x\,.\,\phi$ in the usual way for universal quantification, except that the variable x must be in multiplicative combination with all of the other variables. In the elimination rule we must be careful not to substitute an arbitrary term for x, but only one that is respective of the multiplicative relationship between x and other variables in X. In particular, t cannot contain any of these other variables appearing in X; this requirement is implemented by the linearity restriction.

The form of the quantifier rules is intimately related to the form of axiom sequents in predicate **BI**. In order to permit the combination of bunches of variables, using either "," or ";", axiom sequents have the form

$$\frac{X_1 \vdash \mathrm{p}(X_1) : \mathbf{Prop}\ X_2 \vdash \mathrm{p}(X_2) : \mathbf{Prop}}{(X)\ \mathrm{p}(X_1) \vdash \mathrm{p}(X_2)}\ \mathrm{Axiom}(X_1, X_2),$$

where p is a predicate letter and X is X_1, X_2 or $X_1; X_2$. The side-condition $\mathrm{Axiom}(X_1, X_2)$ is used, in the Substitution and predicate Cut rules, which are not given herein, to ensure that the bunches X_1 and X_2 are treated as being the same in a subsequent derivation. Such side-conditions are also used in the quantifier rules to enforce the maintenance of the usual side-conditions not only for the given x but also for variables constrained to be equal to it via $\mathrm{Axiom}(-,-)$; see [25].

Following on from § 7, predicate **BI** can be presented as a cut-free sequent calculus [25].

A consequence of these observations is that $\forall_{\mathbf{new}}$ should not be read literally as "for all". Rather, we must understand that, in $\forall_{\mathbf{new}} x\,.\,\phi$, the multiplicative relationship between x and other variables must be observed. We read it as "for all x different from the variables in ϕ", or more briefly, for all *new* x.

It is important to avoid the trap of thinking of $\forall_{\mathbf{new}}$ as an infinite multiplicative conjunction; this idea is sometimes mentioned, by analogy with the view of additive quantification as an infinite additive conjunction. Rather, $\forall_{\mathbf{new}}$ is closely allied to the multiplicative implication $-\!\!*$, although a version allied to multiplicative conjunction is possible under restricted circumstances. It would be interesting to formulate a dependent function type, along the lines of [15], which generalizes both of them.

For the additive universal quantifier, one might expect the elimination rule to be

$$\frac{(X)\ \Gamma \vdash \forall x\,.\,\phi \quad X \vdash t : \mathbf{Term}}{(X)\ \Gamma \vdash \phi[t/x]}\ \forall E$$

Such a rule is admissible if we have Contraction for ";" on the level of terms. Similar considerations apply to the additive existential quantifier.

We now extend the Kripke Resource Semantics to quantifiers. The semantics of the additive quantifiers is standard, and the multiplicative quantifiers use the monoid structure, as follows:

$$\begin{array}{l} (X)\ u \mid m \models \forall x\,.\,\phi \ \text{ iff } \ \forall n \sqsubseteq m\ \forall d \in Dn \\ (X;x)\ ([\![X]\!](n \sqsubseteq m)u, d) \mid n \models \phi \\ \\ (X)\ u \mid m \models \exists x\,.\,\phi \ \text{ iff } \ \exists d \in Dm \\ (X;x)\ (u,d) \mid m \models \phi \\ \\ (X)\ u \mid m \models \forall_{\mathbf{new}} x\,.\,\phi \ \text{ iff } \ \forall n\ \forall d \in Dn \\ (X,x)\ [u,d] \mid n \cdot m \models \phi \\ \\ (X)\ u \mid m \models \exists_{\mathbf{new}} x\,.\,\phi \ \text{ iff } \ \exists n\ \exists d \in Dn \\ (X,x)\ [u,d] \mid n \cdot m \models \phi \end{array}$$

In the multiplicative cases, we have used the pairing operation $[u,d]$ for Day's tensor product (§ 5). In each case, it takes an environment $u \in [\![X]\!]m$ and an element $d \in Dn$ and forms the element $[u,d] \in ([\![X]\!] * D)(m \cdot n)$. (From the definition of environment functors, this functor $[\![X]\!] * D$ is equal to $[\![X,x]\!]$, so the definitions are type-correct.)

The difference between $\forall$ and $\forall_{\mathbf{new}}$ is that $\forall$ considers elements $d \in Dn$ only in accessible worlds where $n \sqsubseteq m$, whereas for $\forall_{\mathbf{new}}$ we look to a separate world n, and then use the monoid operation $\cdot$ to combine it with m.

The multiplicative existential is different in another respect from its additive counterpart. The additive $\exists$, from intuitionistic logic, is often described as being "local", in that the definition stays at the same world, where for $\forall$ you travel to accessible worlds to find elements. The multiplicative $\exists_{\mathbf{new}}$ does not stay at the same world when it looks for an element that exists, but neither does it travel along accessible lines; it hops to an arbitrary world, accessible or not, to find an element, then considers the resulting formula in an environment formed by multiplicative combination.

Finally, the BHK-style categorical semantics of proofs can be extended to predicate **BI** by considering a category of DCCs indexed over a DCC; see [24, 25]. The *normalization*, *soundness* and *completeness* theorems extend to **BI**'s predicates and quantifiers for both presheaf and fibred models [24, 25].

