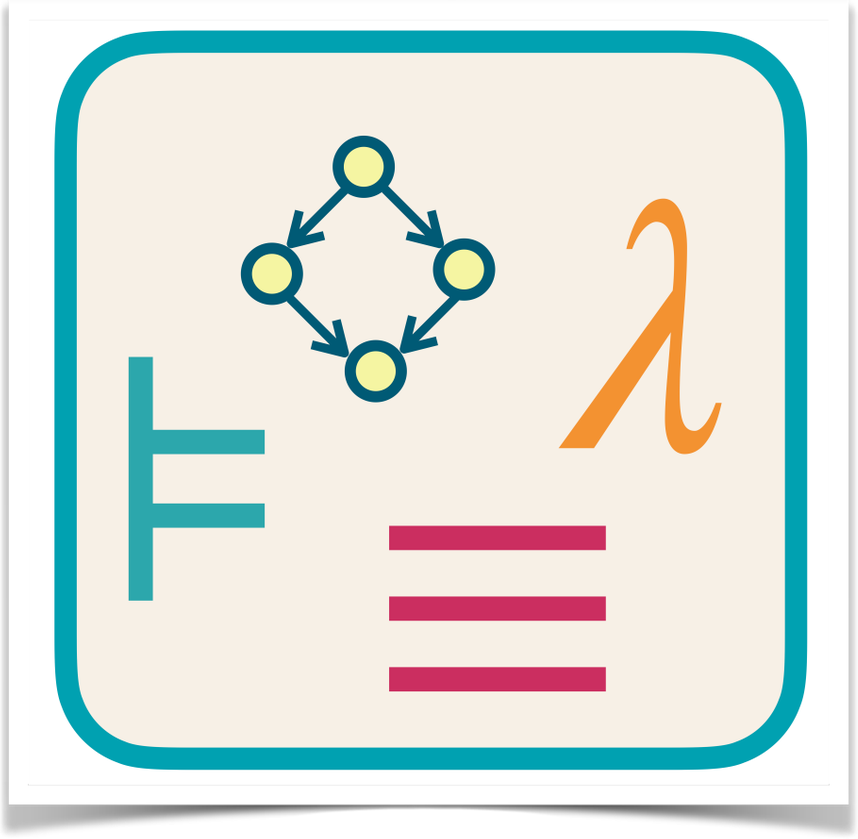


LPP 2026/27 (063AA, 9CFU)

Programming Languages

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13 - HOFL Denotational Semantics

Interpretation Domains

Interpretation Domains

$$D_{int} \triangleq \mathbb{Z}_\perp$$

$$D_{\tau_1 * \tau_2} \triangleq (D_{\tau_1} \times D_{\tau_2})_\perp$$

to distinguish
pair of divergent terms
from divergent pair

$$D_{\tau_1 \to \tau_2} \triangleq [D_{\tau_1} \to D_{\tau_2}]_\perp$$

to distinguish
takes arg and diverge
from divergence without taking arg

Example

$$D_{int*int} \triangleq (\mathbb{Z}_\perp \times \mathbb{Z}_\perp)_\perp$$

$\mathbf{rec}\ p.\ p$	$(\mathbf{rec}\ x.\ x, \mathbf{rec}\ y.\ y)$
$\perp_{D_{int*int}}$	$(\perp_{D_{int}}, \perp_{D_{int}})$

Example

$$D_{int\to int} \triangleq [\mathbb{Z}_\perp \to \mathbb{Z}_\perp]_\perp$$

$\mathbf{rec}\ f.\ f$	$\lambda x.\ \mathbf{rec}\ y.\ y$
$\perp_{D_{int\to int}}$	$\lambda d.\ \perp_{D_{int}}$

Interpretation Domains

$$D_{int} \triangleq \mathbb{Z}_{\perp} \qquad D_{\tau_1 * \tau_2} \triangleq (D_{\tau_1} \times D_{\tau_2})_{\perp} \qquad D_{\tau_1 \to \tau_2} \triangleq [D_{\tau_1} \to D_{\tau_2}]_{\perp}$$

Equivalently: $D_{\tau} \triangleq (V_{\tau})_{\perp}$

$$V_{int} \triangleq \mathbb{Z}$$

$$V_{\tau_1 * \tau_2} \triangleq D_{\tau_1} \times D_{\tau_2} = (V_{\tau_1})_{\perp} \times (V_{\tau_2})_{\perp}$$

$$V_{\tau_1 \to \tau_2} \triangleq [D_{\tau_1} \to D_{\tau_2}] = [(V_{\tau_1})_{\perp} \to (V_{\tau_2})_{\perp}]$$

Interpretation Function

$t : \tau$ $\qquad$ $[\![t]\!]\rho \in D_\tau$

environment $\quad \rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $\quad x : \tau \Rightarrow \rho(x) \in D_\tau$

we define the interpretation function by structural recursion

Denotational Semantics

Constants

$$[\![n]\!]\rho \triangleq \underbrace{\lfloor n \rfloor}_{int}$$

$$\underbrace{\lfloor n \rfloor}_{D_{int} = \mathbb{Z}_\perp} \qquad \underbrace{\lfloor n \rfloor}_{\mathbb{Z}_\perp}$$

Variables

$$\underbrace{[\![x]\!]}_{\tau}\rho \triangleq \rho(x)$$

$$x : \tau \Rightarrow \rho(x) \in D_\tau$$

D_τ D_τ

Arithmetic ops

to prove: $\underline{\mathsf{op}}_\perp$ is monotone and continuous

$$\mathsf{op} \in \{+, -\times\}$$

$$[\![\underbrace{t_1}_{\substack{int \\ D_{int}=\mathbb{Z}_\perp}} \ \mathsf{op} \ \underbrace{t_2}_{\substack{int \\ D_{int}=\mathbb{Z}_\perp}}]\!]\rho \triangleq [\![\underbrace{t_1}_{\substack{int \\ D_{int}=\mathbb{Z}_\perp}}]\!]\rho \ \underline{\mathsf{op}}_\perp \ [\![\underbrace{t_2}_{\substack{int \\ D_{int}=\mathbb{Z}_\perp}}]\!]\rho$$

