Overview

Introduction

Modelling parallel systems

Linear Time Properties

- state-based and linear time view ⟵
- definition of linear time properties
- invariants and safety
- liveness and fairness

Regular Properties

Linear Temporal Logic

Computation-Tree Logic

Equivalences and Abstraction

transition system $\mathcal{T} = (S, Act, \longrightarrow, S_0, AP, L)$

$\downarrow$ abstraction from actions

state graph $G_{\mathcal{T}}$

- set of nodes = state space S
- edges = transitions without action label

Act for modeling interactions/communication and specifying fairness assumptions

AP, L for specifying properties

State-based view of TS

transition system $\mathcal{T} = (S, Act, \longrightarrow, S_0, AP, L)$

↓ abstraction from actions

state graph $G_{\mathcal{T}}$

- set of nodes = state space S
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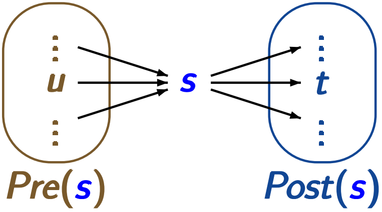
use standard notations for graphs, e.g.,

$$Post(s) = \{t \in S : s \to t\}$$

$$Pre(s) = \{u \in S : u \to s\}$$



Execution and path fragments

execution fragment: sequence of consecutive transitions

$s_0 \xrightarrow{\alpha_0} s_1 \xrightarrow{\alpha_1} \ldots$ infinite or

$s_0 \xrightarrow{\alpha_0} s_1 \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_{n-1}} s_n$ finite

path fragment: sequence of states arising from the projection of an execution fragment to the states

$\pi = s_0\, s_1\, s_2 \ldots$ infinite or $\pi = s_0\, s_1 \ldots s_n$ finite

such that $s_{i+1} \in Post(s_i)$ for all $i < |\pi|$

initial: if $s_0 \in S_0$ = set of initial states

maximal: if infinite or ending in a terminal state

Notations for paths

path fragment: sequence of states

$\pi = s_0\, s_1\, s_2 \ldots$ infinite or $\pi = s_0\, s_1 \ldots s_n$ finite

s.t. $s_{i+1} \in Post(s_i)$ for all $i < |\pi|$

initial: if $s_0 \in S_0$ = set of initial states

maximal: if infinite or ending in terminal state

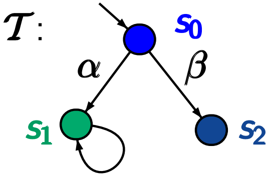
path of TS $\mathcal{T}$ $\widehat{=}$ initial, maximal path fragment

path of state s $\widehat{=}$ maximal path fragment starting in state s

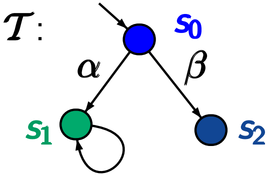
$Paths(\mathcal{T})$ = set of all initial, maximal path fragments

$Paths(s)$ = set of all maximal path fragments starting in state s

Paths of a TS



How many paths are there in $\mathcal{T}$?

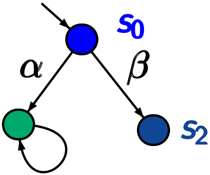


How many paths are there in $\mathcal{T}$?

answer: **2**, namely $s_0\, s_1\, s_1\, s_1 \ldots$ and $s_0\, s_2$

Paths of a TS and its states

$\mathcal{T}$:

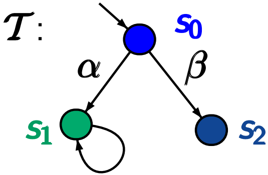


How many paths are there in $\mathcal{T}$?

answer: 2, namely $s_0\, s_1\, s_1\, s_1 \ldots$ and $s_0\, s_2$

$$\begin{aligned} \mathit{Paths}(s_1) &= \text{set of all maximal paths fragments} \\ &\quad\ \text{starting in } s_1 \\ &= \{s_1^\omega\} \text{ where } s_1^\omega = s_1\, s_1\, s_1\, s_1\, \ldots \end{aligned}$$

Paths of a TS and its states



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$$Paths(s_1) = \text{set of all maximal paths fragments starting in } s_1$$
$$= \{s_1^\omega\} \text{ where } s_1^\omega = s_1\, s_1\, s_1\, s_1\, \ldots$$

$$Paths_{fin}(s_1) = \text{set of all finite path fragments starting in } s_1$$
$$= \{s_1^n : n \in \mathbb{N}, n \geq 1\}$$

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Linear-time vs branching-time

Linear-time vs branching-time

transition system

$$\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$$

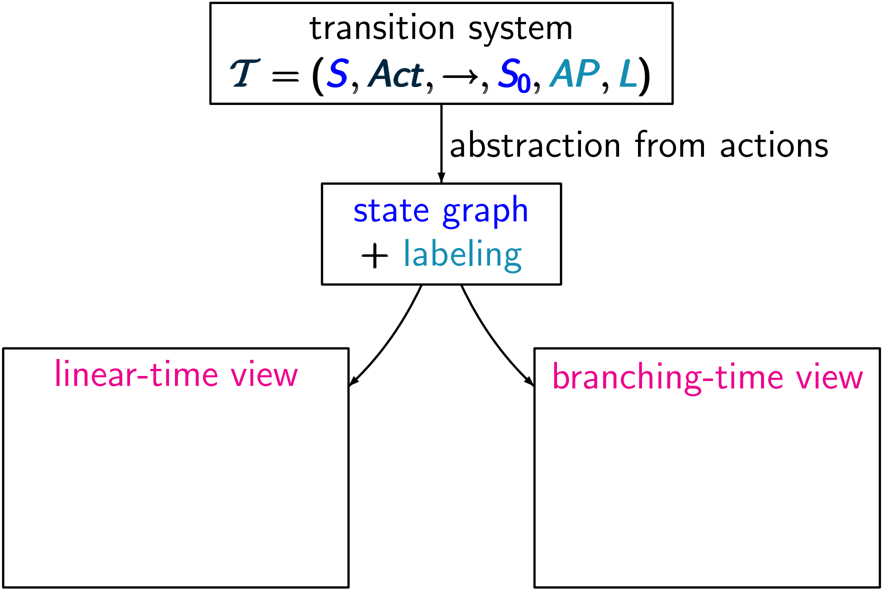
Linear-time vs branching-time

transition system
$\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$

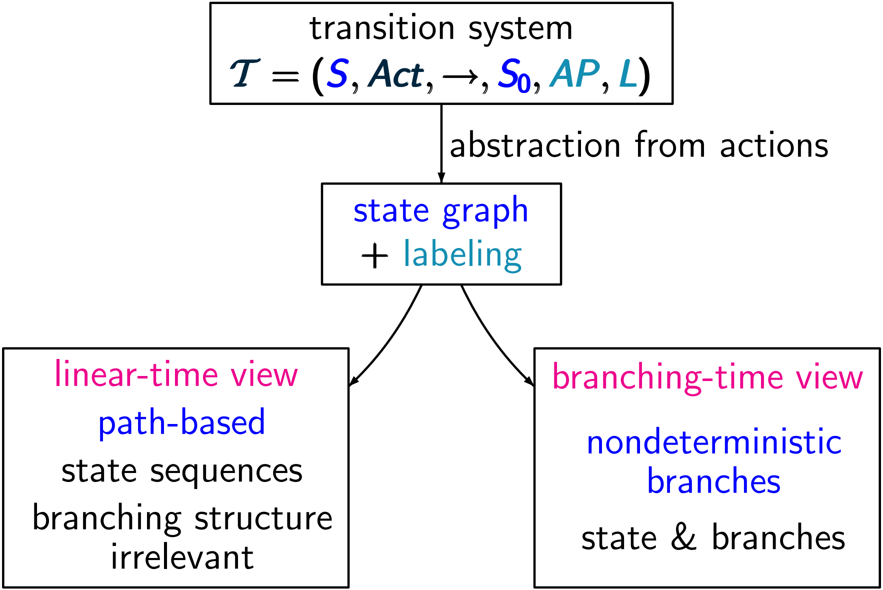
↓ abstraction from actions

state graph
+ labeling

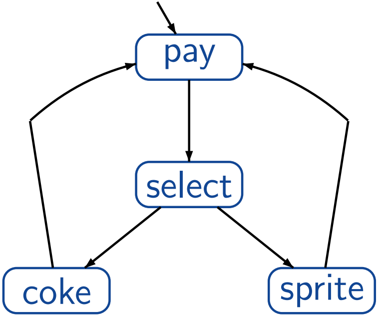
Linear-time vs branching-time



Linear-time vs branching-time

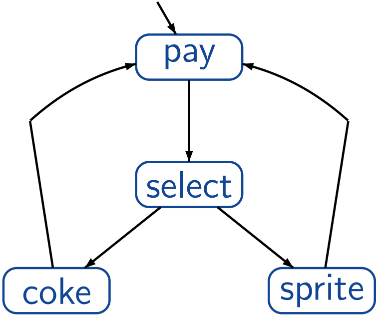


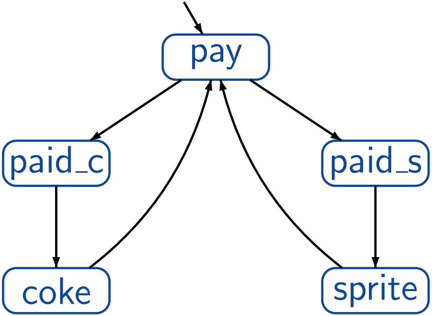
Example: vending machine



vending machine with
1 coin deposit
select drink after
having paid

Example: vending machine





vending machine with
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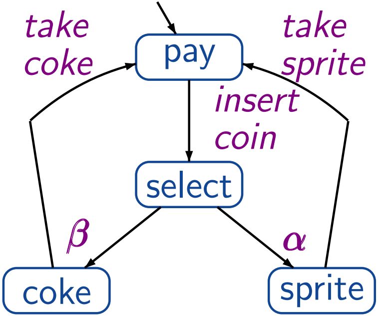
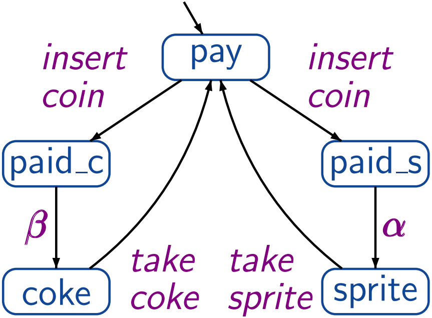
select drink after having paid

vending machine with
2 coin deposits
select drink by inserting the coin

Example: vending machine





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select drink by inserting
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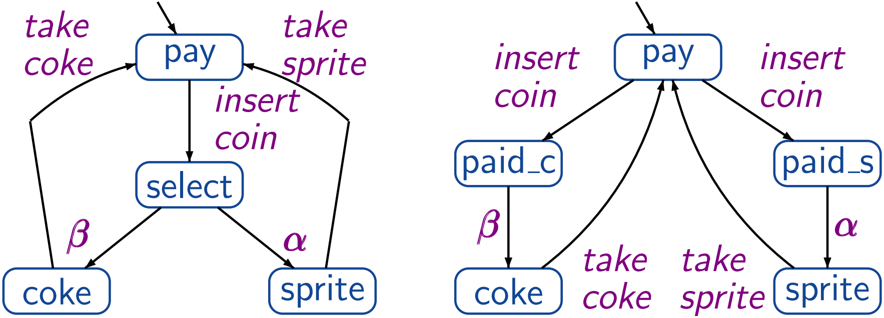
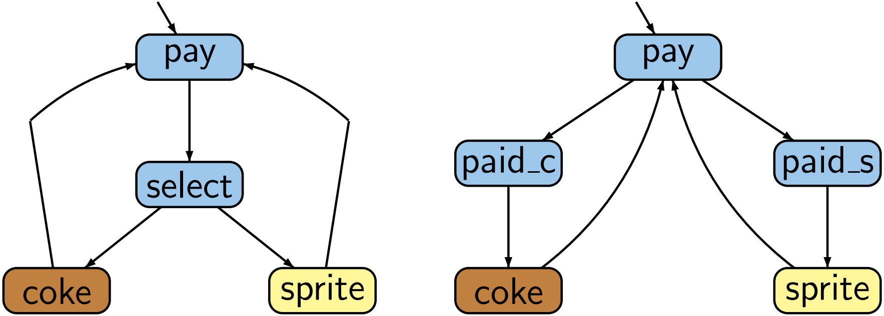
Example: vending machine



state based view: abstracts from actions and projects onto atomic propositions, e.g. $AP = \{coke, sprite\}$

Example: vending machine



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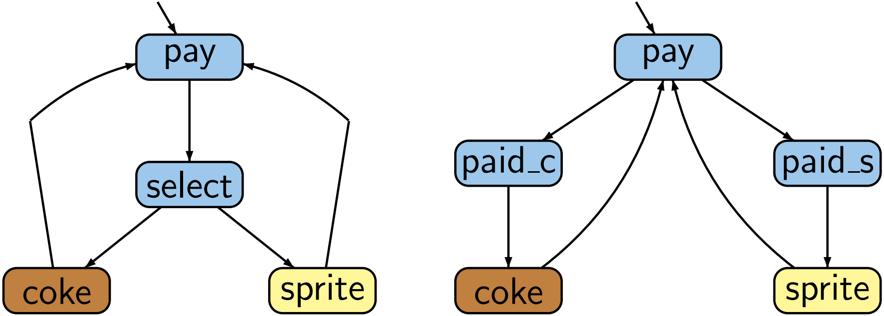
e.g., $L(\text{coke}) = \{coke\}$, $L(\text{pay}) = \varnothing$

Example: vending machine



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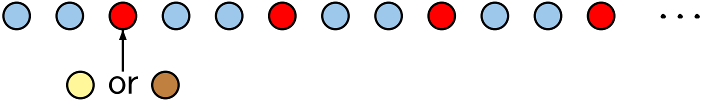
linear time: all observable behaviors are of the form



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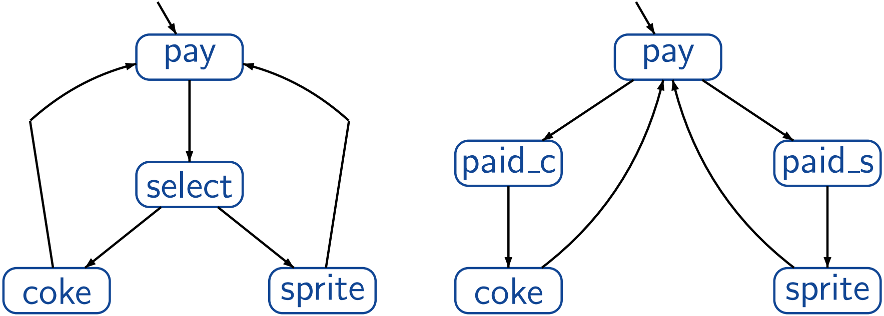

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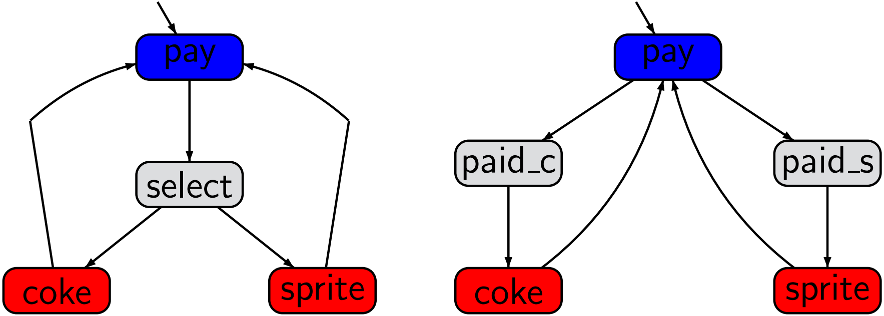