10 BI in context and computation

Following Read's [27] taxonomy, we have interpreted "Relevant Logic" broadly, as referring to a variety of logics which control the use of structural rules, and not necessarily to logics that admit Contraction but not Weakening. Whilst it is clear that **BI** lies within the range of systems "known" to relevantists, and appears to be, up to minor syntactic details, an instance of Belnap's scheme of display logic [6]. But the relevantists have not, to the best of our knowledge, studied it explicitly. Perhaps this is because there was not an indication of why such a logic might be of interest, and particularly of its semantics. It is perhaps surprising, in light of growing interest in "resource-sensitive" logics, that relevant techniques, such as bunches and resource semantics, have not spread further into theoretical computer science.

The study of bunches arose from Dunn's work on a sequent calculus for the relevant logic **R** [4]. The motivation was not implication but distribution of additive conjunction over disjunction. The difficulty is that distribution fails in standard sequent calculi for logics, such as linear logic, lacking Weakening or Contraction, whereas it is present in **R**. Bunches were used to give a more delicate treatment of structural rules, allowing them to be used in some places but not others, a key improvement in the treatment of additives. Since then bunches have been a standard device in relevant logic. See, for example, Read's account of the proof theory of relevant logics [27] and Schroeder-Heister's work on structural frameworks [28].

Distribution of $\vee$ over $\wedge$ has been a source of debate within and without the relevant logic community (see [7] for a lively account). But for our purposes the debate is not a serious issue. Simply, distribution is valid in Kripke resource semantics, which we regard as a natural semantics, and we must accept it when working in a situation justified by the semantics. Similarly, distribution follows at once in the semantics of proofs, using the fact that $E{*}(-)$ and $E\wedge(-)$ are left adjoints. This is not to suggest that one wants distribution in all circumstances; the use of linear typing to provide a type structure for domain theory [21] is a prime example of where one does not.

Turning to linear logic, we briefly compare **BI** and intuitionistic linear logic, with "!", in logical (including categorical) and computational terms.

The linear implication $\phi \multimap \psi$ is often considered to be about using an argument once but the multiplicative implication $\phi \mathrel{-\!\!*} \psi$ does not readily admit such a reading. The sharing-of-resources interpretation [20, 19], however, provides a reading of connectives consistent with the existence of $\mathrel{-\!\!*}$-typed functions using their arguments multiple times.

Bunches are *not* similar to the zoned contexts used in some presentations of linear logic (*e.g.*, [12, 5]). In particular, ";" and "," can be nested in a bunch, and ";" (just like ", ") is internalized as a connective in the logic, while the stoup ";" does not internalize in this way.

Linear logic admits a decomposition of $\phi \to \psi$ as $!\phi \multimap \psi$; **BI** admits no such decomposition.

In common with **R**, **BI** is a distributive logic, where additive disjunction distributes over additive and multiplicative conjunctions. Linear logic does not admit distribution, and so is *incomplete* for Kripke Resource Semantics.

Categorical models of linear logic are based on two closed categories (often one is a Kleisli category), with functors between them [8, 5]. For **BI**, we use a semantics of proofs based on a single category with two closed structures.

The Tarski-style models of **BI**, which explain formulæ in terms of a notion of truth, are obtained by combining directly existing semantics of intuitionistic and relevant logics. We know of no comparable semantics of linear logic.

BI has multiplicative quantifiers $\forall_{\mathbf{new}}$ and $\exists_{\mathbf{new}}$, alongside the additive quantifiers. Despite many rumblings on this topic over the years, we have not been able to locate a worked-out predicate logic with multiplicative quantifiers, apart from Ambler's system for the existential only [2], the formulation of which is somewhat more complex than ours. Also relevant is the theory of multiplicative dependent function types in [15], together with its fibrational semantics [14]; it can be regarded as relying on a version of bunches appropriate to dependent type theory.

Linear logic has a number of computational readings, including: the number-of-uses reading, based on the original coherence space model; an eager and lazy evaluation reading, based on a strict function model; a concurrency reading of proofs [1]; and a logic programming interpretation [26, 13]. In these early days for **BI**, we have just two related computational readings. At the propositional level, **BI**'s λ-calculus, $\alpha\lambda$, can be used to characterize local variables and local state in imperative programming languages such as Idealized Algol and Reynolds' Syntactic Control of Interference [20, 19]. At the predicate level, **BI**'s hereditary Harrop fragment provides a basis for logic programming within which accounts of local variables, modularity and scoping can be developed [20, 23, 22]. This can be seen as the sharing interpretation for answer substitutions-as-resources. To see this, consider the following version of *resolution*, which is available in **BI**:

$$\frac{(X)\Gamma \vdash G \qquad \dfrac{}{(Y)A\sigma \vdash A'\sigma}\ Id \quad (A\sigma = A'\sigma)}{(X;Y)\Gamma;\forall_{\mathbf{new}}y.G \to A \vdash A'}\ Resolution,$$

where A and A' are atoms and G is an hereditary Harrop goal-formula [26, 18]. Here the rule must be read as a *reduction (or search) operator*, from conclusion to premisses. The idea is that the program is given by the antecendent, *i.e.*, the bunch $(X;Y)\ \Gamma;\forall_{\mathbf{new}}y.G \to A$, contains the clause

$\forall_{\mathbf{new}} y.G \rightarrow A$. We suppose that the *local variable* y occurs in A and A' but not in G. The resolution rule codes both an $\rightarrow$E and a $\forall_{\mathbf{new}}$E, both read as reduction operators, the latter rule replacing Y with Y, y *on the right-hand branch only*. Thus the use of $\forall_{\mathbf{new}}$ confines y, and hence the answer substitution σ calculated for it, to the invocation of the clause: y does not persist on the left-hand branch, *i.e.*, the remainder of the computation.

Finally, we conjecture that predicate **BI** provides a suitable basis for a Hoare-style logic for imperative languages with local state.

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