$$\underline{\mathsf{op}}_\perp : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$v_1 \ \underline{\mathsf{op}}_\perp \ v_2 \triangleq \begin{cases} \lfloor n_1 \ \underline{\mathsf{op}} \ n_2 \rfloor & \text{if } v_1 = \lfloor n_1 \rfloor \text{ and } v_2 = \lfloor n_2 \rfloor \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise } (v_1 = \perp_{\mathbb{Z}_\perp} \text{ or } v_2 = \perp_{\mathbb{Z}_\perp}) \end{cases}$$

called *strict* extension

Conditionals

to prove: Cond_τ is monotone and continuous

$$[\![\textbf{if } t \textbf{ then } t_1 \textbf{ else } t_2]\!]\rho \triangleq \mathsf{Cond}_\tau(\ [\![t]\!]\rho\ ,\ [\![t_1]\!]\rho\ ,\ [\![t_2]\!]\rho\ )$$

$$\underbrace{\underbrace{\underbrace{int}_{\mathbb{Z}_\perp}}_{D_{int}=\mathbb{Z}_\perp} \to \underbrace{\underbrace{\tau}_{D_\tau} \to \underbrace{\tau}_{D_\tau}}_{D_\tau}}_{D_\tau} \qquad \underbrace{\underbrace{\underbrace{int}_{\mathbb{Z}_\perp}}_{D_{int}=\mathbb{Z}_\perp} \to \underbrace{\underbrace{\tau}_{D_\tau} \to \underbrace{\tau}_{D_\tau}}_{D_\tau}}_{D_\tau}$$

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = \lfloor 0 \rfloor \\ d_2 & \text{otherwise } (v = \lfloor n \rfloor \text{ with } n \neq 0) \end{cases}$$

Pairing

$$D_{\tau_1 * \tau_2} \triangleq (D_{\tau_1} \times D_{\tau_2})_\perp$$

$$[\![(t_1\ ,\ t_2)]\!]\rho \triangleq \lfloor\ (\ [\![t_1]\!]\rho\ ,\ [\![t_2]\!]\rho\ )\ \rfloor$$

τ_1 τ_2 — $\tau_1 * \tau_2$ — $D_{\tau_1 * \tau_2} = (D_{\tau_1} \times D_{\tau_2})_\perp$

τ_1 τ_2 — D_{τ_1} D_{τ_2} — $D_{\tau_1} \times D_{\tau_2}$ — $D_{\tau_1 * \tau_2} = (D_{\tau_1} \times D_{\tau_2})_\perp$

Projections

Equivalently: $[\![\mathbf{fst}(\ t\ )]\!]\rho \triangleq \mathbf{let}\ d \Leftarrow [\![t]\!]\rho.\ \pi_1(d)$

$[\![\mathbf{fst}(\ t\ )]\!]\rho \triangleq \pi_1^*\ (\ [\![t]\!]\rho\ )$

$\tau_1 * \tau_2$ $\quad D_{\tau_1} \times D_{\tau_2} \to D_{\tau_1}$ $\quad \tau_1 * \tau_2$

τ_1 $\quad D_{\tau_1 * \tau_2} = (D_{\tau_1} \times D_{\tau_2})_\perp$

D_{τ_1} $\quad (D_{\tau_1} \times D_{\tau_2})_\perp \to D_{\tau_1}$

D_{τ_1}

$$[\![\mathbf{snd}(\ t\ )]\!]\rho \triangleq \pi_2^*\ (\ [\![t]\!]\rho\ )$$

Abstraction

$$D_{\tau_1 \to \tau_2} \triangleq [D_{\tau_1} \to D_{\tau_2}]_\perp$$

$$[\![\lambda x.\ t]\!]\rho \triangleq \lfloor\ \lambda d.\ [\![t]\!]\rho[^d/_x]\ \rfloor$$

τ_1 τ_2 → $\tau_1 \to \tau_2$ = $D_{\tau_1 \to \tau_2} = [D_{\tau_1} \to D_{\tau_2}]_\perp$

D_{τ_1} τ_2 → D_{τ_2} → $[D_{\tau_1} \to D_{\tau_2}]$ → $[D_{\tau_1} \to D_{\tau_2}]_\perp$

Application (lazy)

Equivalently:

$$[\![\ t\ t_0\ ]\!]\rho \triangleq (\lambda\varphi.\ \varphi([\![t_0]\!]\rho))^*\ ([\![t]\!]\rho)$$

$$[\![\ t\ t_0\ ]\!]\rho \triangleq \mathbf{let}\ \varphi \Leftarrow [\![t]\!]\rho.\ \varphi([\![t_0]\!]\rho)$$

$\tau_0 \to \tau$ τ_0 — τ — D_τ

$V_{\tau_0\to\tau} = [D_{\tau_0} \to D_\tau]$ $\tau_0 \to \tau$ $[D_{\tau_0} \to D_\tau]$ τ_0 — $D_{\tau_0\to\tau} = (V_{\tau_0\to\tau})_\perp$ — D_{τ_0} — D_τ — D_τ

Recursion

$$[\![\mathbf{rec}\ x.\ t]\!]\rho \triangleq [\![t]\!]\rho[{}^{[\![\mathbf{rec}\ x.\ t]\!]\rho}/_{x}]$$

τ τ τ D_τ

τ D_τ τ D_τ

Recursion

$$[\![\mathbf{rec}\ x.\ t]\!]\rho \triangleq \mathit{fix}\ \lambda d.\ [\![t]\!]\rho[^{d}/_{x}]$$

τ τ — τ — D_τ

$[[D_\tau \to D_\tau] \to D_\tau]$ D_τ — τ D_τ τ — D_τ — $[D_\tau \to D_\tau]$ — D_τ