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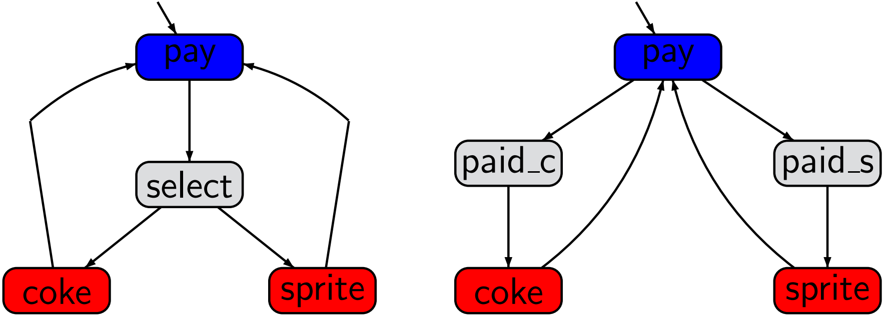
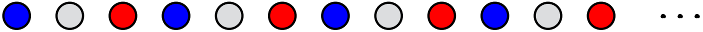


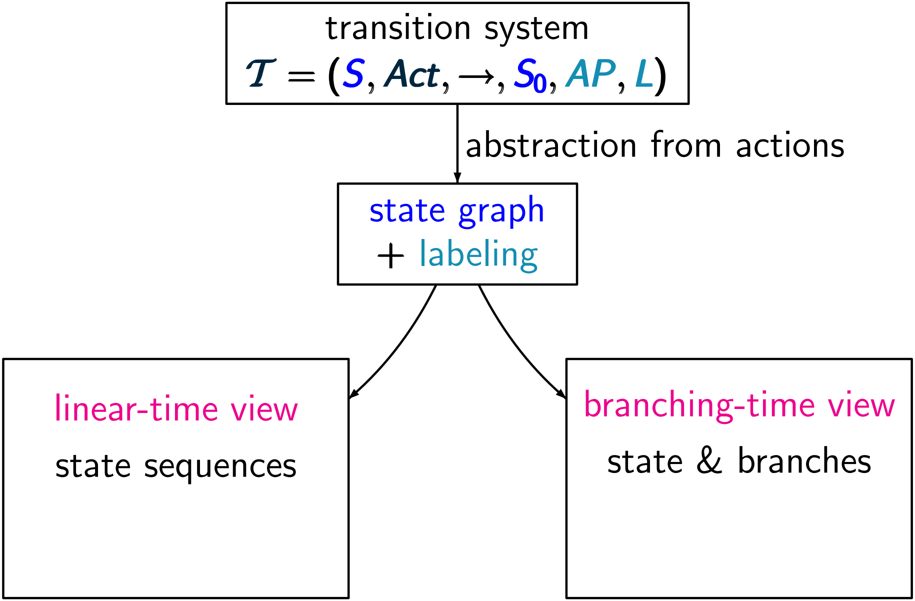
state based view: abstracts from actions and projects on atomc propositions, e.g., $AP = \{pay, drink\}$

linear & branching time:

all observable behaviors have the form



Linear-time vs branching-time



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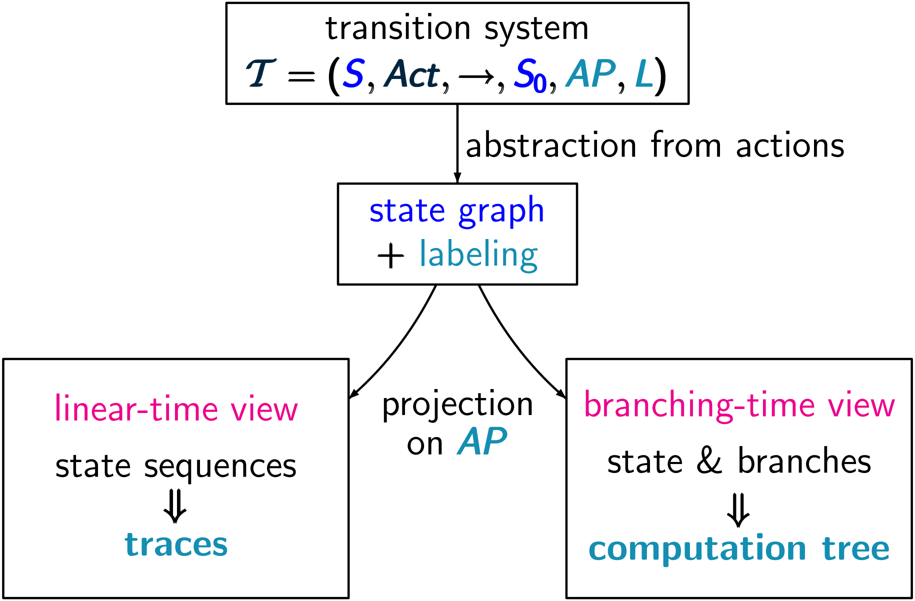

Traces

LTB2.4-4

for TS with labeling function $L : S \to 2^{AP}$

execution: states + actions

$s_0 \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} s_2 \xrightarrow{\alpha_3} \ldots$ infinite or finite

↓

paths: sequences of states

$s_0\, s_1\, s_2 \ldots$ infinite or $s_0\, s_1 \ldots s_n$ finite

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traces: sequences of sets of atomic propositions

$L(s_0)\, L(s_1)\, L(s_2)\, \ldots$

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traces: sequences of sets of atomic propositions

$L(s_0)\, L(s_1)\, L(s_2)\, \ldots \quad \in (2^{AP})^{\omega} \cup (2^{AP})^{+}$

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for simplicity: we often assume that the given TS has **no terminal states**

Traces

for TS with labeling function $L : S \rightarrow 2^{AP}$

execution: states + actions

$s_0 \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} s_2 \xrightarrow{\alpha_3} \ldots$ infinite or ~~finite~~

paths: sequences of states

$s_0\, s_1\, s_2 \ldots$ infinite or ~~$s_0\, s_1 \ldots s_n$ finite~~

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for simplicity: we often assume that the given TS has **no terminal states**

Treatment of terminal states

perform standard graph algorithms to compute the reachable fragment of the given TS

$$Reach(\mathcal{T}) = \left\{ \begin{array}{c} \text{set of states that are reachable} \\ \text{from some initial state} \end{array} \right.$$

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for each reachable terminal state s:

- if s stands for an intended halting configuration then add a transition from s to a trap state:

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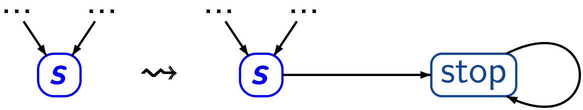

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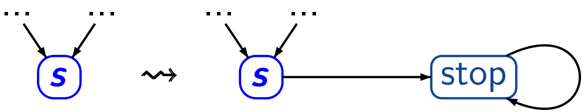

- if s stands for system fault, e.g., deadlock then correct the design before checking further properties

Traces of a transition system

Let $\mathcal{T}$ be a TS

$$Traces(\mathcal{T}) \stackrel{\text{def}}{=} \{trace(\pi) : \pi \in Paths(\mathcal{T})\}$$

$$Traces_{fin}(\mathcal{T}) \stackrel{\text{def}}{=} \{trace(\widehat{\pi}) : \widehat{\pi} \in Paths_{fin}(\mathcal{T})\}$$

Traces of a transition system

Let $\mathcal{T}$ be a TS

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↑ initial, maximal path fragment

$$Traces_{fin}(\mathcal{T}) \stackrel{\text{def}}{=} \{trace(\widehat{\pi}) : \widehat{\pi} \in Paths_{fin}(\mathcal{T})\}$$

↑ initial, finite path fragment

Traces of a transition system

Let $\mathcal{T}$ be a TS ⟵ *without* terminal states

$$Traces(\mathcal{T}) \stackrel{\text{def}}{=} \{trace(\pi) : \pi \in Paths(\mathcal{T})\}$$

↑ initial, infinite path fragment

$$Traces_{fin}(\mathcal{T}) \stackrel{\text{def}}{=} \{trace(\widehat{\pi}) : \widehat{\pi} \in Paths_{fin}(\mathcal{T})\}$$

↑ initial, finite path fragment

Traces of a transition system

Let $\mathcal{T}$ be a TS ⟵ *without* terminal states

$$Traces(\mathcal{T}) \stackrel{\text{def}}{=} \{trace(\pi) : \pi \in Paths(\mathcal{T})\} \subseteq (2^{AP})^{\omega}$$

↑ initial, infinite path fragment

$$Traces_{fin}(\mathcal{T}) \stackrel{\text{def}}{=} \{trace(\widehat{\pi}) : \widehat{\pi} \in Paths_{fin}(\mathcal{T})\} \subseteq (2^{AP})^{*}$$

↑ initial, finite path fragment

Example: traces

Let $\mathcal{T}$ be a TS without terminal states.

$$Traces(\mathcal{T}) \stackrel{\text{def}}{=} \{trace(\pi) : \pi \in Paths(\mathcal{T})\} \subseteq (2^{AP})^{\omega}$$

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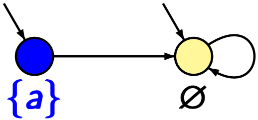

TS $\mathcal{T}$ with a single atomic proposition a

Example: traces

Let $\mathcal{T}$ be a TS without terminal states.

$$Traces(\mathcal{T}) \stackrel{\text{def}}{=} \{trace(\pi) : \pi \in Paths(\mathcal{T})\} \subseteq (2^{AP})^\omega$$

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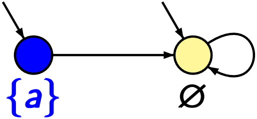
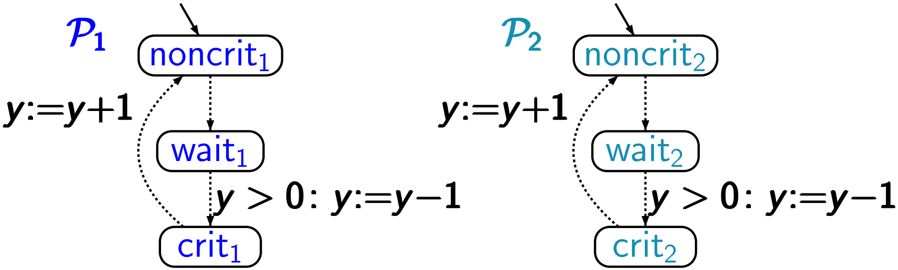


TS $\mathcal{T}$ with a single atomic proposition a

$$Traces(\mathcal{T}) = \{ \{a\}\varnothing^\omega, \varnothing^\omega \}$$

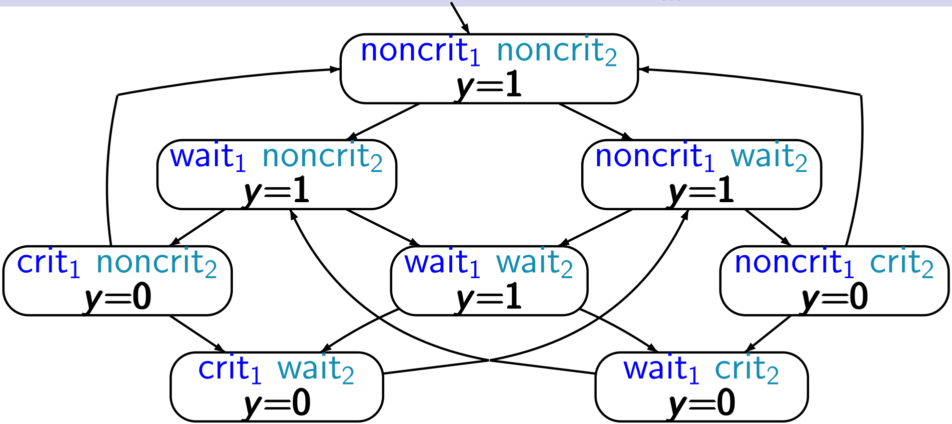
$$Traces_{fin}(\mathcal{T}) = \{\{a\}\varnothing^n : n \geq 0\} \cup \{\varnothing^m : m \geq 1\}$$

Mutual exclusion with semaphore



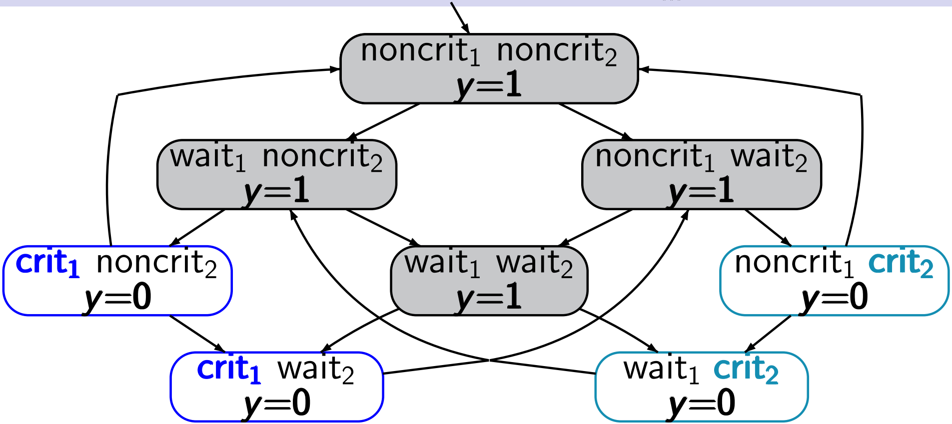
transition system $\mathcal{T}_{\mathcal{P}_1 ||| \mathcal{P}_2}$ arises by unfolding the composite program graph $\mathcal{P}_1 ||| \mathcal{P}_2$

Mutual exclusion with semaphore $\mathcal{T}_{\mathcal{P}_1 ||| \mathcal{P}_2}$



set of atomic propositions $AP = \{\mathbf{crit_1}, \mathbf{crit_2}\}$

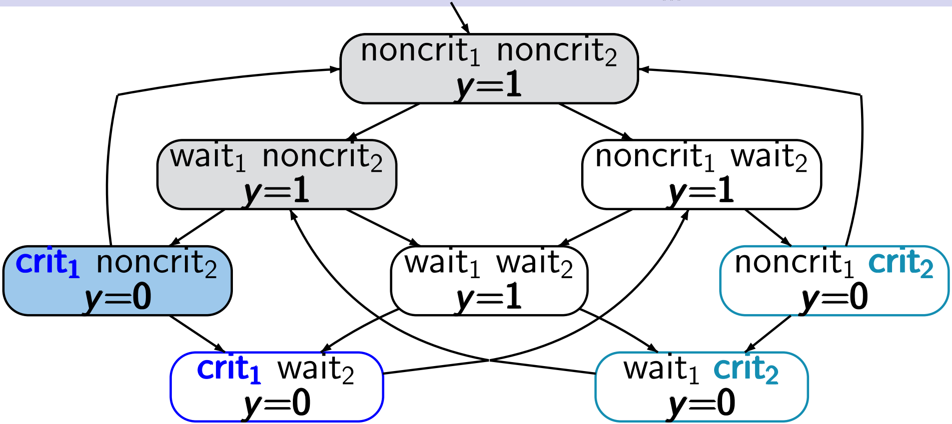
Mutual exclusion with semaphore $\mathcal{T}_{\mathcal{P}_1 ||| \mathcal{P}_2}$



set of atomic propositions $AP = \{\mathsf{crit}_1, \mathsf{crit}_2\}$

e.g., $L(\langle \mathsf{noncrit}_1, \mathsf{noncrit}_2, y{=}1\rangle) =$
$L(\langle \mathsf{wait}_1, \mathsf{noncrit}_2, y{=}1\rangle) = \varnothing$

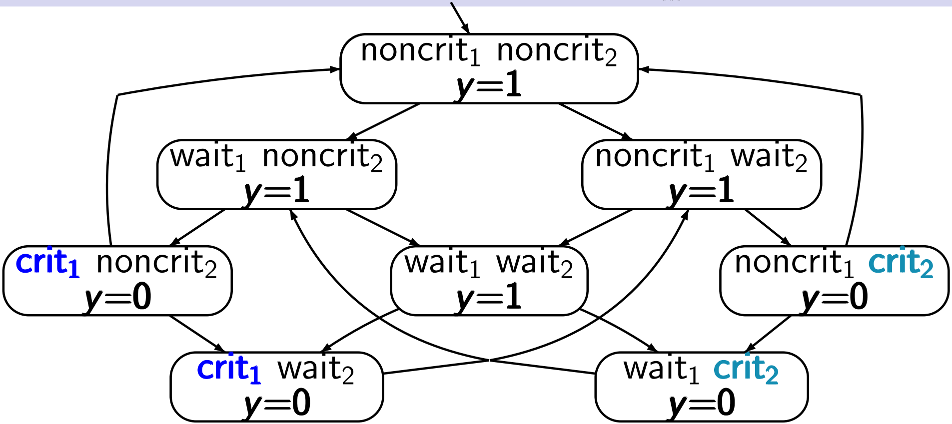
Mutual exclusion with semaphore $\mathcal{T}_{\mathcal{P}_1 ||| \mathcal{P}_2}$



set of atomic propositions $AP = \{crit_1, crit_2\}$

traces, e.g., $\varnothing\ \varnothing\ \{crit_1\}\ \varnothing\ \varnothing\ \{crit_1\}\ \varnothing\ \varnothing\ \{crit_1\} \ldots$

