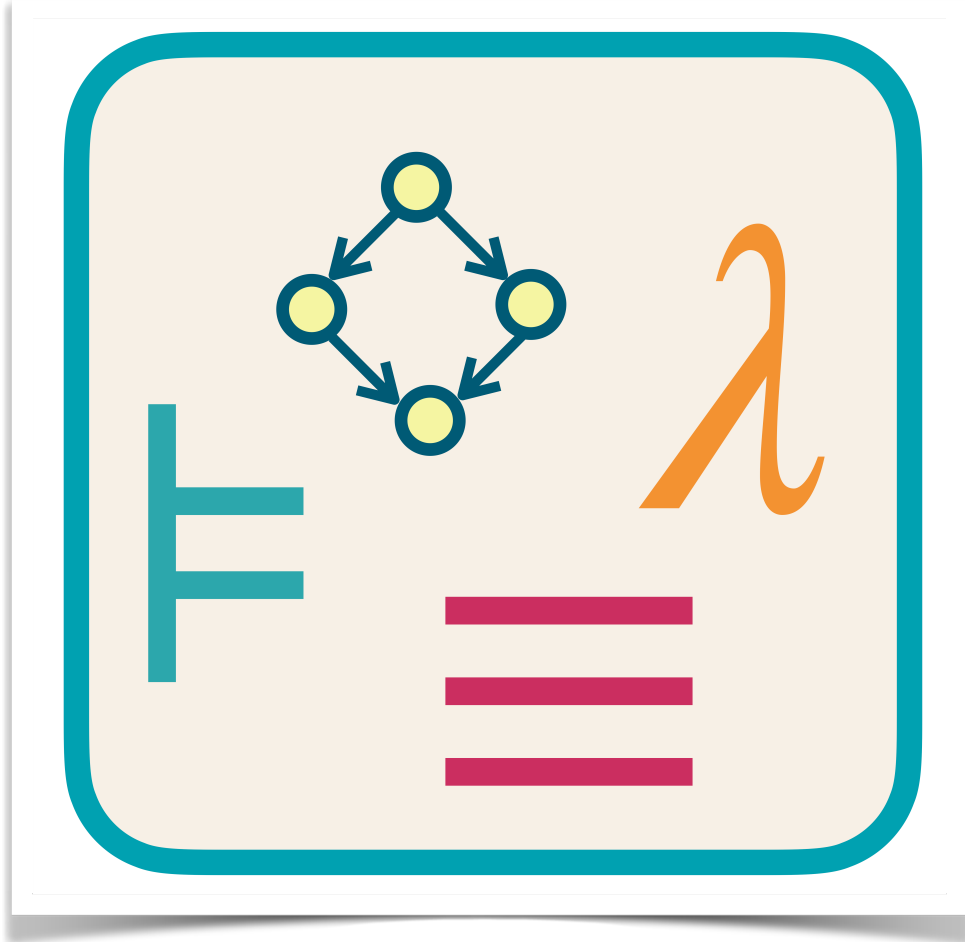




LPL 2026/27 (063AA, 9CFU)