Recap

$$[\![n]\!]\rho \triangleq \lfloor n \rfloor$$

$$[\![x]\!]\rho \triangleq \rho(x)$$

$$[\![t_1 \ \mathsf{op} \ t_2]\!]\rho \triangleq [\![t_1]\!]\rho \ \underline{\mathsf{op}}_{\perp} \ [\![t_2]\!]\rho$$

$$[\![\mathbf{if}\ t\ \mathbf{then}\ t_1\ \mathbf{else}\ t_2]\!]\rho \triangleq \mathsf{Cond}_{\tau}(\ [\![t]\!]\rho\ ,\ [\![t_1]\!]\rho\ ,\ [\![t_2]\!]\rho\ )$$

$$[\![(t_1\ ,\ t_2)]\!]\rho \triangleq \lfloor\ (\ [\![t_1]\!]\rho\ ,\ [\![t_2]\!]\rho\ )\ \rfloor$$

$$[\![\mathbf{fst}(\ t\ )]\!]\rho \triangleq \pi_1^*\ (\ [\![t]\!]\rho\ )$$

$$[\![\mathbf{snd}(\ t\ )]\!]\rho \triangleq \pi_2^*\ (\ [\![t]\!]\rho\ )$$

$$[\![\lambda x.\ t]\!]\rho \triangleq \lfloor\ \lambda d.\ [\![t]\!]\rho[^d/_x]\ \rfloor$$

$$[\![\ t\ t_0\ ]\!]\rho \triangleq \mathbf{let}\ \varphi \Leftarrow [\![t]\!]\rho.\ \varphi([\![t_0]\!]\rho)$$

$$[\![\mathbf{rec}\ x.\ t]\!]\rho \triangleq \mathit{fix}\ \lambda d.\ [\![t]\!]\rho[^d/_x]$$

Example

$$f \stackrel{\text{def}}{=} \lambda x : int.\ 3$$

$$[\![\lambda x.\ t]\!]\rho \triangleq \lfloor\ \lambda d.\ [\![t]\!]\rho[^{d}/_{x}]\ \rfloor \qquad\qquad [\![n]\!]\rho \triangleq \lfloor n \rfloor$$

$$[\![f]\!]\rho = [\![\lambda x.\ 3]\!]\rho = \lfloor \lambda d.\ [\![3]\!]\rho[^{d}/_{x}] \rfloor = \lfloor \lambda d.\ \lfloor 3 \rfloor \rfloor$$

Example

$$g \stackrel{\text{def}}{=} \lambda x : \mathit{int}.\ \textbf{if}\ x\ \textbf{then}\ 3\ \textbf{else}\ 3$$

$$[\![\lambda x.\ t]\!]\rho \triangleq \lfloor\ \lambda d.\ [\![t]\!]\rho[^{d}/_{x}]\ \rfloor$$

$$\begin{aligned}
[\![g]\!]\rho &= [\![\lambda x.\ \textbf{if}\ x\ \textbf{then}\ 3\ \textbf{else}\ 3]\!]\rho \\
&= \lfloor \lambda d.\ [\![\textbf{if}\ x\ \textbf{then}\ 3\ \textbf{else}\ 3]\!]\rho[^{d}/_{x}] \rfloor \\
&= \lfloor \lambda d.\ \mathit{Cond}(d, \lfloor 3 \rfloor, \lfloor 3 \rfloor) \rfloor \\
&= \lfloor \lambda d.\ \textbf{let}\ x \Leftarrow d.\ \lfloor 3 \rfloor \rfloor
\end{aligned}$$

$$[\![f]\!]\rho \neq [\![g]\!]\rho$$

$$\lfloor \lambda d.\ \lfloor 3 \rfloor \rfloor$$

Example

$$h \stackrel{\text{def}}{=} \mathbf{rec}\, y : int \to int.\ \lambda x : int.\ 3$$

$$\begin{aligned}
[\![h]\!]\rho &= [\![\mathbf{rec}\ y.\ \lambda x.\ 3]\!]\rho && [\![\mathbf{rec}\ x.\ t]\!]\rho \triangleq fix\ \lambda d.\ [\![t]\!]\rho[^{d}/_{x}] \\
&= \text{fix}\ \lambda d_y.\ [\![\lambda x.\ 3]\!]\rho[^{d_y}/_{y}] && [\![\lambda x.\ t]\!]\rho \triangleq \lfloor\ \lambda d.\ [\![t]\!]\rho[^{d}/_{x}]\ \rfloor \\
&= \text{fix}\ \lambda d_y.\ \lfloor \lambda d_x.\ [\![3]\!]\rho[^{d_y}/_{y}, ^{d_x}/_{x}] \rfloor \\
&= \text{fix}\ \lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor
\end{aligned}$$

$$\Gamma_h = \lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor$$

$$\begin{aligned}
d_0 &= \Gamma_h^0(\bot_{[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]_\bot}) = \bot_{[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]_\bot} \\
d_1 &= \Gamma_h(d_0) = (\lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor)\bot = \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor \\
d_2 &= \Gamma_h(d_1) = (\lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor)\lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor = \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor = d_1
\end{aligned}$$

Example

$$h \stackrel{\text{def}}{=} \mathbf{rec}\, y : int \to int.\ \lambda x : int.\ 3$$

$$\begin{aligned}
[\![h]\!]\rho &= [\![\mathbf{rec}\ y.\ \lambda x.\ 3]\!]\rho \\
&= \text{fix}\ \lambda d_y.\ [\![\lambda x.\ 3]\!]\rho[{}^{d_y}/_{y}] \\
&= \text{fix}\ \lambda d_y.\ \lfloor \lambda d_x.\ [\![3]\!]\rho[{}^{d_y}/_{y}, {}^{d_x}/_{x}] \rfloor \\
&= \text{fix}\ \lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor
\end{aligned}$$

$$\Gamma_h = \lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor$$

$$d_0 = \Gamma_h^0(\bot_{[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]_\bot}) = \bot_{[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]_\bot}$$

$$d_1 = \Gamma_h(d_0) = (\lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor)\bot = \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor$$

Maximal element in $[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]_\bot$
we could already stop here

Example

$$h \stackrel{\text{def}}{=} \mathbf{rec}\, y : int \to int.\ \lambda x : int.\ 3$$

$$\begin{aligned}
[\![h]\!]\rho &= [\![\mathbf{rec}\ y.\ \lambda x.\ 3]\!]\rho \\
&= \text{fix}\ \lambda d_y.\ [\![\lambda x.\ 3]\!]\rho[{}^{d_y}/_{y}] \\
&= \text{fix}\ \lambda d_y.\ \lfloor \lambda d_x.\ [\![3]\!]\rho[{}^{d_y}/_{y}, {}^{d_x}/_{x}] \rfloor \\
&= \text{fix}\ \lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor
\end{aligned}$$

$$[\![h]\!]\rho = \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor = [\![f]\!]\rho$$