Mutual exclusion with semaphore $\mathcal{T}_{\mathcal{P}_1 ||| \mathcal{P}_2}$



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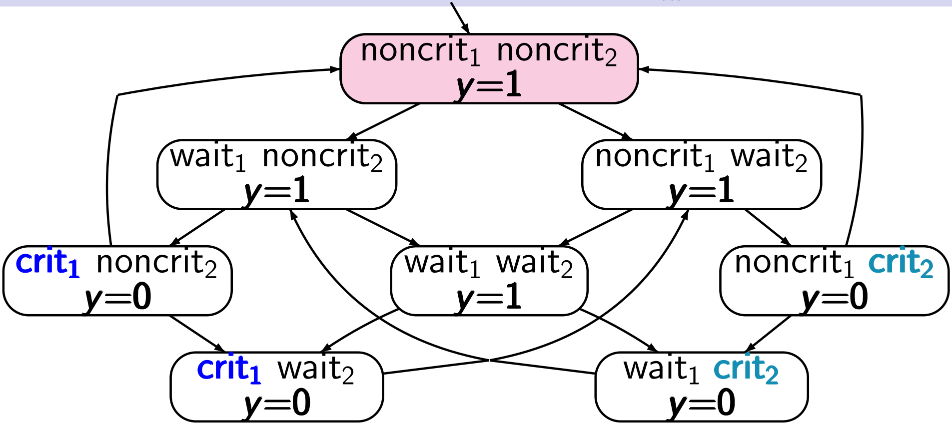
Mutual exclusion with semaphore $\mathcal{T}_{\mathcal{P}_1 ||| \mathcal{P}_2}$



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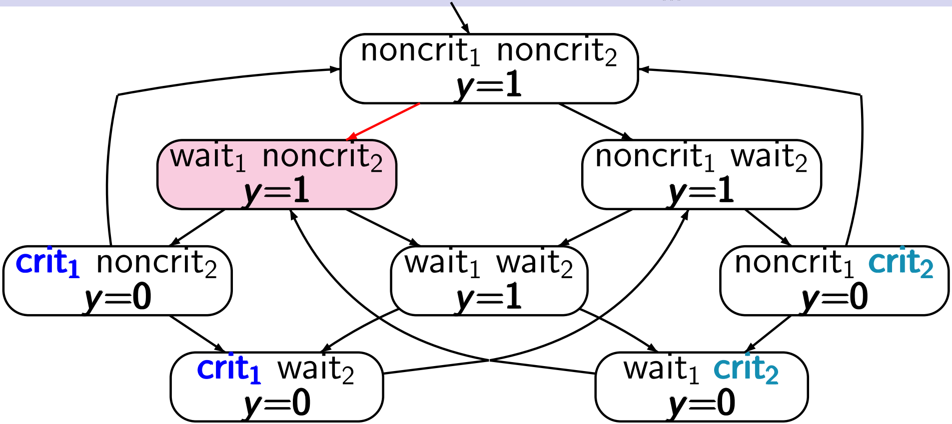
Mutual exclusion with semaphore $\mathcal{T}_{\mathcal{P}_1 ||| \mathcal{P}_2}$



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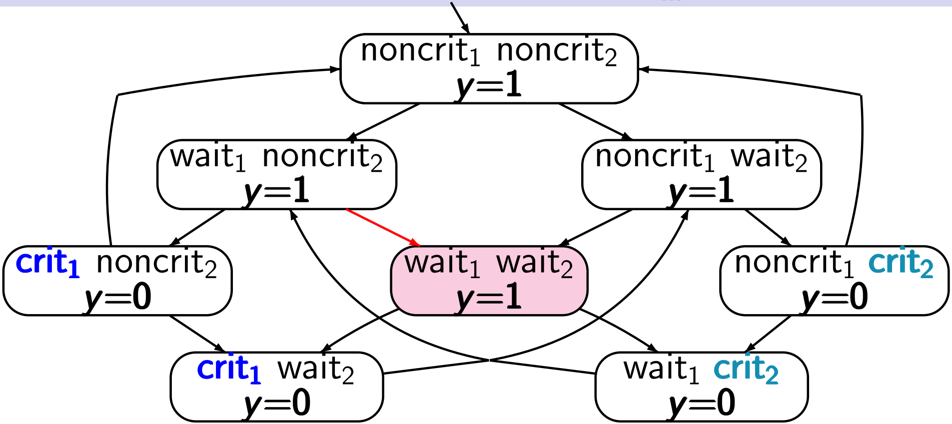
Mutual exclusion with semaphore $\mathcal{T}_{\mathcal{P}_1 ||| \mathcal{P}_2}$



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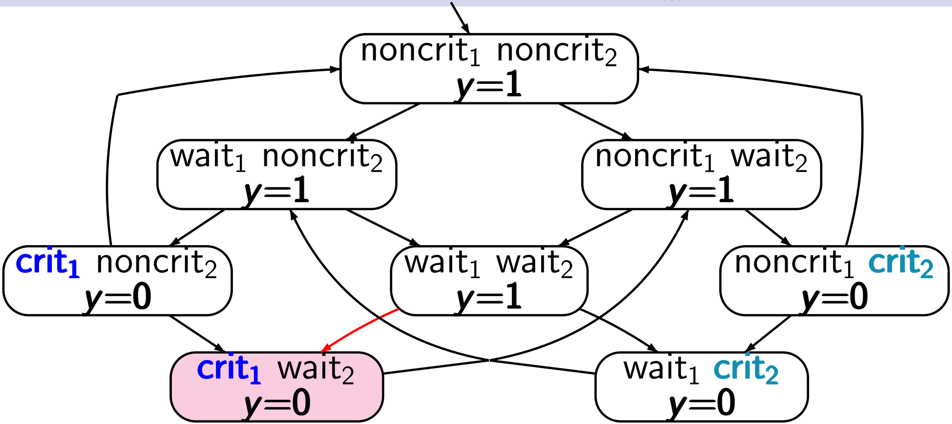
Mutual exclusion with semaphore $\mathcal{T}_{\mathcal{P}_1 ||| \mathcal{P}_2}$



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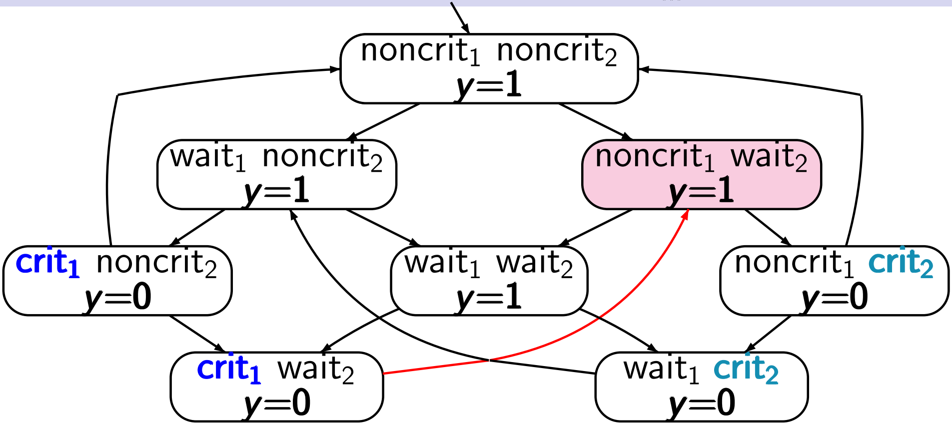

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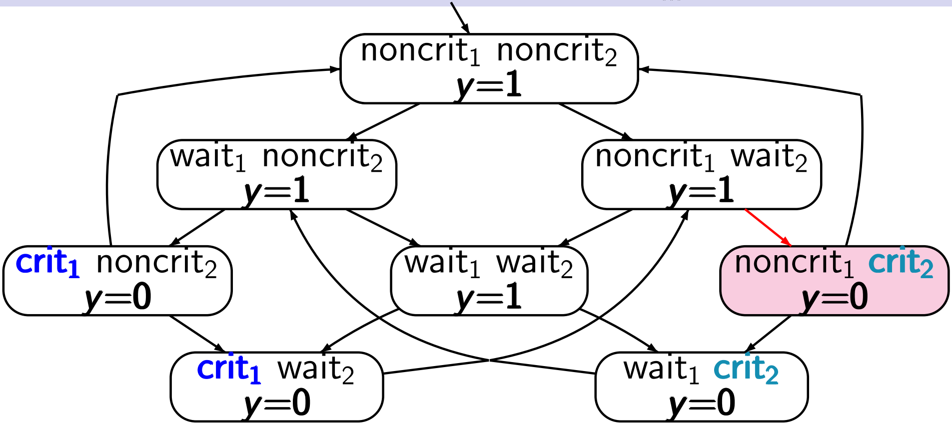

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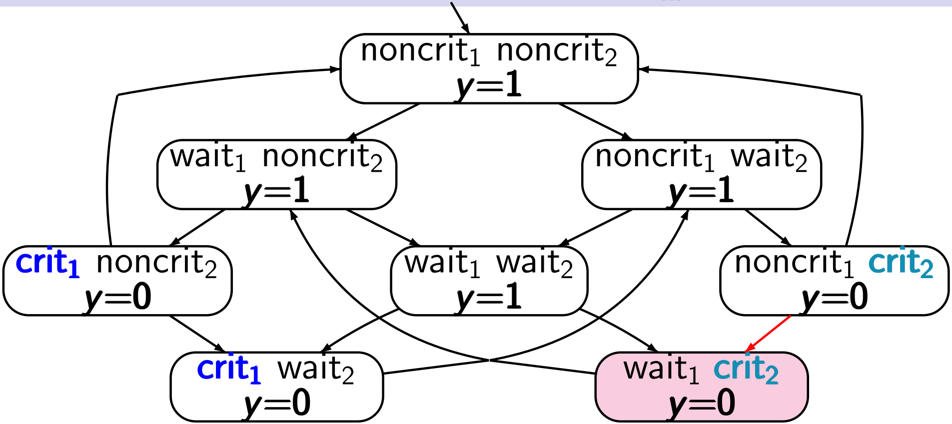

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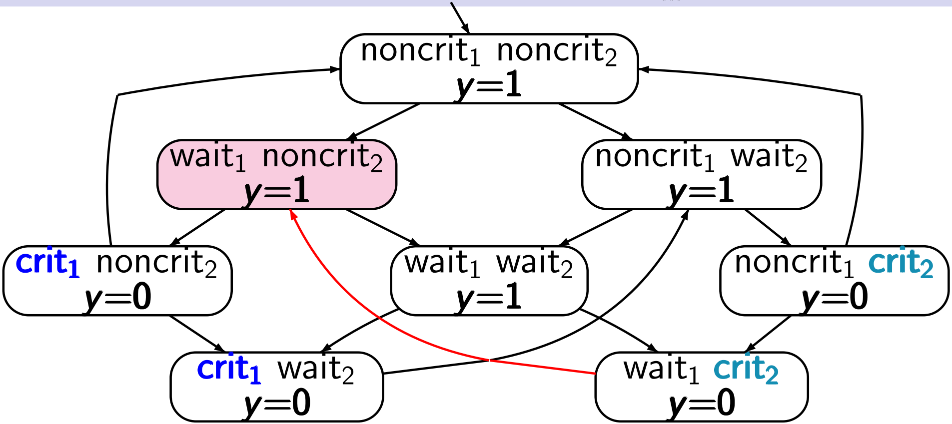

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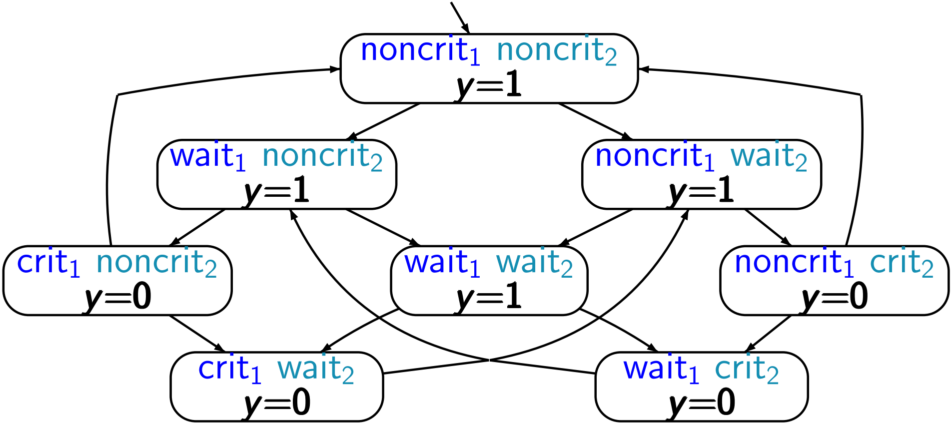

set of propositions $AP = \{\mathbf{wait_1}, \mathbf{crit_1}, \mathbf{wait_2}, \mathbf{crit_2}\}$

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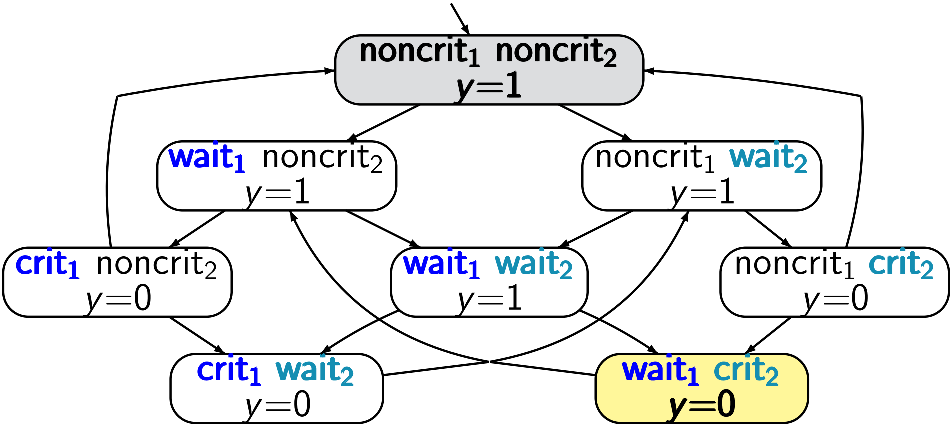
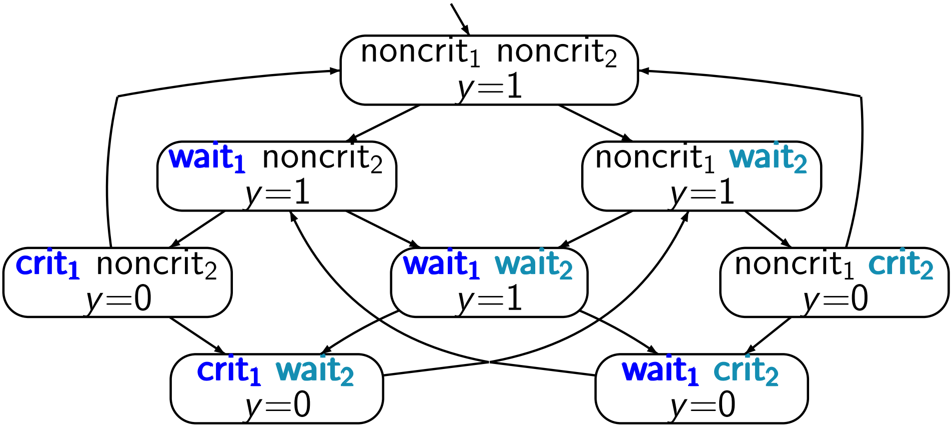


set of propositions $AP = \{wait_1, crit_1, wait_2, crit_2\}$

e.g., $L(\langle noncrit_1, noncrit_2, y{=}1\rangle) = \varnothing$

$L(\langle wait_1, crit_2, y{=}1\rangle) = \{wait_1, crit_2\}$

Mutual exclusion with semaphor $\mathcal{T}_{\mathcal{P}_1 ||| \mathcal{P}_2}$



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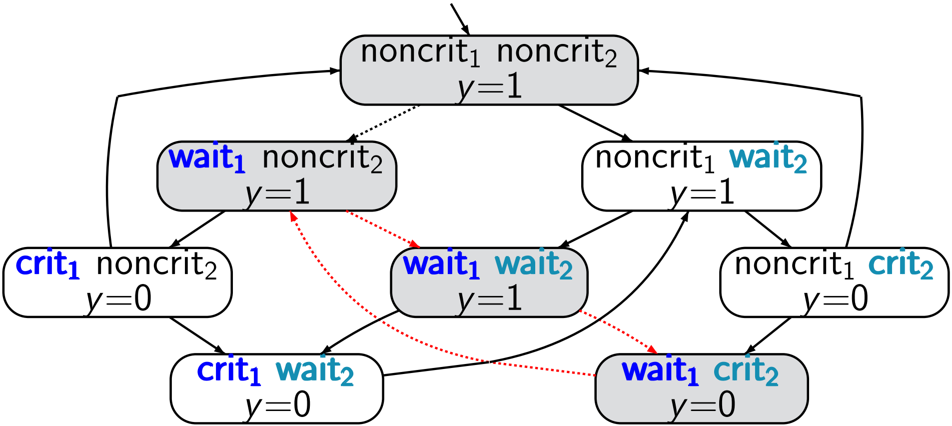