Programming Languages

Fabio Gadducci

and Laboratory

Vincenzo Ciancia

08b - Kleene's fixed point theorem

Monotone functions

Monotone function

$(D, \sqsubseteq_D)$ PO

$(E, \sqsubseteq_E)$ PO

$f : D \rightarrow E$

f is monotone if

Monotone function

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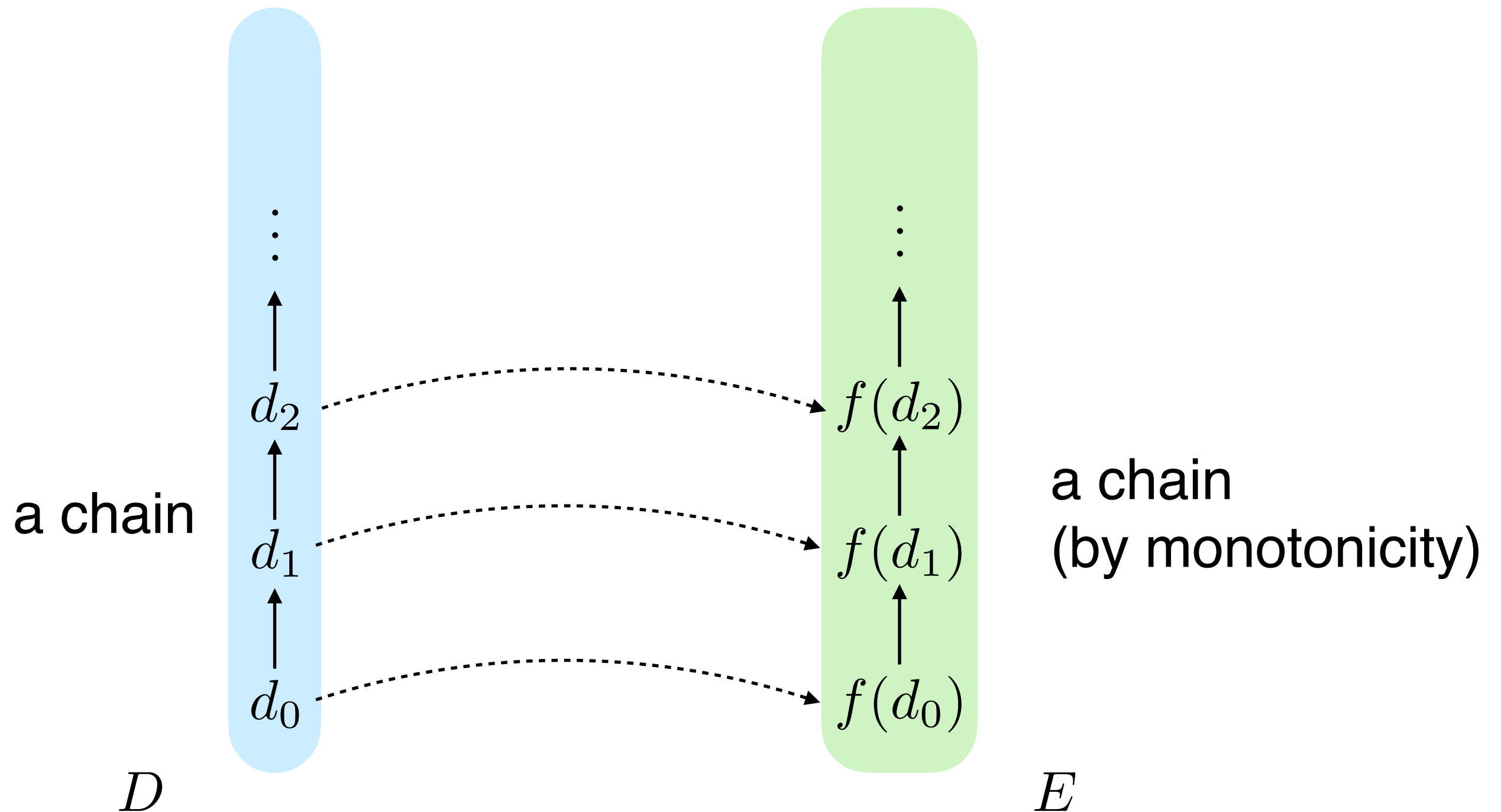
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When $D = E$ we say $f : D \rightarrow D$ is a function on D