Example

$$x : \tau$$

$$[\![\mathbf{rec}\ x.\ x]\!]\rho = \mathit{fix}\ \lambda d_x.\ [\![x]\!]\rho[^{d_x}/_x]$$

$$= \mathit{fix}\ \lambda d_x.\ d_x$$

$$d_0 = \bot_{D_\tau}$$

$$d_1 = (\lambda d_x.d_x)\ d_0 = d_0 = \bot_{D_\tau}$$

$$[\![\mathbf{rec}\ x.\ x]\!]\rho = \bot_{D_\tau}$$

$x : \mathit{int} \to \mathit{int}$ $\qquad [\![\mathbf{rec}\ x.\ x]\!]\rho = \bot_{[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]_\bot}$

$x : \mathit{int} * \mathit{int}$ $\qquad [\![\mathbf{rec}\ x.\ x]\!]\rho = \bot_{(\mathbb{Z}_\bot \times \mathbb{Z}_\bot)_\bot}$

Example

$$y : \tau_1 \qquad z : \tau_2$$

$$[\![\lambda y.\ \mathbf{rec}\ z.\ z]\!]\rho = \lfloor \lambda d_y.\ [\![\mathbf{rec}\ z.\ z]\!]\rho[{}^{d_y}/_{y}] \rfloor$$

$$= \lfloor \lambda d_y.\ \bot_{D_{\tau_2}} \rfloor$$

$$= \lfloor \bot_{[D_{\tau_1} \to D_{\tau_2}]} \rfloor$$

$$= \lfloor \bot_{V_{\tau_1 \to \tau_2}} \rfloor$$

$$\neq \bot_{D_{\tau_1 \to \tau_2}} \quad = \bot_{(V_{\tau_1 \to \tau_2})_\bot}$$

$x : int \to int$	$[\![\mathbf{rec}\ x.\ x]\!]\rho = \bot_{[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]_\bot}$	diverges
$y : int, z : int$	$[\![\lambda y.\ \mathbf{rec}\ z.\ z]\!]\rho = \lfloor \bot_{[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]} \rfloor$	waits arg and diverges

Exercise

$$x : int * int \ , \ y : int \ , \ z : int$$

$$[\![\ \mathbf{rec}\ x.\ x \ ]\!]\rho \quad \stackrel{?}{=} \quad [\![\ (\ \mathbf{rec}\ y.\ y \ , \ \mathbf{rec}\ z.\ z \ ) \ ]\!]\rho$$



diverges

$$\bot_{D_{int*int}}$$

a pair
of diverging computations

$$\lfloor (\bot_{D_{int}}, \bot_{D_{int}}) \rfloor$$

Lazy vs Eager

Eager Application

returns $\bot$ when $[\![t]\!]\rho = \bot$

lazy $\quad [\![\ t\ t_0\ ]\!]\rho \triangleq \mathbf{let}\ \varphi \Leftarrow [\![t]\!]\rho.\ \varphi([\![t_0]\!]\rho)$

eager $\quad [\![\ t\ t_0\ ]\!]\rho \triangleq \mathbf{let}\ \varphi \Leftarrow [\![t]\!]\rho.\ \mathbf{let}\ d \Leftarrow [\![t_0]\!]\rho.\ \varphi(\lfloor d \rfloor)$

returns $\bot$ when $[\![t]\!]\rho = \bot$ or $[\![t_0]\!]\rho = \bot$

Well-given definitions: continuity theorems

Well-definedness

We must guarantee that all functions we have used are monotone and continuous,
so that Kleene's fix point theory is applicable

$\pi_1\ \pi_2\ (\cdot)^*$ already considered

$\mathbf{let}$

$apply\ \ fix\ \ \underline{\mathsf{op}}_\perp\ \ \mathsf{Cond}_\tau\ \ \lambda$ to be checked

TH. $(D, \sqsubseteq_D)$ $(E, \sqsubseteq_E)$ $(F, \sqsubseteq_F)$ CPO $f : D \times E \to F$

$$f_d : E \to F \qquad f_d \triangleq \lambda e.\ f(d, e)$$

$$f_e : D \to F \qquad f_e \triangleq \lambda d.\ f(d, e)$$

f is continuous iff $\forall d \in D.\ f_d$ are continuous

$\forall e \in E.\ f_e$ are continuous

proof. $\Rightarrow$) assume f is continuous

take a generic $d \in D$

we want to prove f_d is continuous

take a chain $\{e_i\}_{i \in \mathbb{N}}$ in E

we prove $f_d \left(\bigsqcup_{i \in \mathbb{N}} e_i \right) = \bigsqcup_{i \in \mathbb{N}} f_d(e_i)$

$e \in E$

f_e

(omitted)

(see next slide)

(continue) $f_d\left(\bigsqcup_{i\in\mathbb{N}} e_i\right) = \bigsqcup_{i\in\mathbb{N}} f_d(e_i)$

$$f_d\left(\bigsqcup_{i\in\mathbb{N}} e_i\right) = f\left(d, \bigsqcup_{i\in\mathbb{N}} e_i\right) \qquad \text{by def of } f_d$$

$$= f\left(\bigsqcup_{i\in\mathbb{N}} d, \bigsqcup_{i\in\mathbb{N}} e_i\right) \qquad \text{by lub of constant chain}$$

$$= f\left(\bigsqcup_{i\in\mathbb{N}} (d, e_i)\right) \qquad \text{by lub of pairs}$$

$$= \bigsqcup_{i\in\mathbb{N}} f(d, e_i) \qquad \text{by continuity of } f$$

$$= \bigsqcup_{i\in\mathbb{N}} f_d(e_i) \qquad \text{by def of } f_d$$

TH. $(D, \sqsubseteq_D)$ $(E, \sqsubseteq_E)$ $(F, \sqsubseteq_F)$ CPO $f : D \times E \to F$

$$f_d : E \to F \qquad f_d \triangleq \lambda e.\ f(d, e)$$

$$f_e : D \to F \qquad f_e \triangleq \lambda d.\ f(d, e)$$

f is continuous iff $\forall d \in D.\ f_d$ are continuous

$\forall e \in E.\ f_e$ are continuous

$\Leftarrow$) assume f_d, f_e are continuous for all d, e

we want to prove f is continuous

take a chain $\{(d_k, e_k)\}_{k \in \mathbb{N}}$ in $D \times E$

we prove $f\left(\bigsqcup_{k \in \mathbb{N}} (d_k, e_k)\right) = \bigsqcup_{k \in \mathbb{N}} f(d_k, e_k)$