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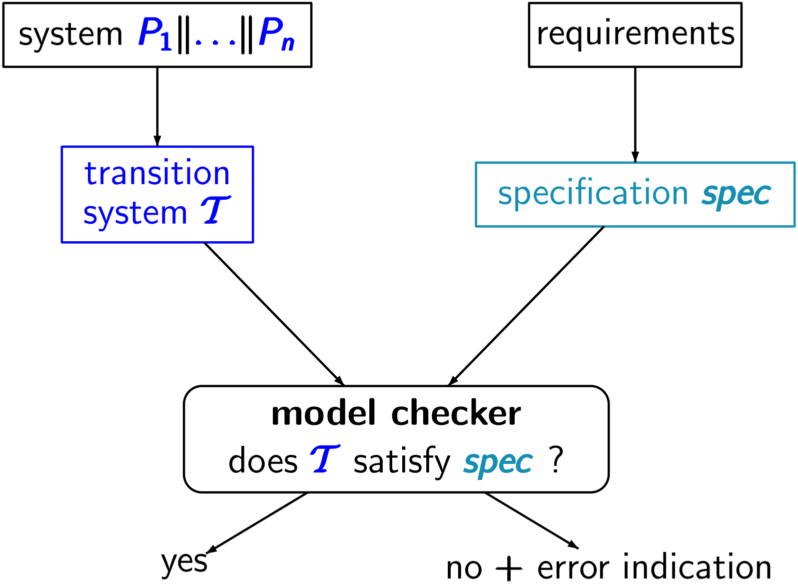
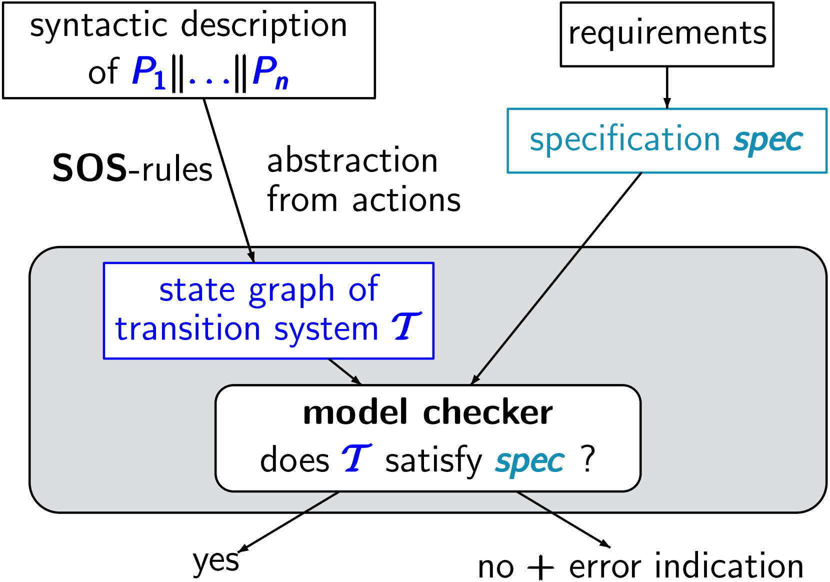
Overview

Model checking



Model checking



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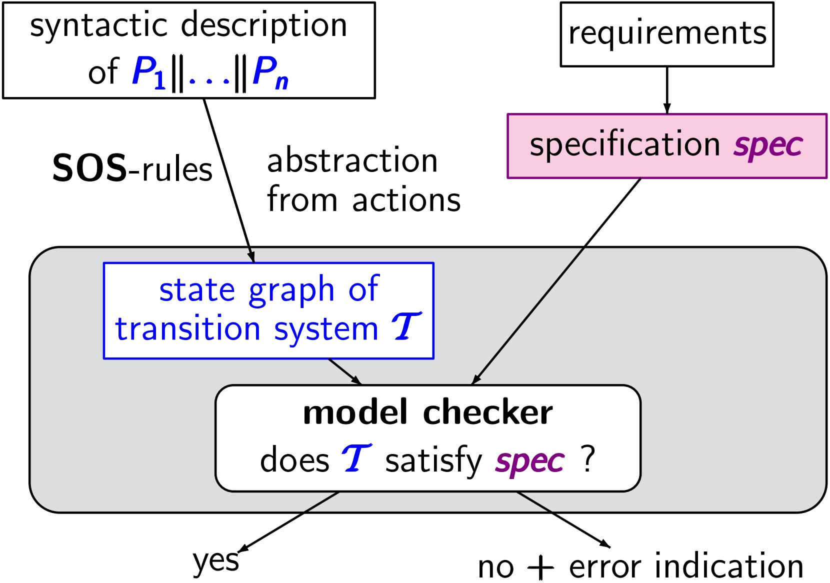

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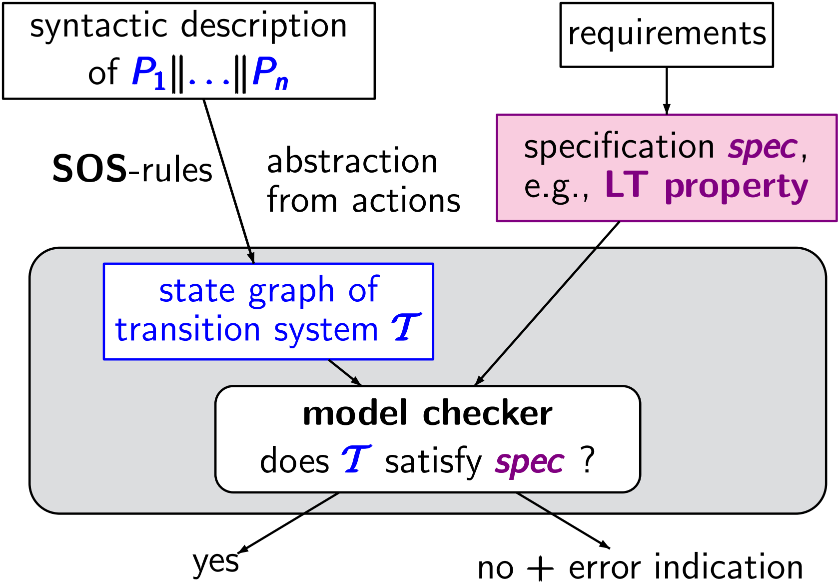

Linear-time properties (LT properties)

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for TS over AP without terminal states

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E.g., for mutual exclusion problems and
$AP = \{crit_1, crit_2, \ldots\}$

safety:

$MUTEX =$ set of all infinite words $A_0\, A_1\, A_2 \ldots$ over 2^{AP} such that for all $i \in \mathbb{N}$:
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LT properties for mutual exclusion protocols

$AP = \{wait_1, crit_1, wait_2, crit_2\}$

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$\varnothing \{wait_1\} \{crit_1\} \varnothing \{wait_1\} \{crit_1\} \ldots \quad \in MUTEX$

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$$\varnothing\ \{wait_1\}\ \{crit_1\}\ \varnothing\ \{wait_1\}\ \{crit_1\}\ \ldots \qquad \in MUTEX$$

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liveness (starvation freedom):

$LIVE$ = set of all infinite words $A_0\, A_1\, A_2 \ldots$ s.t.

$$\overset{\infty}{\exists}\, i \in \mathbb{N}.wait_1 \in A_i \implies \overset{\infty}{\exists}\, i \in \mathbb{N}.crit_1 \in A_i$$

$$\wedge\ \overset{\infty}{\exists}\, i \in \mathbb{N}.wait_2 \in A_i \implies \overset{\infty}{\exists}\, i \in \mathbb{N}.crit_2 \in A_i$$

Satisfaction relation for LT properties

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Satisfaction relation $\models$ for TS:

If $\mathcal{T}$ is a TS (without terminal states) over AP and E an LT property over AP then

$$\mathcal{T} \models E \quad \text{iff} \quad Traces(\mathcal{T}) \subseteq E$$

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Satisfaction relation $\models$ for TS and states:

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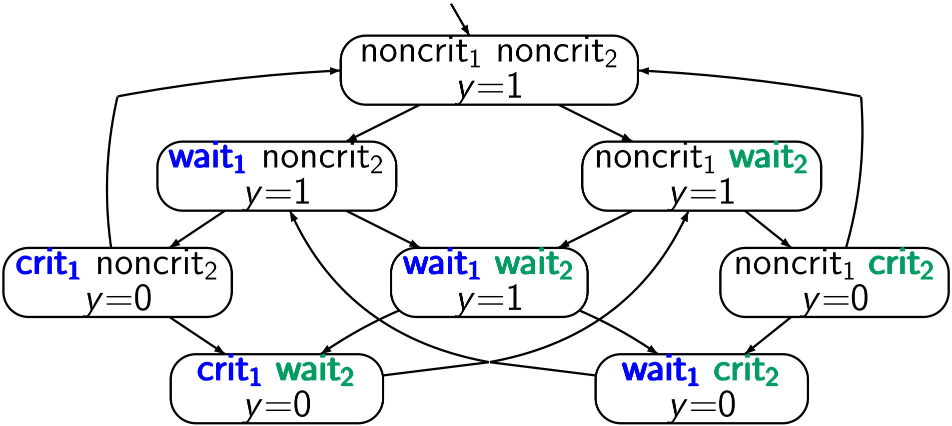
If s is a state in $\mathcal{T}$ then

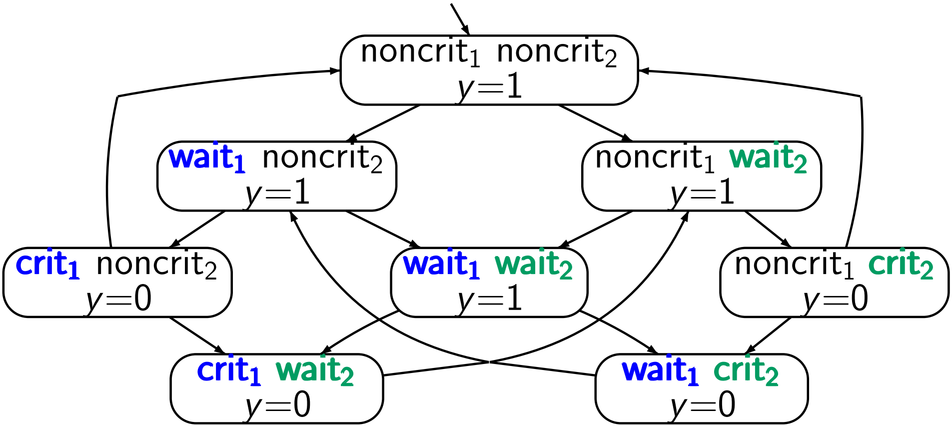
$$s \models E \quad \text{iff} \quad Traces(s) \subseteq E$$

Mutual exclusion with semaphore



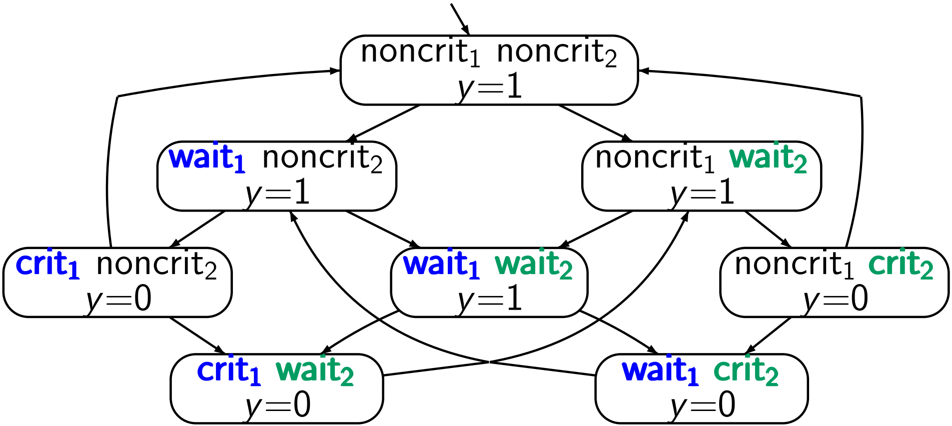
$\mathcal{T}_{Sem} \models \mathit{MUTEX}$

Mutual exclusion with semaphore



$\mathcal{T}_{Sem} \models MUTEX, \quad \mathcal{T}_{Sem} \models LIVE$?