Monotonicity illustrated

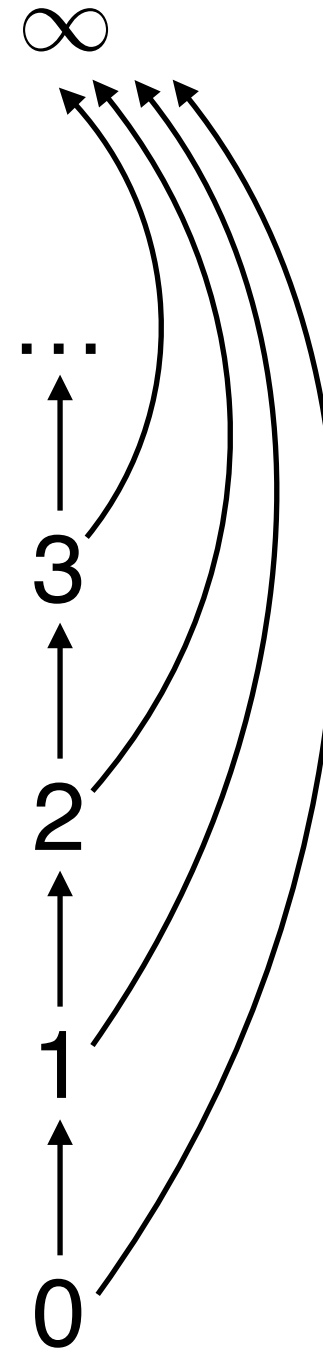
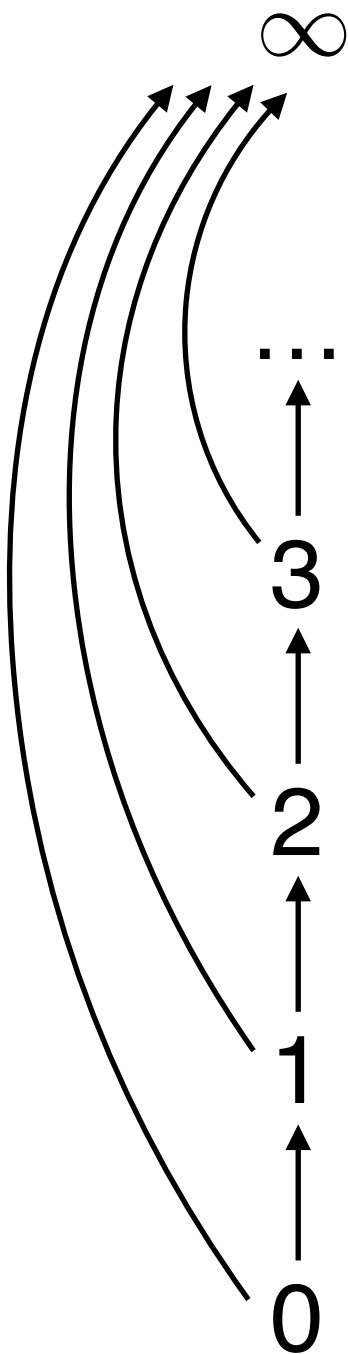


Example

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = n + 1$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$