(see next slide)

(continue) $f\left(\bigsqcup_{k\in\mathbb{N}}(d_k, e_k)\right) = \bigsqcup_{k\in\mathbb{N}} f(d_k, e_k)$

$$
\begin{aligned}
f(\sqcup_k(d_k, e_k)) &= f(\sqcup_i d_i, \sqcup_j e_j) && \text{by def of lub of pairs} \\
&= f_d(\sqcup_j e_j) && \text{by def of } f_d \text{ with } d \triangleq \sqcup_i d_i \\
&= \sqcup_j f_d(e_j) && \text{by continuity of } f_d \\
&= \sqcup_j f(d, e_j) && \text{by def of } f_d \\
&= \sqcup_j f_{e_j}(d) && \text{by def of } f_{e_j} \\
&= \sqcup_j f_{e_j}(\sqcup_i d_i) && \text{by def of } d \triangleq \sqcup_i d_i \\
&= \sqcup_j \sqcup_i f_{e_j}(d_i) && \text{by continuity of } f_{e_j} \\
&= \sqcup_j \sqcup_i f(d_i, e_j) && \text{by def of } f_{e_j} \\
&= \sqcup_k f(d_k, e_k) && \text{by switch lemma (applicable?)}
\end{aligned}
$$

(continue) $f\left(\bigsqcup_{k\in\mathbb{N}}(d_k, e_k)\right) = \bigsqcup_{k\in\mathbb{N}} f(d_k, e_k)$

if $i \leq n \wedge j \leq m$ then $f(d_i, e_j) \sqsubseteq f(d_n, e_m)$?

$$\Downarrow$$

$$d_i \sqsubseteq_D d_n \wedge e_j \sqsubseteq_E e_m$$

$$f(d_i, e_j) = f_{d_i}(e_j) \sqsubseteq f_{d_i}(e_m) = f(d_i, e_m) = f_{e_m}(d_i) \sqsubseteq f_{e_m}(d_n) = f(d_n, e_m)$$

f_{d_i} monotone

f_{e_m} monotone

$= \sqcup_j \sqcup_i f(d_i, e_j)$

$= \sqcup_k f(d_k, e_k)$ by switch lemma (applicable?)

Apply

$(D, \sqsubseteq_D)$
$(E, \sqsubseteq_E)$ CPO

$$apply : [D \to E] \times D \to E$$
$$apply(f, d) \triangleq f(d)$$

TH. $apply$ is monotone

(try to prove on your own)

TH. $apply$ is continuous

proof. from a previous theorem, we prove continuity on each parameter separately $apply_f$ $apply_d$

1. for any $f \in [D \to E]$ $apply_f \triangleq \lambda d.\ f(d)$ is continuous

2. for any $d \in D$ $apply_d \triangleq \lambda f.\ f(d)$ is continuous

(see next slides)

1. for any $f \in [D \to E]$ $\quad apply_f \triangleq \lambda d.\ f(d)$ is continuous

take $f \in [D \to E]$ and a chain $\{d_i\}_{i \in \mathbb{N}}$ in D

we want to prove $\quad apply_f \left(\bigsqcup_i d_i \right) = \bigsqcup_i apply_f(d_i)$

$apply_f(\sqcup_i d_i) = apply(f, \sqcup_i d_i)$ by def of $apply_f$

$= f(\sqcup_i d_i)$ by def of $apply$

$= \sqcup_i f(d_i)$ by continuity of f

$= \sqcup_i apply(f, d_i)$ by def of $apply$

$= \sqcup_i apply_f(d_i)$ by def of $apply_f$

2. for any $d \in D$ $\quad apply_d \triangleq \lambda f.\ f(d)$ is continuous

take $d \in D$ and a chain $\{f_i\}_{i \in \mathbb{N}}$ in $[D \to E]$

we want to prove $\quad apply_d \left(\bigsqcup_i f_i \right) = \bigsqcup_i apply_d(f_i)$

$apply_d(\sqcup_i f_i) = apply(\sqcup_i f_i, d)$ by def of $apply_d$

$= (\sqcup_i f_i)(d)$ by def of $apply$

$= \sqcup_i f_i(d)$ by def of lub of functions

$= \sqcup_i apply(f_i, d)$ by def of $apply$

$= \sqcup_i apply_d(f_i)$ by def of $apply_d$

Apply: recap

$(D, \sqsubseteq_D)$
$(E, \sqsubseteq_E)$ CPO

$$apply : [D \to E] \times D \to E$$
$$apply(f, d) \triangleq f(d)$$

$$apply \in [[D \to E] \times D \to E]$$

Fix

$(D, \sqsubseteq_D)\ \mathrm{CPO}_\perp$

$$fix : [D \to D] \to D$$
$$fix \triangleq \lambda f.\ \bigsqcup_{n \in \mathbb{N}} f^n(\perp_D)$$

TH. fix is monotone

(try to prove on your own)

TH. fix is continuous

proof. $fix \triangleq \lambda f.\ \bigsqcup_{n \in \mathbb{N}} f^n(\perp_D) = \bigsqcup_{n \in \mathbb{N}} \lambda f.\ f^n(\perp_D)$

by def of lub in functional domains

we prove that $\forall n.\ \lambda f.\ f^n(\perp_D)$ is continuous

(by mathematical induction on n)

then fix is continuous because lub of continuous functions

(see next slides)

(continue) $\forall n.\ \lambda f.\ f^n(\bot_D)$

base case: $\lambda f.\ f^0(\bot_D) = \lambda f.\ \bot_D$

is a constant function (continuous)

inductive case: assume $g \triangleq \lambda f.\ f^n(\bot_D)$ is continuous

we want to prove $h \triangleq \lambda f.\ f^{n+1}(\bot_D)$ is continuous

take a chain $\{f_i\}_{i \in \mathbb{N}}$ in $[D \to D]$

we want to prove $h\left(\bigsqcup_{i \in \mathbb{N}} f_i\right) = \bigsqcup_{i \in \mathbb{N}} h(f_i)$

(see next slide)