Mutual exclusion with semaphore



$$\mathcal{T}_{Sem} \models \mathit{MUTEX}, \quad \mathcal{T}_{Sem} \not\models \mathit{LIVE}$$

$$\varnothing\ \{wait_1\}\ \big(\ \{wait_1, wait_2\}\ \{crit_1, wait_2\}\{wait_2\}\ \big)^{\omega} \notin \mathit{LIVE}$$

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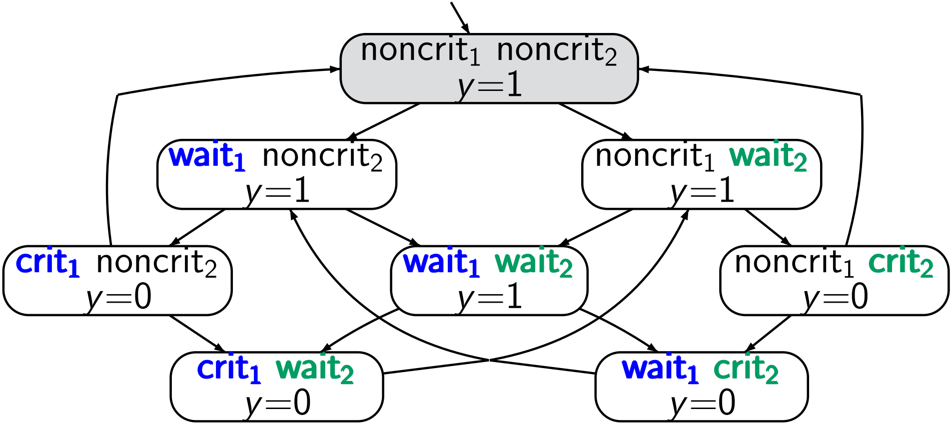

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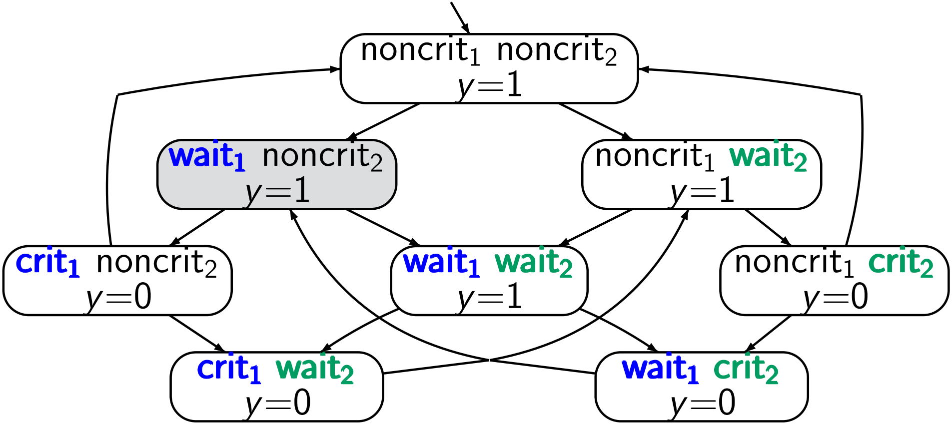

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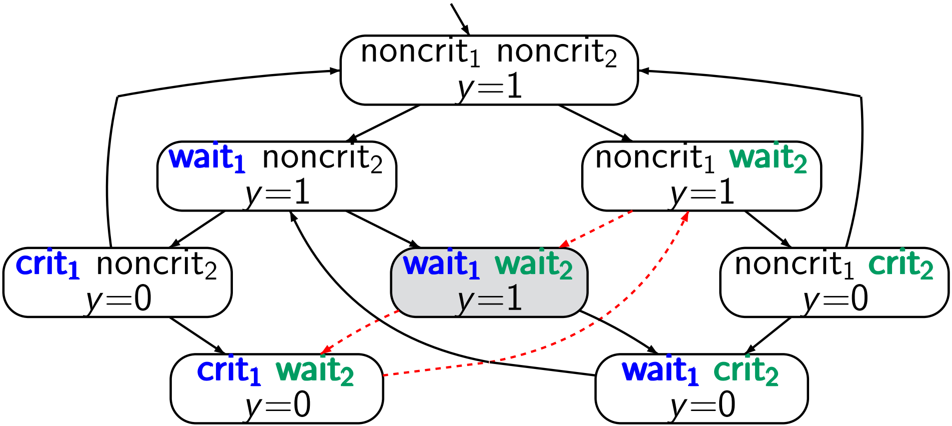

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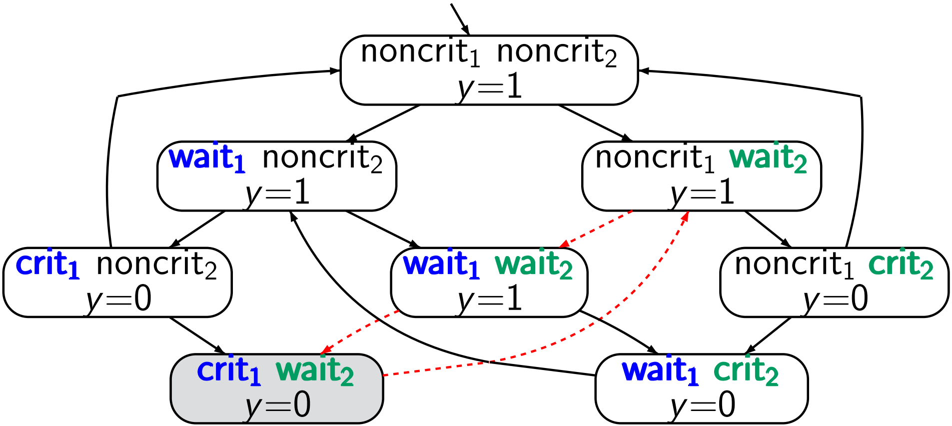

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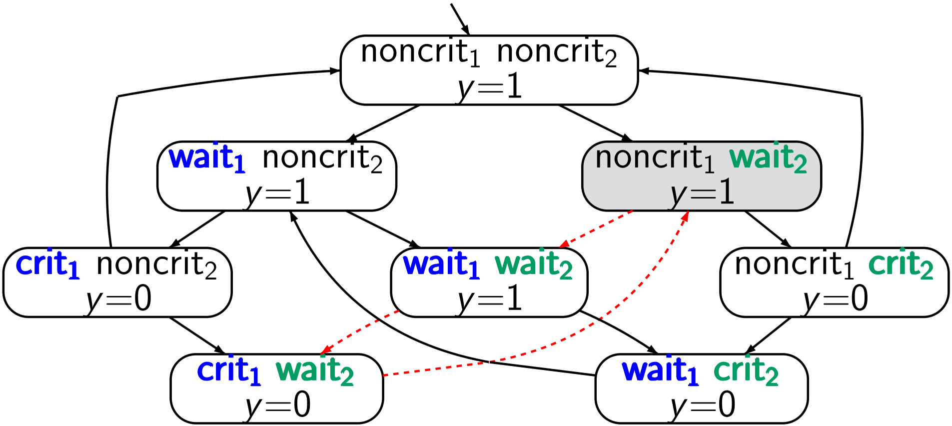

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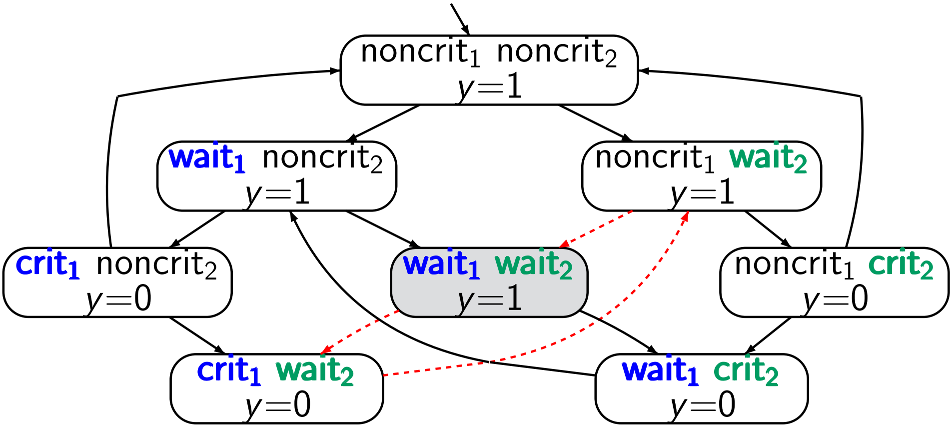

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Peterson's mutual exclusion algorithm

LTB2.4-17

Peterson's mutual exclusion algorithm

for competing processes $\mathcal{P}_1$ and $\mathcal{P}_2$,

using three additional shared variables

$b_1, b_2 \in \{0,1\}$, $x \in \{1,2\}$

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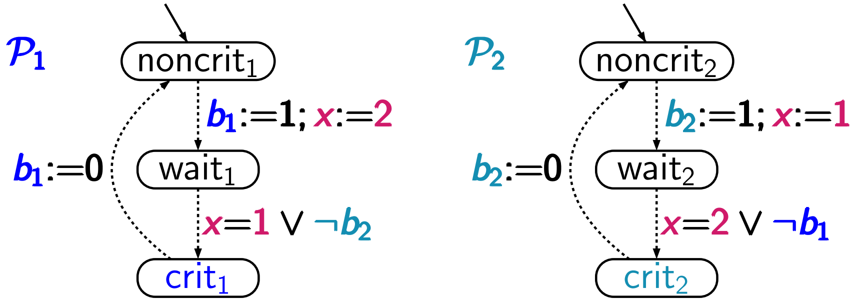

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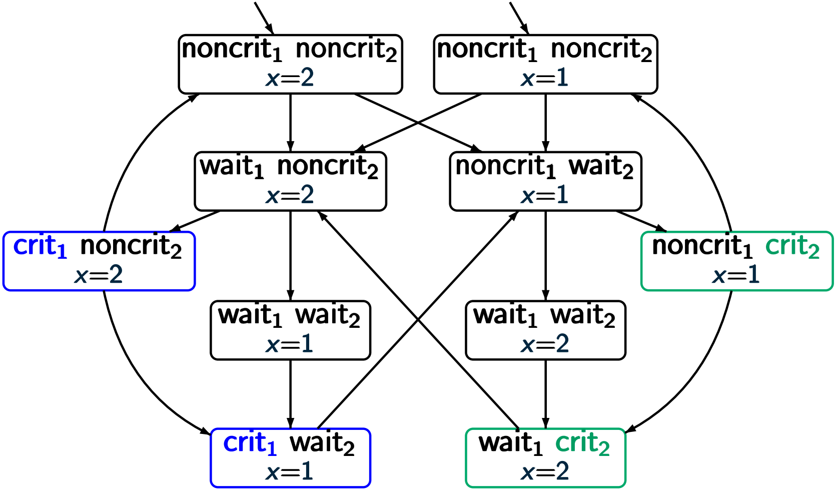
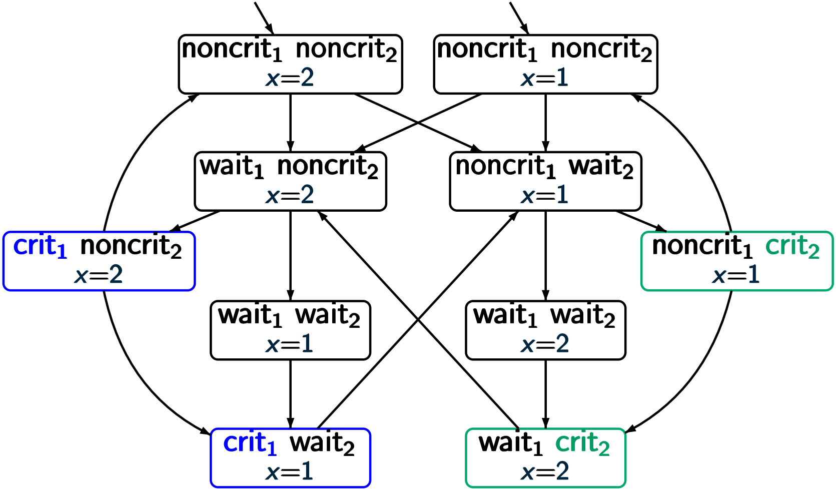


$\mathcal{T}_{Pet} \models \mathit{MUTEX}$

Peterson's mutual exclusion algorithm



$\mathcal{T}_{Pet} \models \mathit{MUTEX}$ and $\mathcal{T}_{Pet} \models \mathit{LIVE}$

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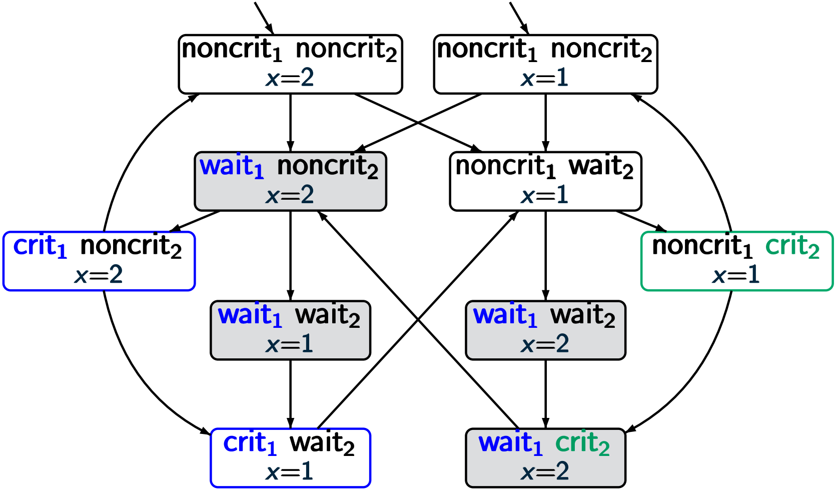

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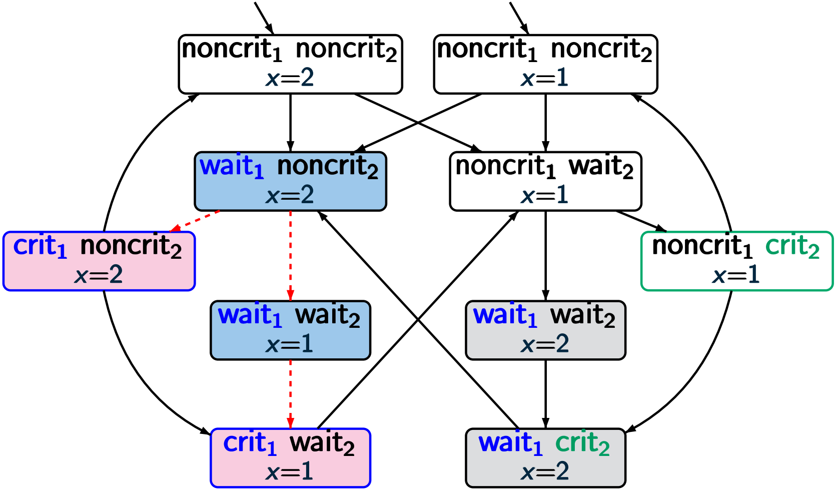

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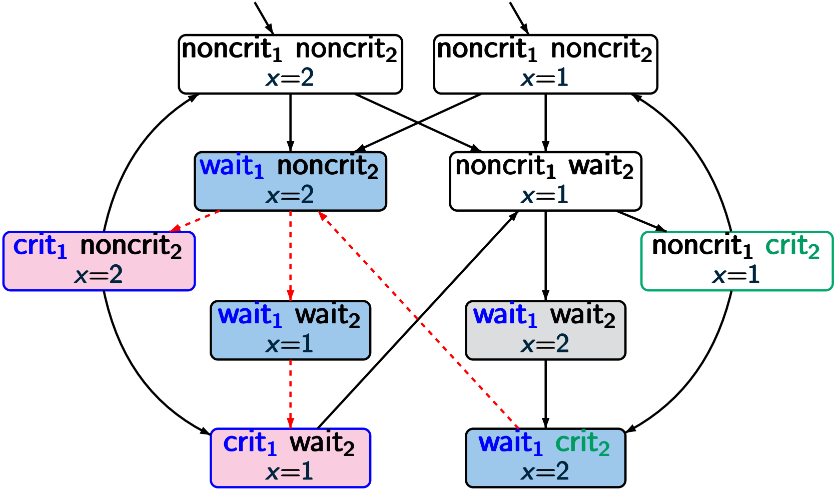

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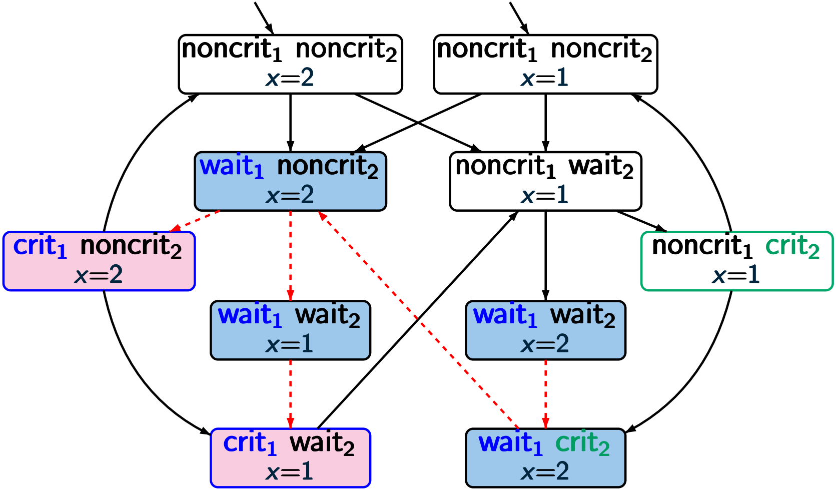

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LT properties and trace inclusion

An LT property over AP is a language E of infinite words over the alphabet $\Sigma = 2^{AP}$, i.e., $E \subseteq \left(2^{AP}\right)^{\omega}$.

If $\mathcal{T}$ is a TS over AP then $\mathcal{T} \models E$ iff $Traces(\mathcal{T}) \subseteq E$.