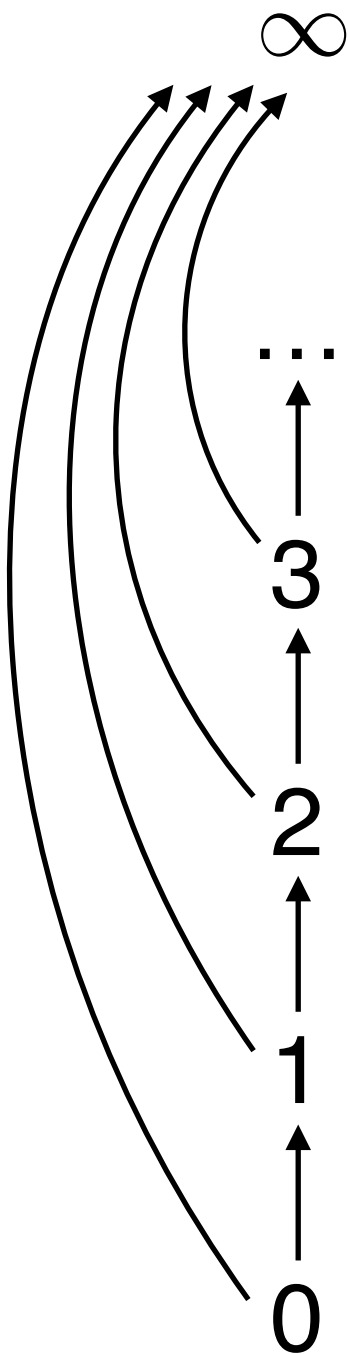


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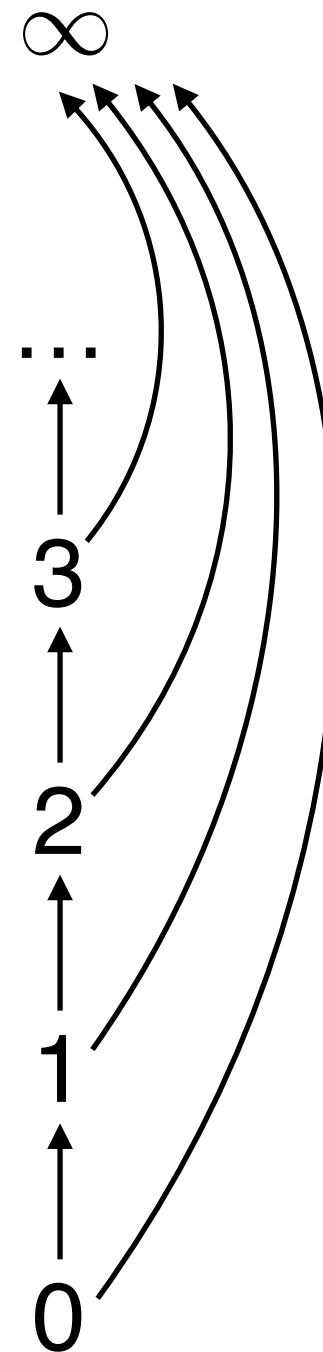
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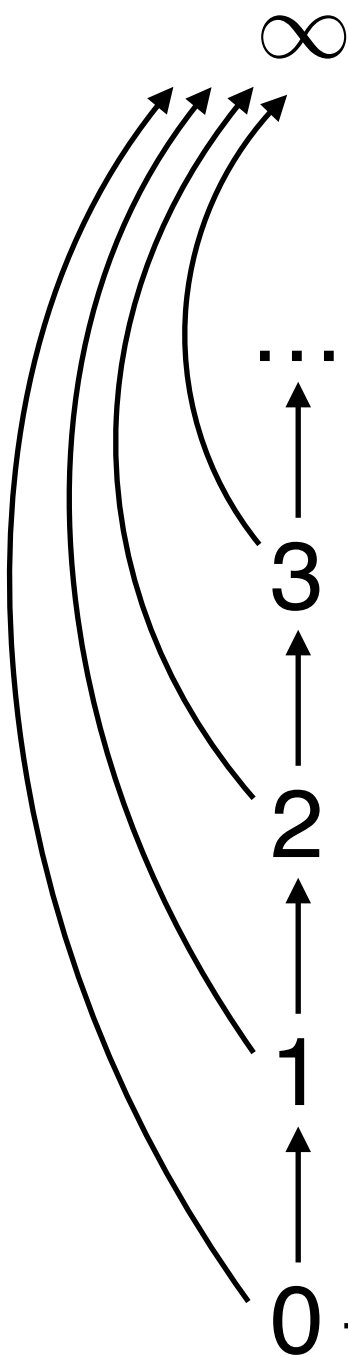