(continue) $\forall n.\ \lambda f.\ f^n(\bot_D)$ $\quad g \triangleq \lambda f.\ f^n(\bot_D)$ $\quad h \triangleq \lambda f.\ f^{n+1}(\bot_D)$ $\quad h\left(\bigsqcup_{i\in\mathbb{N}} f_i\right) = \bigsqcup_{i\in\mathbb{N}} h(f_i)$

$$
\begin{aligned}
h(\sqcup_i f_i) &= (\sqcup_i f_i)^{n+1}(\bot_D) && \text{by def of } h \\
&= (\sqcup_j f_j)((\sqcup_i f_i)^n(\bot_D)) && \text{by def of } (\cdot)^{n+1} \\
&= (\sqcup_j f_j)(g(\sqcup_i f_i)) && \text{by def of } g \\
&= (\sqcup_j f_j)(\sqcup_i g(f_i)) && \text{by ind. hyp } (g \text{ continuous}) \\
&= (\sqcup_j f_j)(\sqcup_i f_i^n(\bot_D)) && \text{by def of } g \\
&= \sqcup_j f_j(\sqcup_i f_i^n(\bot_D)) && \text{by def of lub in functional CPO} \\
&= \sqcup_j \sqcup_i f_j(f_i^n(\bot_D)) && \text{by continuity of } f_j \\
&= \sqcup_k f_k(f_k^n(\bot_D)) && \text{by switch lemma} \\
&= \sqcup_k f_k^{n+1}(\bot_D) && \text{by def of } (\cdot)^{n+1} \\
&= \sqcup_k h(f_k) && \text{by def of } h
\end{aligned}
$$

Fix: recap

$(D, \sqsubseteq_D)\ \mathrm{CPO}_\perp$

$$fix : [D \to D] \to D$$
$$fix \triangleq \lambda f.\ \bigsqcup_{n \in \mathbb{N}} f^n(\perp_D)$$

$$fix \in [[D \to D] \to D]$$

TH. $\underline{\mathsf{op}}_\perp$ is monotone and continuous

$$\underline{\mathsf{op}}_\perp : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$v_1 \ \underline{\mathsf{op}}_\perp \ v_2 \triangleq \begin{cases} \lfloor n_1 \ \underline{\mathsf{op}} \ n_2 \rfloor & \text{if } v_1 = \lfloor n_1 \rfloor \text{ and } v_2 = \lfloor n_2 \rfloor \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise } (v_1 = \perp_{\mathbb{Z}_\perp} \text{ or } v_2 = \perp_{\mathbb{Z}_\perp}) \end{cases}$$

We omit monotonicity check

Since the domain has only finite chains, it is also continuous

TH. Cond_τ is monotone and continuous

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = \lfloor 0 \rfloor \\ d_2 & \text{otherwise } (v = \lfloor n \rfloor \text{ with } n \neq 0) \end{cases}$$

We omit monotonicity check

We prove continuity on each parameter separately

The first parameter is in $\mathbb{Z}_\perp$

only finite chains are possible, hence continuity is guaranteed

We prove continuity over the second parameter (next slides)

For the third parameter the proof is analogous and omitted

(continue)

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = \lfloor 0 \rfloor \\ d_2 & \text{otherwise } (v = \lfloor n \rfloor \text{ with } n \neq 0) \end{cases}$$

Continuity over the second parameter

take $v \in \mathbb{Z}_\perp, d \in D_\tau, \{d_i\}_{i \in \mathbb{N}} \subseteq D_\tau$

we want to prove $\quad \mathsf{Cond}_\tau \left(v, \bigsqcup_{i \in \mathbb{N}} d_i, d \right) = \bigsqcup_{i \in \mathbb{N}} \mathsf{Cond}_\tau(v, d_i, d)$

we proceed by case analysis on v $\begin{cases} \perp_{\mathbb{Z}_\perp} \\ \lfloor 0 \rfloor \\ \lfloor n \rfloor \ , \ n \neq 0 \end{cases}$

(continue)

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = \lfloor 0 \rfloor \\ d_2 & \text{otherwise } (v = \lfloor n \rfloor \text{ with } n \neq 0) \end{cases}$$

$v = \perp_{\mathbb{Z}_\perp}$

$$\mathsf{Cond}_\tau \left(\perp_{\mathbb{Z}_\perp}, \bigsqcup_{i \in \mathbb{N}} d_i, d \right) = \perp_{D_\tau} = \bigsqcup_{i \in \mathbb{N}} \perp_{D_\tau} = \bigsqcup_{i \in \mathbb{N}} \mathsf{Cond}_\tau(\perp_{\mathbb{Z}_\perp}, d_i, d)$$

(continue)

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = \lfloor 0 \rfloor \\ d_2 & \text{otherwise } (v = \lfloor n \rfloor \text{ with } n \neq 0) \end{cases}$$

$v = \lfloor 0 \rfloor$

$$\mathsf{Cond}_\tau \left(\lfloor 0 \rfloor, \bigsqcup_{i \in \mathbb{N}} d_i, d \right) = \bigsqcup_{i \in \mathbb{N}} d_i = \bigsqcup_{i \in \mathbb{N}} \mathsf{Cond}_\tau(\lfloor 0 \rfloor, d_i, d)$$

(continue)

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = \lfloor 0 \rfloor \\ d_2 & \text{otherwise } (v = \lfloor n \rfloor \text{ with } n \neq 0) \end{cases}$$

$v = \lfloor n \rfloor \ , \ n \neq 0$

$$\mathsf{Cond}_\tau \left(\lfloor n \rfloor, \bigsqcup_{i \in \mathbb{N}} d_i, d \right) = d = \bigsqcup_{i \in \mathbb{N}} d = \bigsqcup_{i \in \mathbb{N}} \mathsf{Cond}_\tau(\lfloor n \rfloor, d_i, d)$$

TH. lambda abstraction is monotone and continuous

$t : \tau$ $\qquad\qquad$ $\lambda d.\ [\![t]\!]\rho[^{d}/_{x}]$ is continuous

we focus on the stronger property

$\lambda \widetilde{d}.\ [\![t]\!]\rho[^{\widetilde{d}}/_{\widetilde{x}}]$ is continuous

the proof is by structural induction on t

(try on your own)

Corollary $t : \tau_0 \to \tau$ $\qquad$ $fix\ \lambda d.\ [\![t]\!]\rho[^{d}/_{x}]$ is continuous