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Consequence of these definitions:

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are TS over AP then for all LT properties E over AP:

$$Traces(\mathcal{T}_1) \subseteq Traces(\mathcal{T}_2) \wedge \mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$$

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note: $Traces(\mathcal{T}_1) \subseteq Traces(\mathcal{T}_2) \subseteq E$

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If $\mathcal{T}_1$ and $\mathcal{T}_2$ are TS over AP then the following statements are equivalent:

(1) $Traces(\mathcal{T}_1) \subseteq Traces(\mathcal{T}_2)$

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$(1) \Longrightarrow (2)$: $\checkmark$

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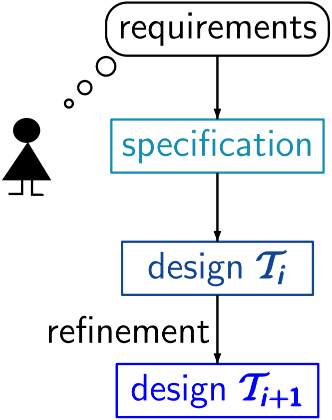
$(2) \Longrightarrow (1)$: consider $E = Traces(\mathcal{T}_2)$

Relevance of trace inclusion

Trace inclusion appears naturally

- as an implementation/refinement relation
- when resolving nondeterminism
- in the context of abstractions

Software design cycle



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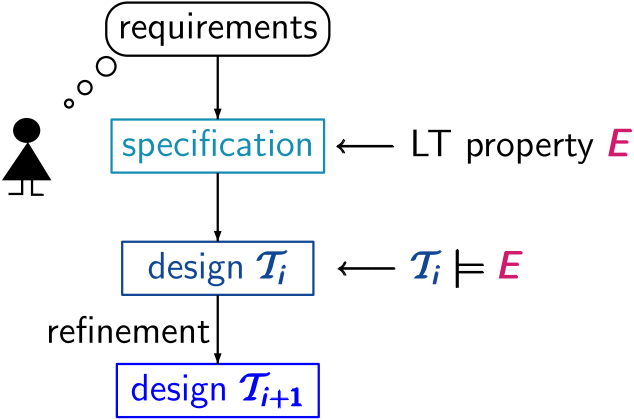

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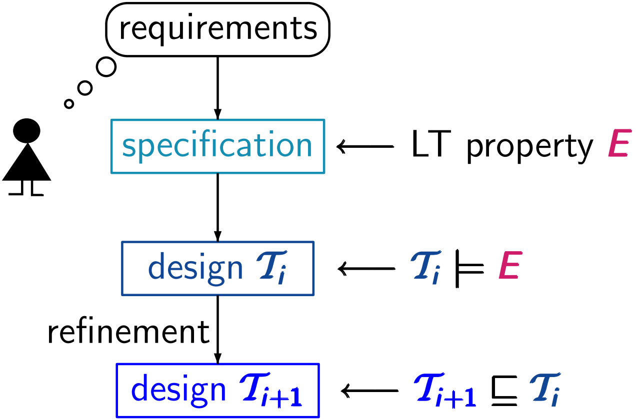


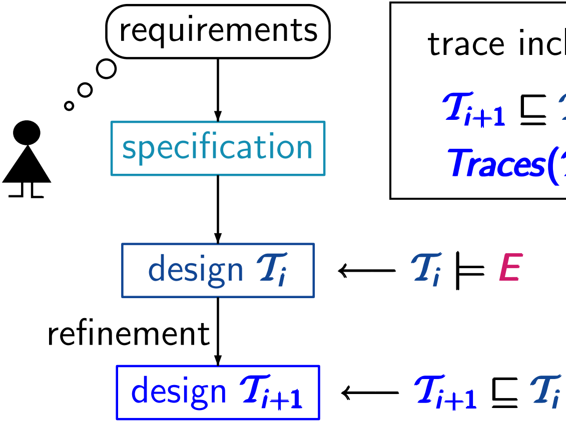
implementation/refinement relation $\sqsubseteq$:

$\mathcal{T}_{i+1} \sqsubseteq \mathcal{T}_i$ iff "$\mathcal{T}_{i+1}$ correctly implements $\mathcal{T}_i$"

trace inclusion

$\mathcal{T}_{i+1} \sqsubseteq \mathcal{T}_i$ iff

$Traces(\mathcal{T}_{i+1}) \subseteq Traces(\mathcal{T}_i)$



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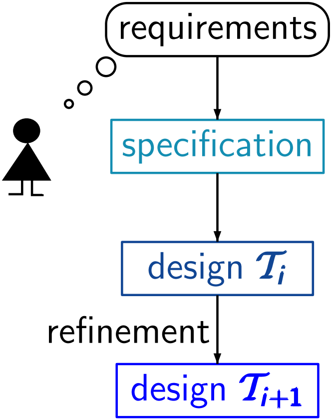
Trace inclusion as an implementation relation



trace inclusion

$\mathcal{T}_{i+1} \sqsubseteq \mathcal{T}_i$ iff

$\mathit{Traces}(\mathcal{T}_{i+1}) \subseteq \mathit{Traces}(\mathcal{T}_i)$

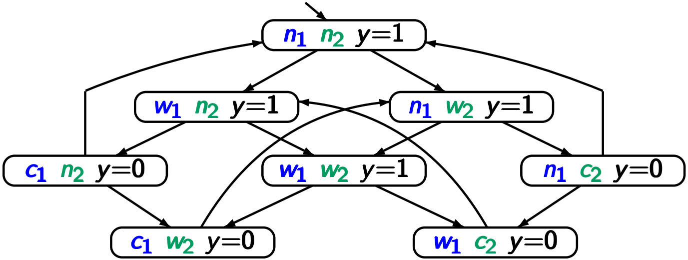
$\longleftarrow \mathcal{T}_i \models E$

$\longleftarrow \mathcal{T}_{i+1} \sqsubseteq \mathcal{T}_i$ implies $\mathcal{T}_{i+1} \models E$

implementation/refinement relation $\sqsubseteq$:

$\mathcal{T}_{i+1} \sqsubseteq \mathcal{T}_i$ iff "$\mathcal{T}_{i+1}$ correctly implements $\mathcal{T}_i$"

Mutual exclusion with semaphore



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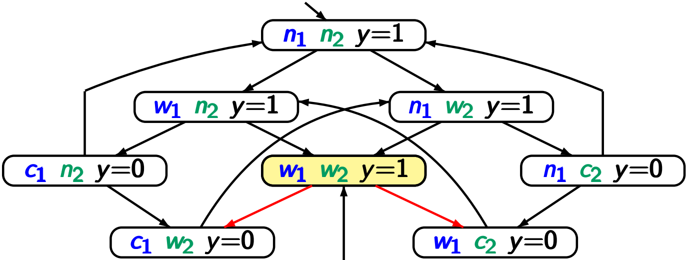
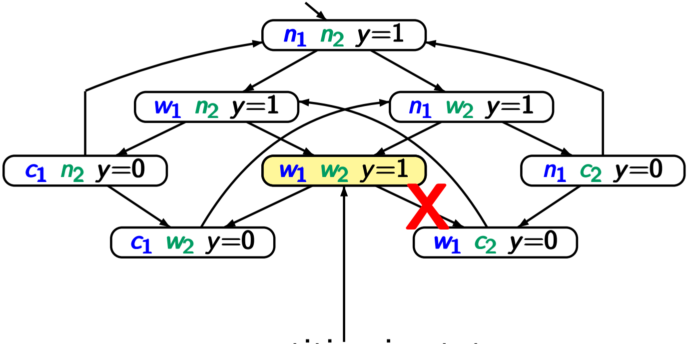


competition in state
$\langle wait_1 \; wait_2 \; y=1 \rangle$

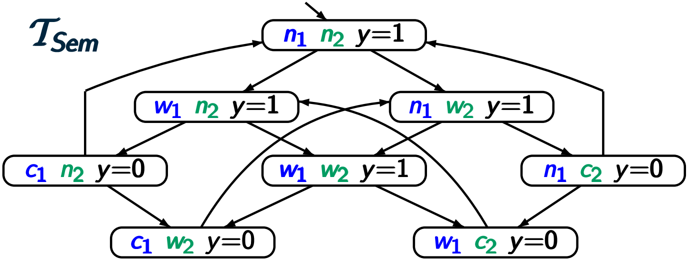
Mutual exclusion with semaphore

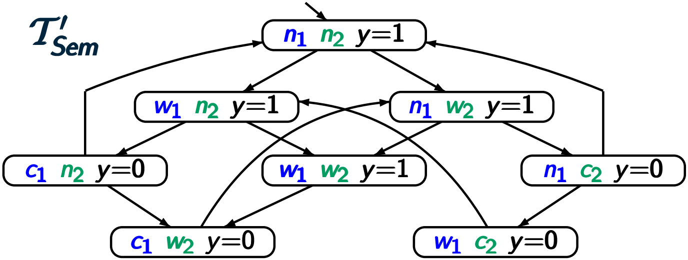


competition in state
$\langle wait_1 \; wait_2 \; y{=}1 \rangle$

resolve the nondeterminism by giving priority to process P_1

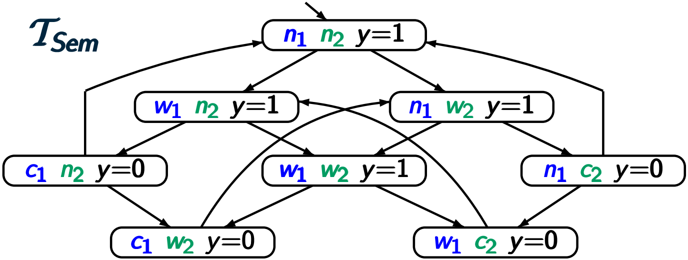
Mutual exclusion with semaphore



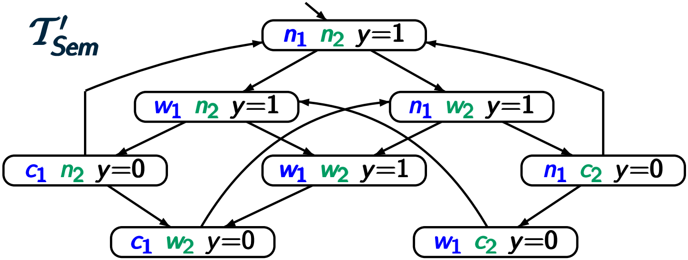


Mutual exclusion with semaphore

$\mathcal{T}_{Sem}$



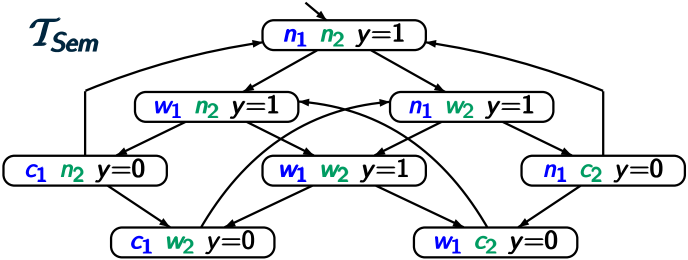
$\mathcal{T}'_{Sem}$



$$Paths(\mathcal{T}'_{Sem}) \subseteq Paths(\mathcal{T}_{Sem})$$

Mutual exclusion with semaphore

$\mathcal{T}_{Sem}$



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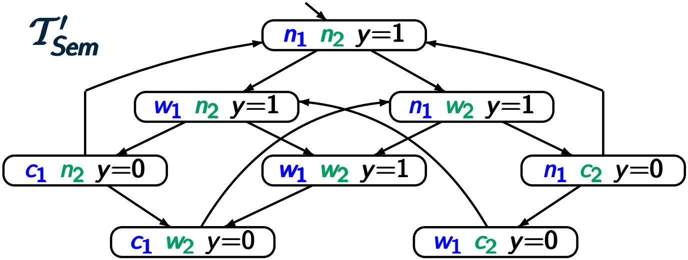
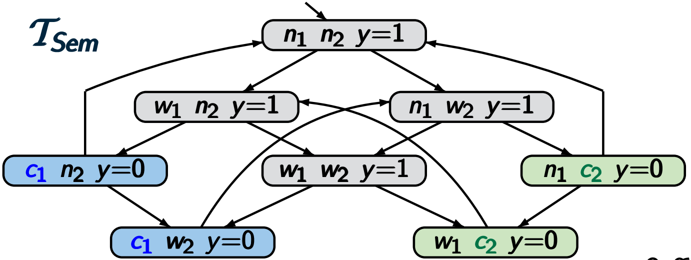


$Traces(\mathcal{T}'_{Sem}) \subseteq Traces(\mathcal{T}_{Sem})$ for any AP

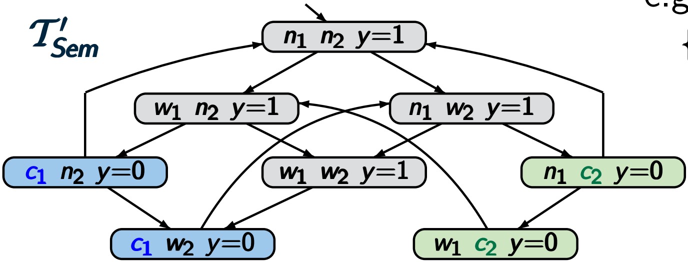
Mutual exclusion with semaphore

$\mathcal{T}_{Sem}$



e.g., for $AP = \{crit_1, crit_2\}$

$\mathcal{T}'_{Sem}$



$Traces(\mathcal{T}_{Sem}) \models E$ implies $Traces(\mathcal{T}'_{Sem}) \models E$ for any E

Relevance of trace inclusion

Trace inclusion appears naturally

- as an implementation/refinement relation
- when resolving nondeterminism $\longleftarrow$

  e.g., $\mathit{Traces}(\mathcal{T}'_{\mathit{Sem}}) \subseteq \mathit{Traces}(\mathcal{T}_{\mathit{Sem}})$
- in the context of abstractions

Trace inclusion appears naturally

- as an implementation/refinement relation
- when resolving nondeterminism ↑

> whenever $\mathcal{T}'$ results from $\mathcal{T}$ by a scheduling policy for resolving nondeterministic choices in $\mathcal{T}$ then
>
> $$Traces(\mathcal{T}') \subseteq Traces(\mathcal{T})$$

- in the context of abstractions

Relevance of trace inclusion

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- as an implementation/refinement relation
- when resolving nondeterminism
- in the context of abstractions $\longleftarrow$

Trace inclusion and data abstraction

```
  ⋮
x:=7;  y:=5;
WHILE x>0 DO
      x:=x−1;
      y:=y+1
OD
  ⋮
```

Trace inclusion and data abstraction


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does $\ell_2 \wedge odd(y)$
never hold ?

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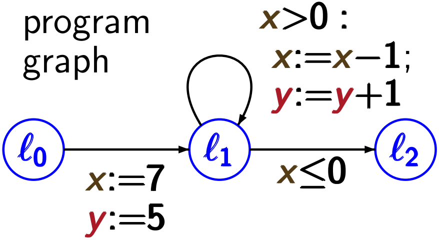
program graph



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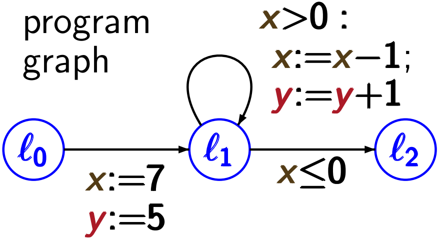

let $\mathcal{T}$ be the associated TS

does $\ell_2 \wedge odd(y)$ never hold ?

$\longleftarrow$ $\mathcal{T} \models$ "never $\ell_2 \wedge odd(y)$" ?