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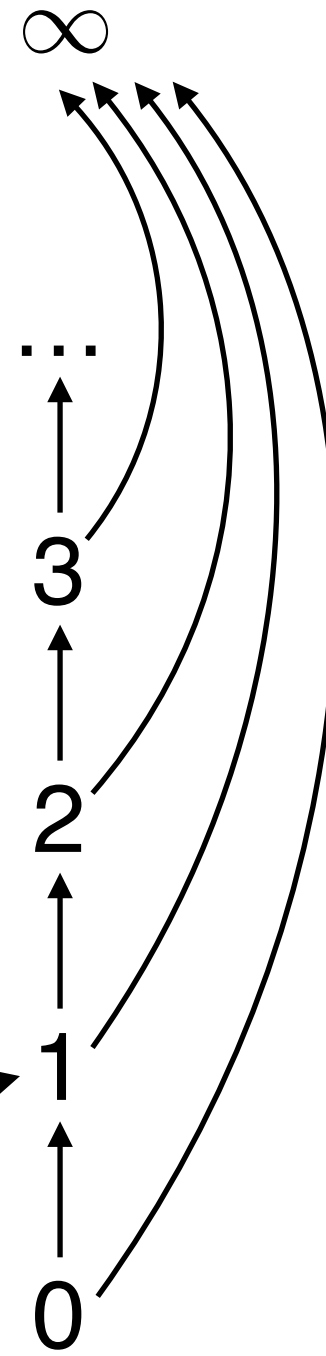
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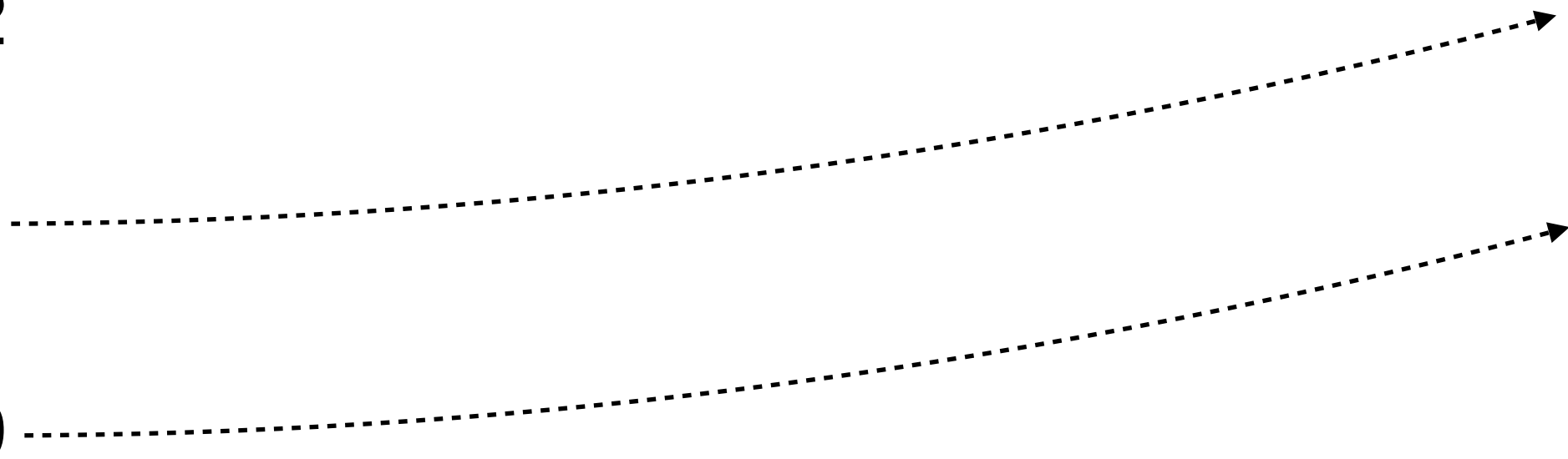