(the limit of continuous functions is continuous)

More continuity theorems

TH. $(D, \sqsubseteq_D)$ $(E_i, \sqsubseteq_{E_i})$ CPO $f : D \to E_1 \times E_2$ $g_i \triangleq \pi_i \circ f$

f is continuous iff g_1, g_2 are continuous

proof. $\Rightarrow)$ f is continuous $\Rightarrow$ g_i is continuous
π_i is continuous

$\Leftarrow)$ note that $\forall d \in D.\ f(d) = (g_1(d), g_2(d))$
assume g_1, g_2 are continuous
we want to prove f is continuous
take a chain $\{d_i\}_{i \in \mathbb{N}}$ in D
we must prove $f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = \bigsqcup_{i \in \mathbb{N}} f(d_i)$

(see next slide)

(continue) $f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) = \bigsqcup_{i\in\mathbb{N}} f(d_i)$

$$f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) = \left(g_1\left(\bigsqcup_{i\in\mathbb{N}} d_i\right), g_2\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)\right) \quad \text{by def } g_1, g_2$$

$$= \left(\bigsqcup_{i\in\mathbb{N}} g_1(d_i), \bigsqcup_{i\in\mathbb{N}} g_2(d_i)\right) \quad g_1, g_2 \text{ are continuous}$$

$$= \bigsqcup_{i\in\mathbb{N}} (g_1(d_i), g_2(d_i)) \quad \text{by def of lub of pairs}$$

$$= \bigsqcup_{i\in\mathbb{N}} f(d_i) \quad \text{by def } g_1, g_2$$

Curry

$(D, \sqsubseteq_D)$
$(E, \sqsubseteq_E)$ CPO
$(F, \sqsubseteq_F)$

$$curry : (D \times E \to F) \to D \to E \to F$$
$$curry\ f\ d\ e \triangleq f(d, e)$$

TH. f continuous $\Rightarrow$ $curry(f)$ continuous

(try to prove on your own)

Curry

$(D, \sqsubseteq_D)$
$(E, \sqsubseteq_E)$ CPO
$(F, \sqsubseteq_F)$

$$curry : (D \times E \to F) \to D \to E \to F$$
$$curry\ f\ d\ e \triangleq f(d, e)$$

TH. f continuous $\Rightarrow$ $curry(f)$ continuous

$h \triangleq curry(f) : D \to E \to F$

given $\{d_i\}_{i \in \mathbb{N}}$ $\quad h\ (\sqcup_i d_i) =^? \sqcup_i (h\ d_i) : E \to F$

take $e \in E$ $\quad h\ (\sqcup_i d_i)\ e =^? (\sqcup_i (h\ d_i))\ e$

$$h\ (\sqcup_i d_i)\ e = curry\ f\ (\sqcup_i d_i)\ e = f((\sqcup_i d_i), e) = f((\sqcup_i d_i), (\sqcup_i e))$$

$$= f(\sqcup_i (d_i, e)) = \sqcup_i f(d_i, e) = \sqcup_i (curry\ f\ d_i\ e)$$

$$= \sqcup_i (h\ d_i\ e) = (\sqcup_i (h\ d_i))\ e$$

Uncurry

$(D, \sqsubseteq_D)$
$(E, \sqsubseteq_E)$ CPO
$(F, \sqsubseteq_F)$

$$uncurry : (D \to E \to F) \to (D \times E) \to F$$
$$uncurry\ f\ (d, e) \triangleq f\ d\ e$$

TH. f continuous $\Rightarrow$ $uncurry(f)$ continuous

(try to prove on your own)

TH. $uncurry$ is the inverse of $curry$

given $g : (D \times E) \to F$ $\quad uncurry(curry(g)) =^{?} g$

take $(d, e) \in (D \times E)$ $\quad (uncurry(curry(g)))(d, e) =^{?} g(d, e)$

$$(uncurry(curry(g)))(d, e) = (curry(g))\ d\ e = g(d, e)$$

Uncurry

$(D, \sqsubseteq_D)$
$(E, \sqsubseteq_E)$ CPO
$(F, \sqsubseteq_F)$

$$uncurry : (D \to E \to F) \to (D \times E) \to F$$
$$uncurry\ f\ (d, e) \triangleq f\ d\ e$$

TH. f continuous $\Rightarrow$ $uncurry(f)$ continuous

(try to prove on your own)

TH. $uncurry$ is the inverse of $curry$

given $f : D \to E \to F$ $\quad curry(uncurry(f)) =^? f$

take $d \in D, e \in E$ $\quad (curry(uncurry(f)))\ d\ e =^? f\ d\ e$

$$(curry(uncurry(f)))\ d\ e = (uncurry(f))(d, e) = f\ d\ e$$

Main properties

Substitution lemma

$$x, t_0 : \tau_0 \qquad t : \tau$$

$$[\![t[^{t_0}/_x]]\!]\rho = [\![t]\!]\rho[^{[\![t_0]\!]\rho}/_x]$$

syntactic substitution

environment update

the proof is by structural induction on t

(try on your own)

Compositionality

The substitution lemma $[\![t[^{t_0}/_x]]\!]\rho = [\![t]\!]\rho[^{[\![t_0]\!]\rho}/_x]$ is important: as it guarantees the compositionality of the denotational semantics

TH. $\quad [\![t_1]\!]\rho = [\![t_2]\!]\rho \quad \Rightarrow \quad [\![t[^{t_1}/_x]]\!]\rho = [\![t[^{t_2}/_x]]\!]\rho$

proof. assume $[\![t_1]\!]\rho = [\![t_2]\!]\rho$

$$[\![t[^{t_1}/_x]]\!]\rho = [\![t]\!]\rho[^{[\![t_1]\!]\rho}/_x] = [\![t]\!]\rho[^{[\![t_2]\!]\rho}/_x] = [\![t[^{t_2}/_x]]\!]\rho$$

subs lemma — $[\![t_1]\!]\rho = [\![t_2]\!]\rho$ — subs lemma

Only free variables matter

TH. $t : \tau$

$\forall x \in \mathsf{fv}(t).\ \rho(x) = \rho'(x) \quad \Rightarrow \quad [\![t]\!]\rho = [\![t]\!]\rho'$

the proof is by structural induction on t

(try on your own)