Trace inclusion and data abstraction

$$
\begin{array}{ll}
 & \vdots \\
\ell_0 & x{:=}7;\ \ y{:=}5; \\
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 & \qquad x{:=}x{-}1; \\
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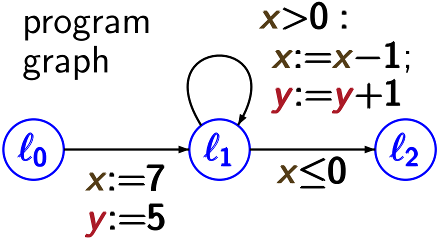
\ell_2 & \vdots
\end{array}
$$

program graph



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data abstraction w.r.t. the predicates

$x{>}0$, $x{=}0$, $x \equiv_2 y$

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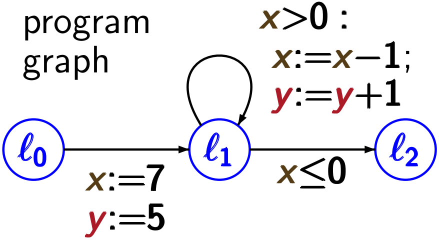

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data abstraction w.r.t. the predicates

$x>0$, $x=0$, $x \equiv_2 y$ $\longleftarrow$ i.e., $x-y$ is even

Trace inclusion and data abstraction


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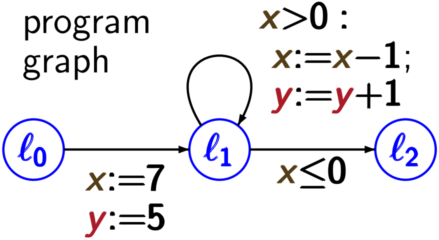

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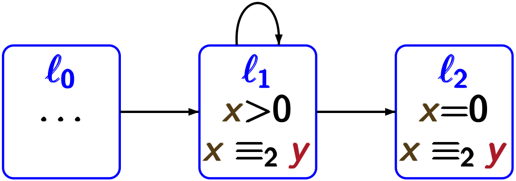

abstract transition system $\mathcal{T}'$

Trace inclusion and data abstraction


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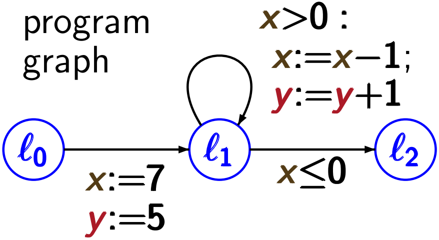

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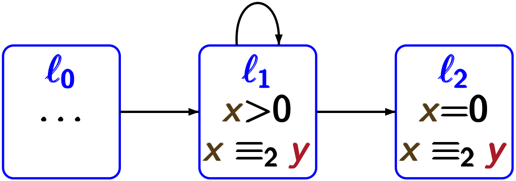

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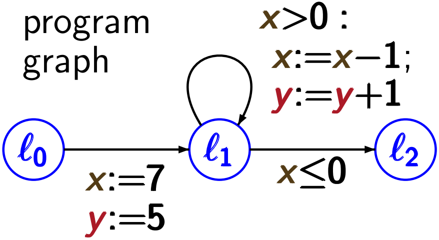

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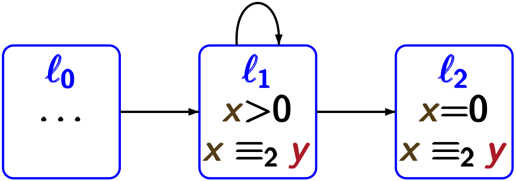

data abstraction w.r.t. the predicates

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$Traces(\mathcal{T}) \subseteq Traces(\mathcal{T}')$

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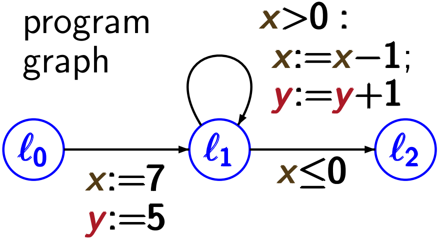

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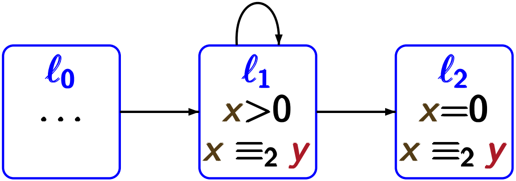

$\mathcal{T} \models$ "never $\ell_2 \wedge odd(y)$"
$\left\{ \begin{array}{l} \mathcal{T}' \models \text{"never } \ell_2 \wedge odd(y)\text{"} \\ Traces(\mathcal{T}) \subseteq Traces(\mathcal{T}') \end{array} \right.$

Trace equivalence

Transition systems $\mathcal{T}_1$ and $\mathcal{T}_2$ over the same set AP of atomic propositions are called trace equivalent iff

$$Traces(\mathcal{T}_1) \; = \; Traces(\mathcal{T}_2)$$

i.e., trace equivalence requires trace inclusion in both directions

Trace equivalent TS satisfy the **same LT properties**

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be TS over AP.

The following statements are equivalent:

(1) $Traces(\mathcal{T}_1) \subseteq Traces(\mathcal{T}_2)$

(2) for all LT-properties E: $\mathcal{T}_2 \models E \Longrightarrow \mathcal{T}_1 \models E$

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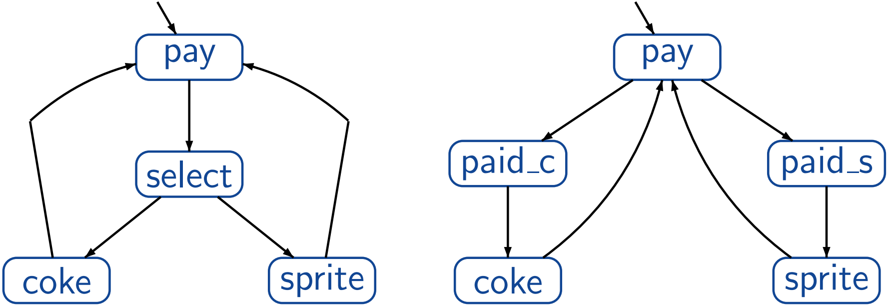
Trace equivalent beverage machines



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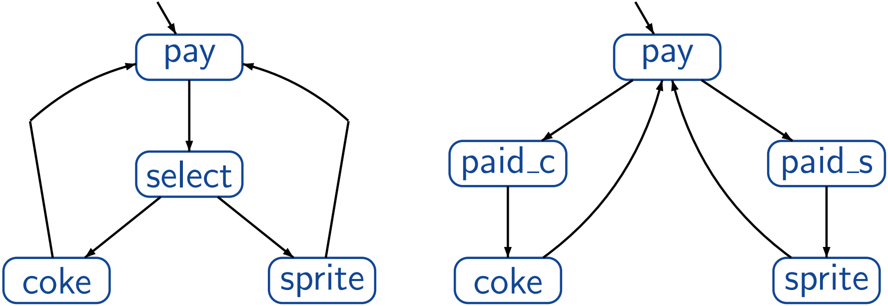
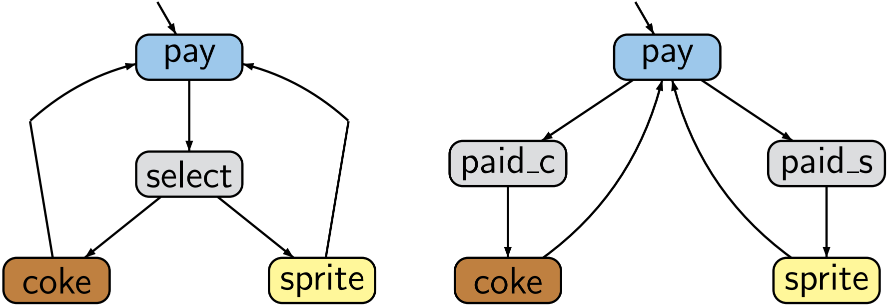


set of atomic propositions $AP = \{pay, coke, sprite\}$

Trace equivalent beverage machines



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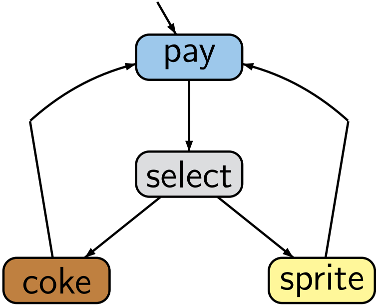
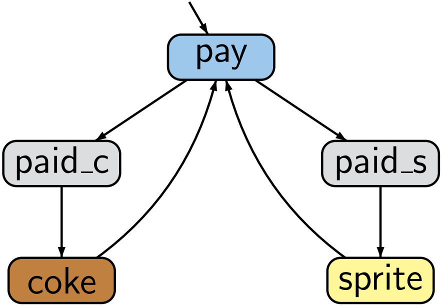




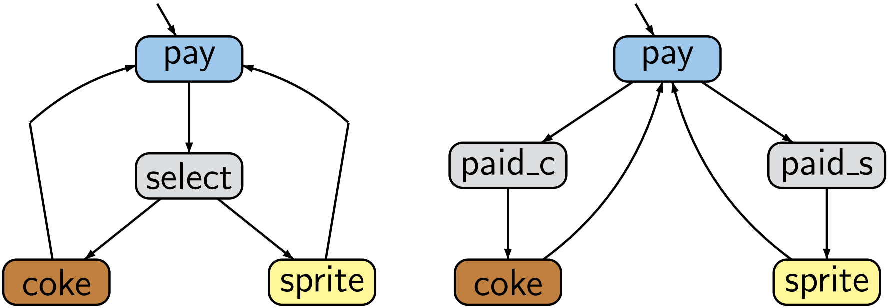
set of atomic propositions $AP = \{pay, coke, sprite\}$

$Traces(\mathcal{T}_1) = Traces(\mathcal{T}_2) =$ set of all infinite words

$\{pay\} \varnothing \{drink_1\} \{pay\} \varnothing \{drink_2\} \ldots$

where $drink_1, drink_2, \ldots \in \{coke, sprite\}$

Trace equivalent beverage machines



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$\{pay\}\varnothing\{drink_1\}\{pay\}\varnothing\{drink_2\}\ldots$

$\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy the same LT-properties over AP

Overview

Classification of LT-properties

safety properties *"nothing bad will happen"*

liveness properties *"something good will happen"*

Classification of LT-properties

safety properties *"nothing bad will happen"*

examples:

- mutual exclusion
- deadlock freedom
- "every red phase is preceded by a yellow phase"

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Classification of LT-properties

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examples:

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} special case: **invariants** *"no bad state will be reached"*

- "every red phase is preceded by a yellow phase"

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- "each waiting process will eventually enter its critical section"
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Propositional logic

$$\Phi \;::=\; \textit{true} \;\Big|\; a \;\Big|\; \Phi_1 \wedge \Phi_2 \;\Big|\; \neg\Phi \;\Big|\; \Phi_1 \vee \Phi_2 \;\Big|\; \Phi_1 \rightarrow \Phi_2 \;\Big|\; \ldots$$

↑

atomic proposition, i.e., $a \in AP$

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atomic proposition, i.e., $a \in AP$

semantics: interpretation over a subsets of AP

Propositional logic

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semantics: Let $A \subseteq AP$

$A \models \mathit{true}$		
$A \models a$	iff	$a \in A$
$A \models \Phi_1 \wedge \Phi_2$	iff	$A \models \Phi_1$ and $A \models \Phi_2$
$A \models \neg\Phi$	iff	$A \not\models \Phi$

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e.g., $\{a, b\} \not\models (a \rightarrow \neg b) \vee c \qquad \{a, b\} \models a \vee c$

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$A \models \neg\Phi$	iff	$A \not\models \Phi$

for state s of a TS over AP: $s \models \Phi$ iff $L(s) \models \Phi$

Invariant

Let E be an LT property over AP.

E is called an invariant if there exists a propositional formula Φ over AP such that

$$E = \left\{ A_0 A_1 A_2 \ldots \in \left(2^{AP}\right)^{\omega} : \forall i \geq 0.\, A_i \models \Phi \right\}$$

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Φ is called the invariant condition of E.

Examples for invariants

mutual exclusion (safety):

$MUTEX$ = set of all infinite words $A_0 \, A_1 \, A_2 \ldots$ s.t. $\forall i \in \mathbb{N}.\ \ \mathsf{crit}_1 \notin A_i$ or $\mathsf{crit}_2 \notin A_i$

here: $AP = \{\mathsf{crit}_1, \mathsf{crit}_2, \ldots\}$

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invariant condition: $\Phi = \neg \mathsf{crit}_1 \vee \neg \mathsf{crit}_2$

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invariant condition: $\Phi = \neg \mathsf{crit}_1 \vee \neg \mathsf{crit}_2$

deadlock freedom for 5 dining philosophers:

$$\mathit{DF} = \begin{array}{c}\text{set of all infinite words } A_0\, A_1\, A_2 \ldots \text{ s.t.} \\ \forall i \in \mathbb{N}\ \exists j \in \{0,1,2,3,4\}.\ \mathsf{wait}_j \notin A_i\end{array}$$

invariant condition:

$$\Phi = \neg \mathsf{wait}_0 \vee \neg \mathsf{wait}_1 \vee \neg \mathsf{wait}_2 \vee \neg \mathsf{wait}_3 \vee \neg \mathsf{wait}_4$$

here: $AP = \{\mathsf{wait}_j : 0 \leq j \leq 4\} \cup \{\ldots\}$

Satisfaction of invariants

Let E be an LT property over AP. E is called an invariant if there exists a propositional formula Φ s.t.

$$E = \{ A_0 A_1 A_2 \ldots \in \left(2^{AP}\right)^\omega : \forall i \geq 0.\, A_i \models \Phi \}$$

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iff $s \models \Phi$ for all states $s \in Reach(\mathcal{T})$

↑ set of reachable states in $\mathcal{T}$

Satisfaction of invariants

Let E be an LT property over AP. E is called an invariant if there exists a propositional formula Φ s.t.

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Let $\mathcal{T}$ be a TS over AP without terminal states. Then:

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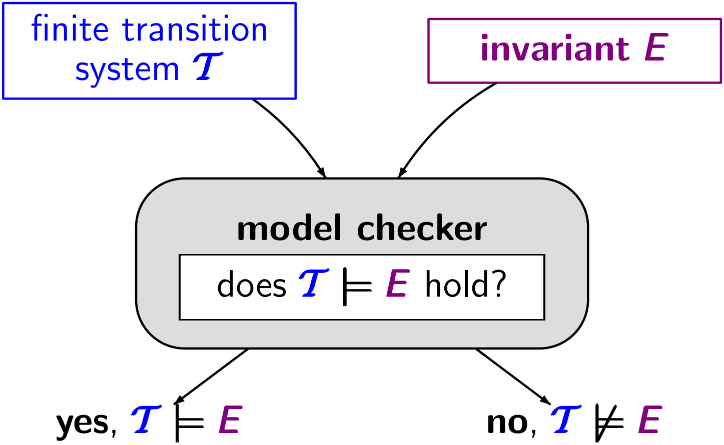
iff $s \models \Phi$ for all states s on a path of $\mathcal{T}$

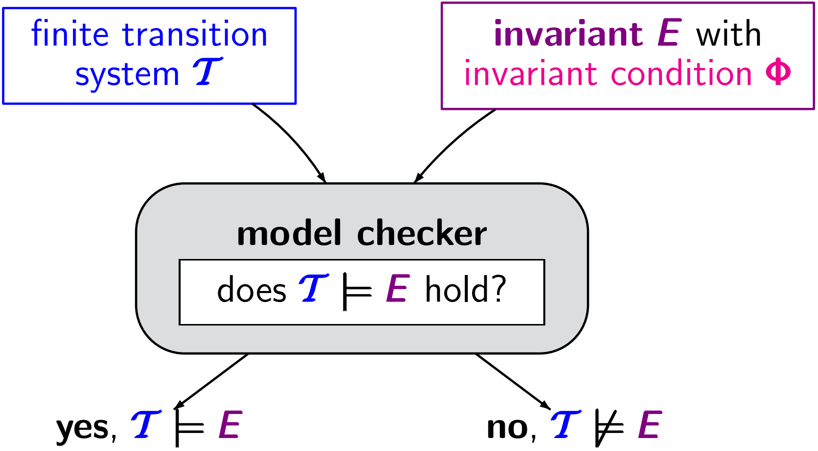
iff $s \models \Phi$ for all states $s \in Reach(\mathcal{T})$

i.e., Φ holds in all initial states and is invariant under all transitions

Invariant checking

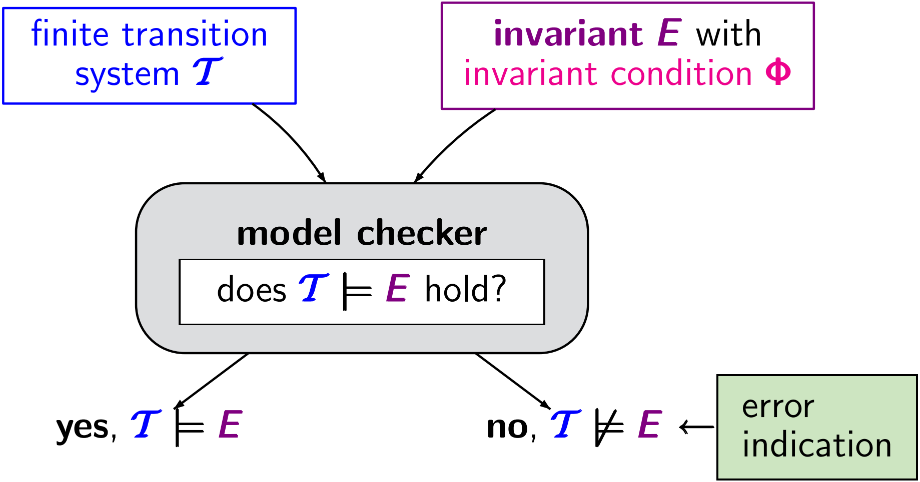


Invariant checking



perform a graph analysis (**DFS** or **BFS**) to check whether $s \models \Phi$ for all $s \in Reach(\mathcal{T})$