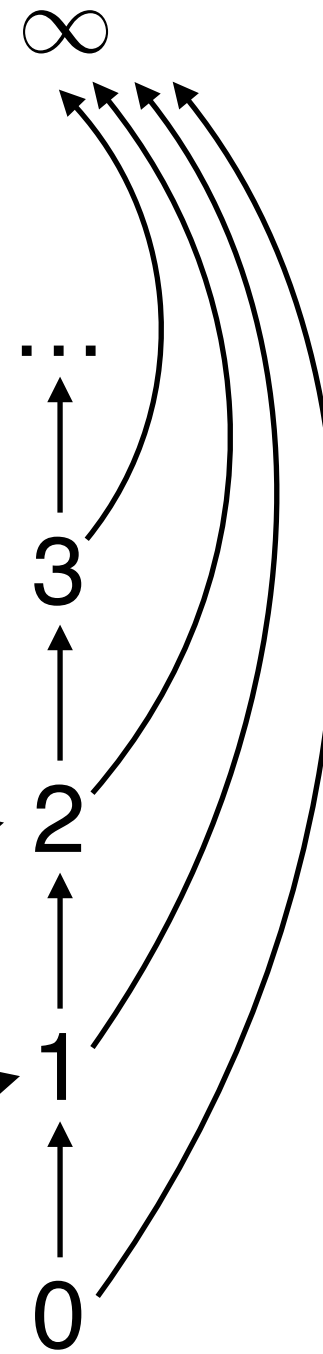
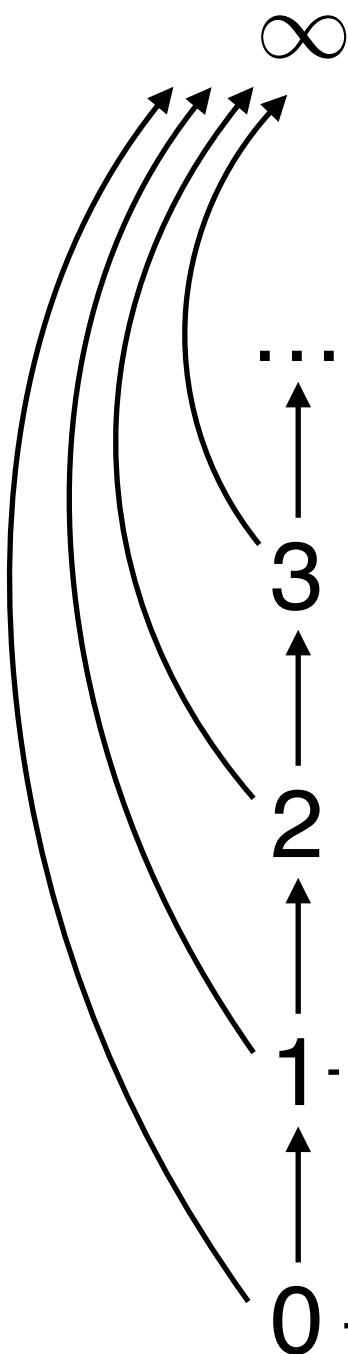
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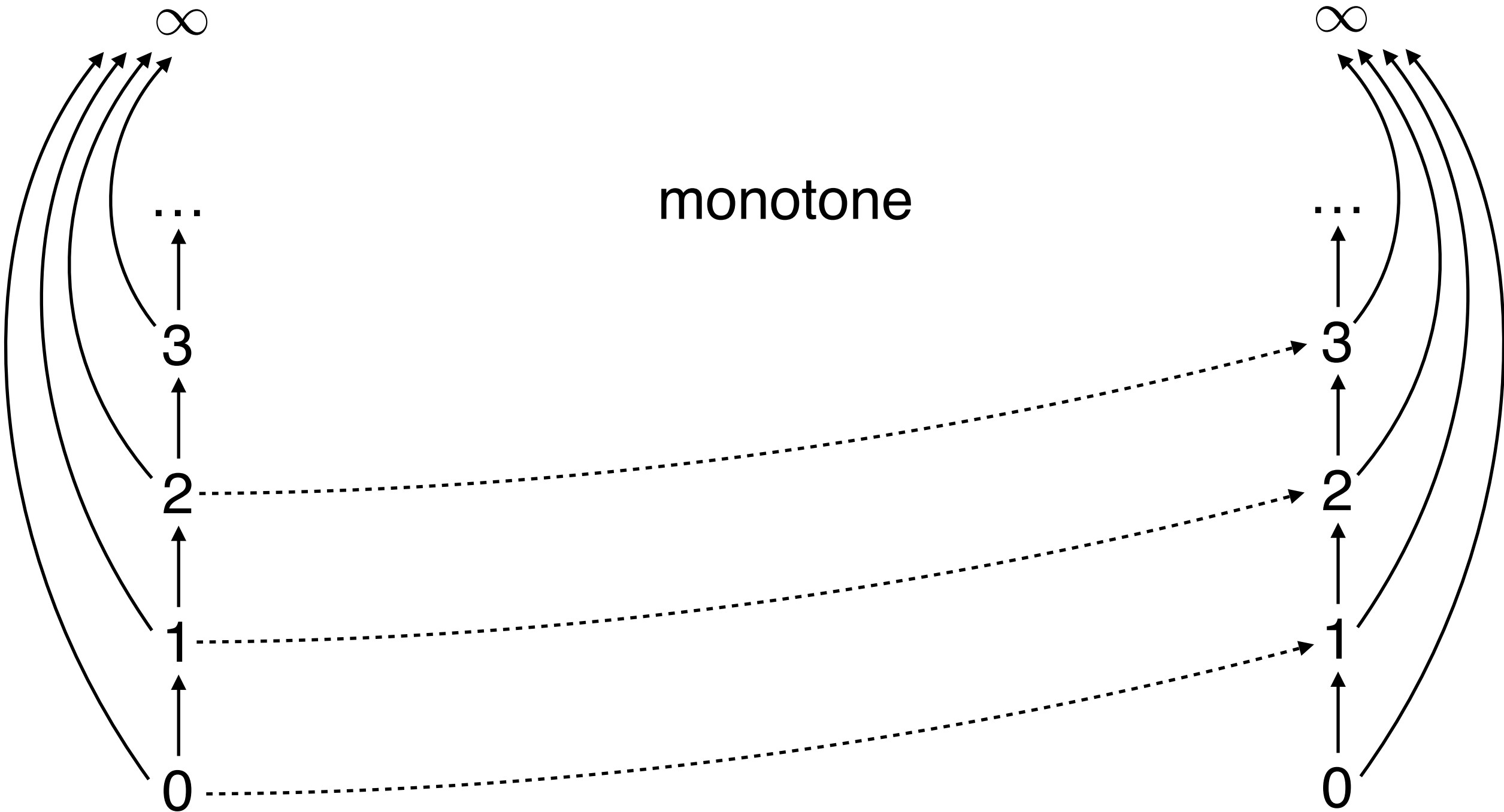