Corollary t closed $\Rightarrow$ $\forall \rho, \rho'.\ [\![t]\!]\rho = [\![t]\!]\rho'$

TH. Canonical terms are not bottom

$$c \in C_\tau \;\Rightarrow\; \forall \rho.\, [\![c]\!]\rho \neq \bot_{D_\tau}$$

proof. by rule induction on the rules for canonical terms

$$P(c \in C_\tau) \triangleq \forall \rho.\, [\![c]\!]\rho \neq \bot_{D_\tau}$$

$$\frac{}{n \in C_{int}} \qquad [\![n]\!]\rho = \lfloor n \rfloor \neq \bot_{D_{int}}$$

$$\frac{t_0 : \tau_0 \quad t_1 : \tau_1 \quad t_0, t_1 \text{ closed}}{(t_0, t_1) \in C_{\tau_0 * \tau_1}} \qquad [\![(t_0, t_1)]\!]\rho = \lfloor ([\![t_0]\!]\rho, [\![t_1]\!]\rho) \rfloor \neq \bot_{D_{\tau_0 * \tau_1}}$$

$$\frac{\lambda x.\, t : \tau_0 \to \tau_1 \quad \lambda x.\, t \text{ closed}}{\lambda x.\, t \in C_{\tau_0 \to \tau_1}} \qquad [\![\lambda x.\ t]\!]\rho = \lfloor \lambda d.\ [\![t]\!]\rho[^d/_x] \rfloor \neq \bot_{D_{\tau_0 \to \tau_1}}$$

Additional exercises

Exercise a, part 1

Prove the implication

$$[\![t_1]\!]\rho = [\![t_2]\!]\rho \Rightarrow [\![t_1\ x]\!]\rho = [\![t_2\ x]\!]\rho$$

Assume $x : \tau$ and thus $t_1, t_2 : \tau \to \sigma$

$$[\![t_1\ x]\!]\rho = \text{let } \varphi \Leftarrow [\![t_1]\!]\rho\ .\ \varphi([\![x]\!]\rho)$$

$$= \text{let } \varphi \Leftarrow [\![t_2]\!]\rho\ .\ \varphi([\![x]\!]\rho)$$

$$= [\![t_2\ x]\!]\rho$$

Exercise a, part 2

Give a counterexample to

$$[\![t_1]\!]\rho = [\![t_2]\!]\rho \Leftarrow [\![t_1\ x]\!]\rho = [\![t_2\ x]\!]\rho$$

Take $x : int$

$t_1 = (\lambda y : int\,.\ \mathsf{rec}\ z : int\,.\ z) : int \to int$

$t_2 = (\mathsf{rec}\ f : int \to int\,.\ f) : int \to int$

Then $[\![t_i\ x]\!]\rho = \mathsf{let}\ \varphi \Leftarrow [\![t_i]\!]\rho\,.\ \varphi([\![x]\!]\rho) = \bot$

But $[\![t_1]\!]\rho = \lfloor \lambda d\,.\ \bot \rfloor \neq \bot = [\![t_2]\!]\rho$

Exercise b

Compute the denotational semantics of the HOFL term

$$t \triangleq \text{rec } f\,.\ \lambda x\,.\ \text{if } x \text{ then } 1 \text{ else } (f\ (x - x))$$

$$[\![t]\!]\rho = fix\ \lambda\varphi\,.\ [\![\lambda x\,.\ \text{if } x \text{ then } 1 \text{ else } (f\ (x - x))]\!]\rho[^{\varphi}/_{f}]$$

$$= fix\ \lambda\varphi\,.\ \lfloor \lambda d\,.\ [\![\text{if } x \text{ then } 1 \text{ else } (f\ (x - x))]\!]\rho[^{\varphi}/_{f}, ^{d}/_{x}] \rfloor$$

$$= fix\ \lambda\varphi\,.\ \lfloor \lambda d\,.\ \text{Cond}(d, \lfloor 1 \rfloor, [\![(f\ (x - x))]\!]\rho[^{\varphi}/_{f}, ^{d}/_{x}]) \rfloor$$

$$= fix\ \lambda\varphi\,.\ \lfloor \lambda d\,.\ \text{Cond}(d, \lfloor 1 \rfloor, \text{let } \psi \Leftarrow \varphi\,.\ \psi(d -_{\perp} d)) \rfloor$$

Exercise b (ctd.)

Compute the denotational semantics of the HOFL term

$$t \triangleq \text{rec } f.\ \lambda x.\ \text{if } x \text{ then } 1 \text{ else } (f\ (x - x))$$

$$[\![t]\!]\rho = fix\ \lambda\varphi.\ \lfloor \lambda d.\ \text{Cond}(d, \lfloor 1 \rfloor, \text{let } \psi \Leftarrow \varphi\ .\ \psi(d -_\perp d)) \rfloor$$

$$\varphi^0 = \perp$$

$$\varphi^1 = \lfloor \lambda d.\ \text{Cond}(d, \lfloor 1 \rfloor, \text{let } \psi \Leftarrow \varphi^0\ .\ \psi(d -_\perp d)) \rfloor$$

$$= \lfloor \lambda d.\ \text{Cond}(d, \lfloor 1 \rfloor, \perp\ ) \rfloor$$

$$\varphi^2 = \lfloor \lambda d.\ \text{Cond}(d, \lfloor 1 \rfloor, \text{let } \psi \Leftarrow \varphi^1\ .\ \varphi(d -_\perp d)) \rfloor$$

$$= \lfloor \lambda d.\ \text{Cond}(d, \lfloor 1 \rfloor, \text{Cond}(d -_\perp d, \lfloor 1 \rfloor, \perp\ )) \rfloor$$

$$= \lfloor \lambda d.\ \text{Cond}(d, \lfloor 1 \rfloor, \text{Cond}(\lfloor 0 \rfloor, \lfloor 1 \rfloor, \perp\ )) \rfloor$$

$$= \lfloor \lambda d.\ \text{Cond}(d, \lfloor 1 \rfloor, \lfloor 1 \rfloor) \rfloor$$

$$\varphi^3 = \lfloor \lambda d.\ \text{Cond}(d, \lfloor 1 \rfloor, \text{let } \psi \Leftarrow \varphi^2\ .\ \varphi(d -_\perp d)) \rfloor = \varphi^2$$