Invariant checking



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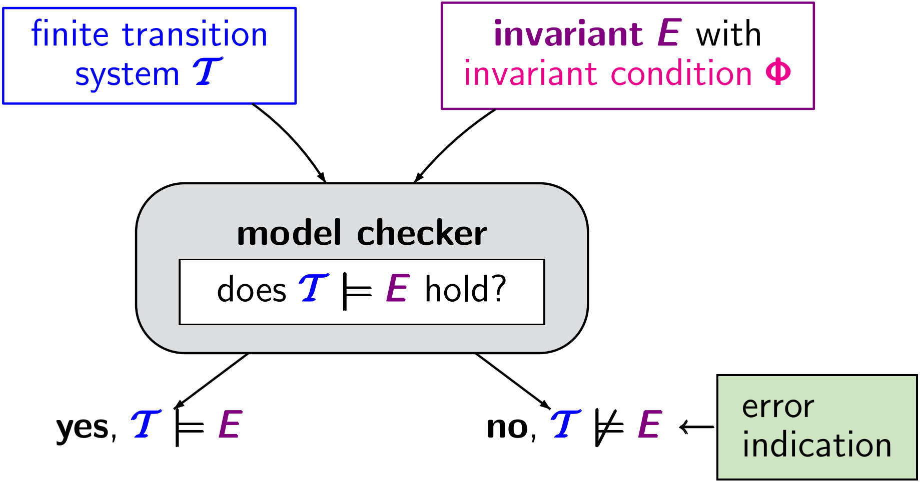
Invariant checking



error indication: initial path fragment $s_0\, s_1 \ldots s_{n-1} s_n$
such that $s_i \models \Phi$ for $0 \leq i < n$ and $s_n \not\models \Phi$

DFS-based invariant checking

input: finite transition system $\mathcal{T}$, invariant condition Φ

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```
FOR ALL s_0 ∈ S_0 DO
      IF DFS(s_0, Φ) THEN
            return "no"
      FI
OD
return "yes"
```

DFS-based invariant checking

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```
FOR ALL s0 ∈ S0 DO
      IF DFS(s0, Φ) THEN
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$DFS(s_0, \Phi)$ returns "true" iff depth-first search from state s_0 leads to some state t with $t \not\models \Phi$

DFS-based invariant checking

input: finite transition system $\mathcal{T}$, invariant condition Φ

```
π := ∅  ⟵  stack for error indication
FOR ALL s₀ ∈ S₀ DO
      IF DFS(s₀, Φ) THEN
            return "no" and reverse(π)
      FI
OD
return "yes"
```

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DFS-based invariant checking

input: finite transition system $\mathcal{T}$, invariant condition Φ

π := $\varnothing$ ⟵ stack for error indication

```
FOR ALL s0 ∈ S0 DO
      IF DFS(s0, Φ) THEN
            return "no" and reverse(π)
      FI
OD
return "yes"
```

s_n | ⋮ | s_1 | s_0 $\quad s_n = t$

$DFS(s_0, \Phi)$ returns "true" iff depth-first search from state s_0 leads to some state t with $t \not\models \Phi$

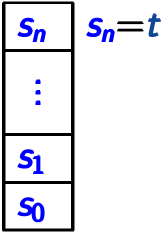
DFS-based invariant checking

input: finite transition system $\mathcal{T}$, invariant condition Φ

$U := \varnothing \longleftarrow$ stores the "processed" states

$\pi := \varnothing \longleftarrow$ stack for error indication

FOR ALL $s_0 \in S_0$ DO
    IF $DFS(s_0, \Phi)$ THEN
        return "no" and $reverse(\pi)$
    FI
OD
return "yes"



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Recursive algorithm $DFS(s, \Phi)$

"searches" for a path fragment $s \ldots t$ with $t \not\models \Phi$

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```
IF s ∉ U THEN
    IF s ⊭ Φ THEN return "true" FI
    IF s ⊨ Φ THEN

            ⋮

    FI
FI
return "false"
```

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```
IF s ∉ U THEN
     IF s ⊭ Φ THEN return "true" FI
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          insert s in U;



     FI
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IF s ∉ U THEN
    IF s ⊭ Φ THEN return "true" FI
    IF s ⊨ Φ THEN
        insert s in U;
        FOR ALL s' ∈ Post(s) DO
            IF DFS(s', Φ) THEN
                return "true" FI
        OD
    FI
FI
return "false"
```

Recursive algorithm $DFS(s, \Phi)$

"searches" for a path fragment $s \ldots t$ with $t \not\models \Phi$

```
Push(π, s);
IF s ∉ U THEN
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    IF s ⊨ Φ THEN
        insert s in U;
        FOR ALL s' ∈ Post(s) DO
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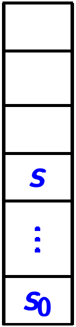
        OD
    FI
FI
Pop(π); return "false"
```

Recursive algorithm $DFS(s, \Phi)$

"searches" for a path fragment $s \ldots t$ with $t \not\models \Phi$

$Push(\pi, s)$;
IF $s \notin U$ THEN
  IF $s \not\models \Phi$ THEN return "true" FI
  IF $s \models \Phi$ THEN
    insert s in U;
    FOR ALL $s' \in Post(s)$ DO
      IF $DFS(s', \Phi)$ THEN
        return "true" FI
    OD
  FI
FI
$Pop(\pi)$; return "false"



initial state

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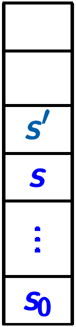

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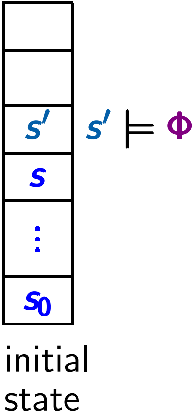

Recursive algorithm $DFS(s, \Phi)$

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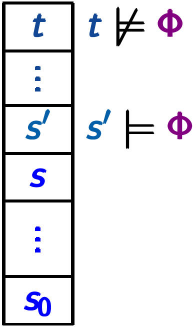
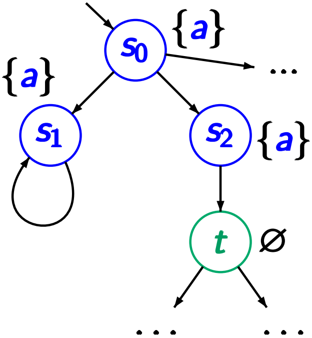


initial state

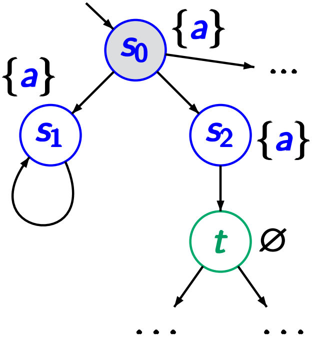
Example: invariant checking



invariant
condition a

$$\begin{aligned} s_0, s_1, s_2 &\models a \\ t &\not\models a \end{aligned}$$

Example: invariant checking



invariant condition a

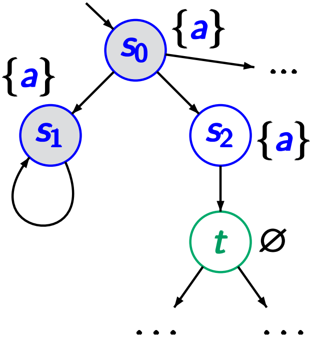
$$s_0, s_1, s_2 \models a$$
$$t \not\models a$$

$DFS(s_0, a)$

stack π

s_0

Example: invariant checking



$DFS(s_0, a)$

$DFS(s_1, a)$

stack π

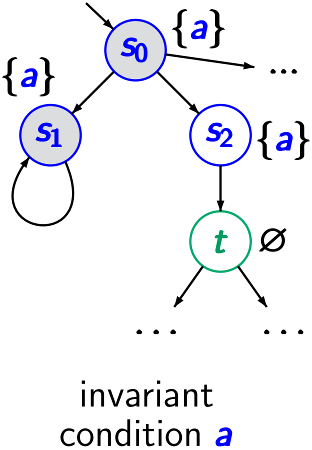
s_1
s_0

invariant condition a

$s_0, s_1, s_2 \models a$

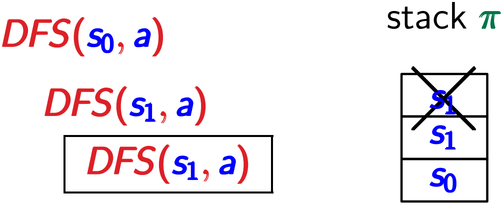
$t \not\models a$

Example: invariant checking



$s_0, s_1, s_2 \models a$

$t \not\models a$



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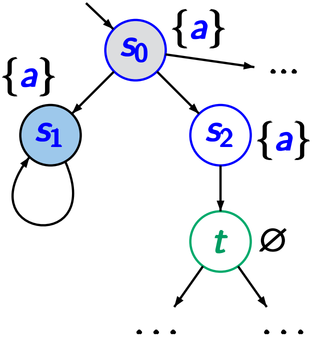


invariant
condition a

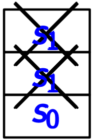
$s_0, s_1, s_2 \models a$
$t \not\models a$

$DFS(s_0, a)$

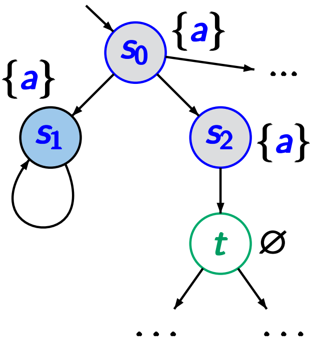
$DFS(s_1, a)$

$DFS(s_1, a)$

stack π

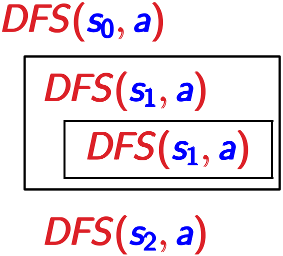


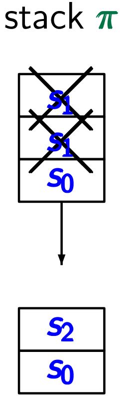
Example: invariant checking



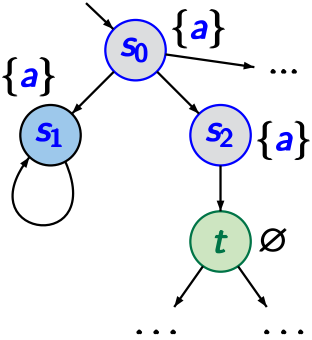
invariant condition a

$s_0, s_1, s_2 \models a$
$t \not\models a$





Example: invariant checking



invariant condition a

$s_0, s_1, s_2 \models a$

$t \not\models a$

$DFS(s_0, a)$

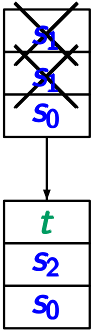
$DFS(s_1, a)$

$DFS(s_1, a)$

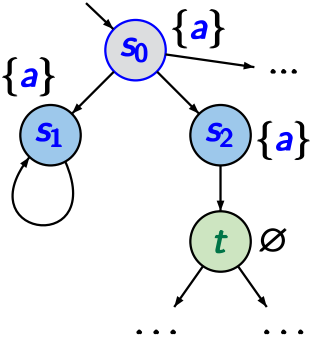
$DFS(s_2, a)$

$DFS(t, a)$

stack π



Example: invariant checking



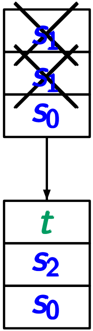
invariant
condition a

$s_0, s_1, s_2 \models a$
$t \not\models a$

$DFS(s_0, a)$

- $DFS(s_1, a)$
  - $DFS(s_1, a)$
- $DFS(s_2, a)$
  - $DFS(t, a)$

stack π



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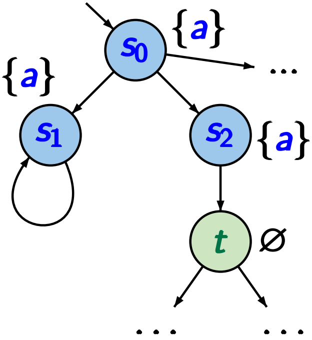
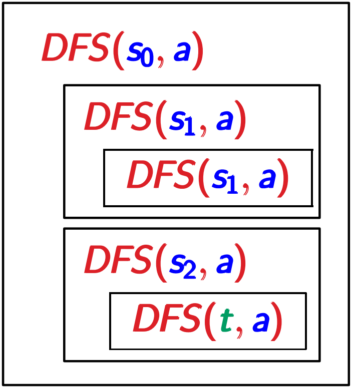


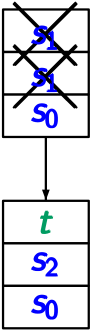
invariant condition a

$s_0, s_1, s_2 \models a$

$t \not\models a$



stack π



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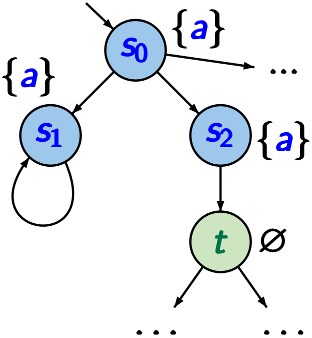
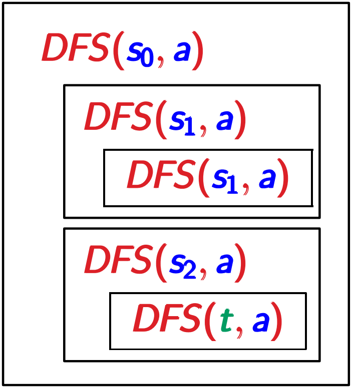
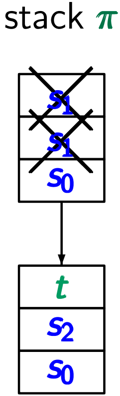




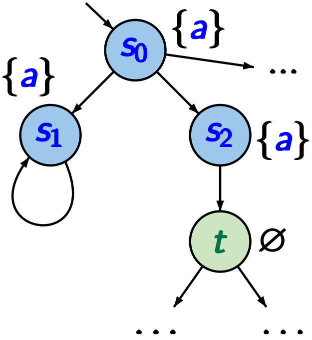


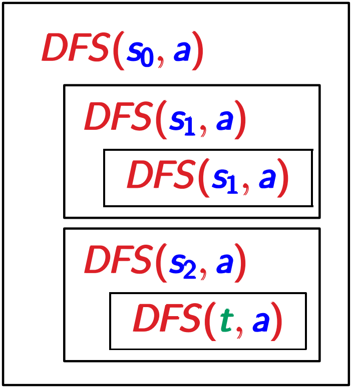
invariant condition a

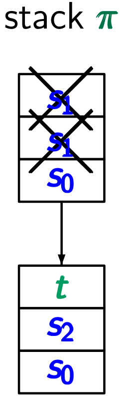
$$s_0, s_1, s_2 \models a$$
$$t \not\models a$$

$s_0 \not\models$ "always a"

Example: invariant checking







invariant condition a

$$s_0, s_1, s_2 \models a$$
$$t \not\models a$$

$s_0 \not\models$ "always a" $\longleftarrow$ error indication: $s_0\, s_2\, t$