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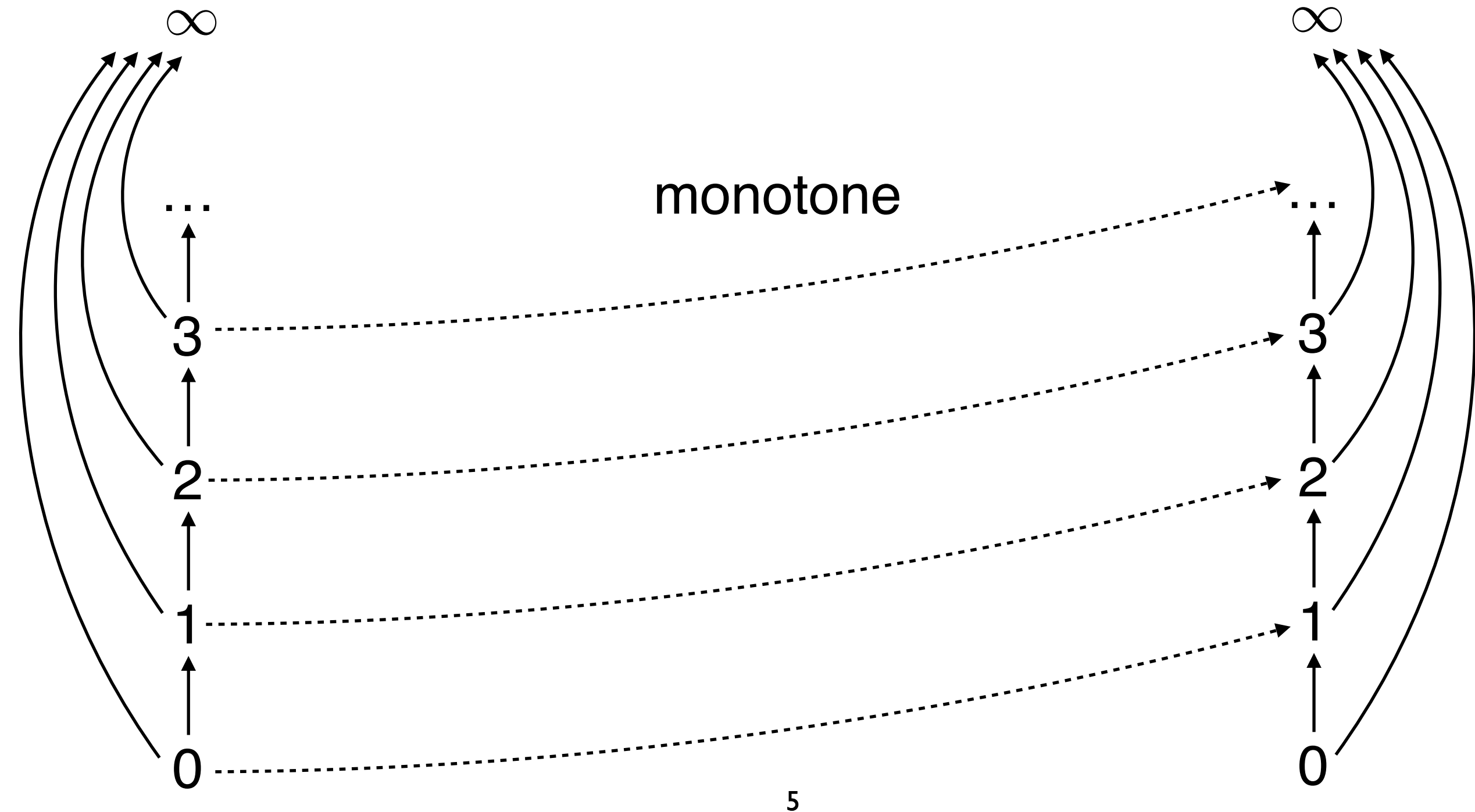


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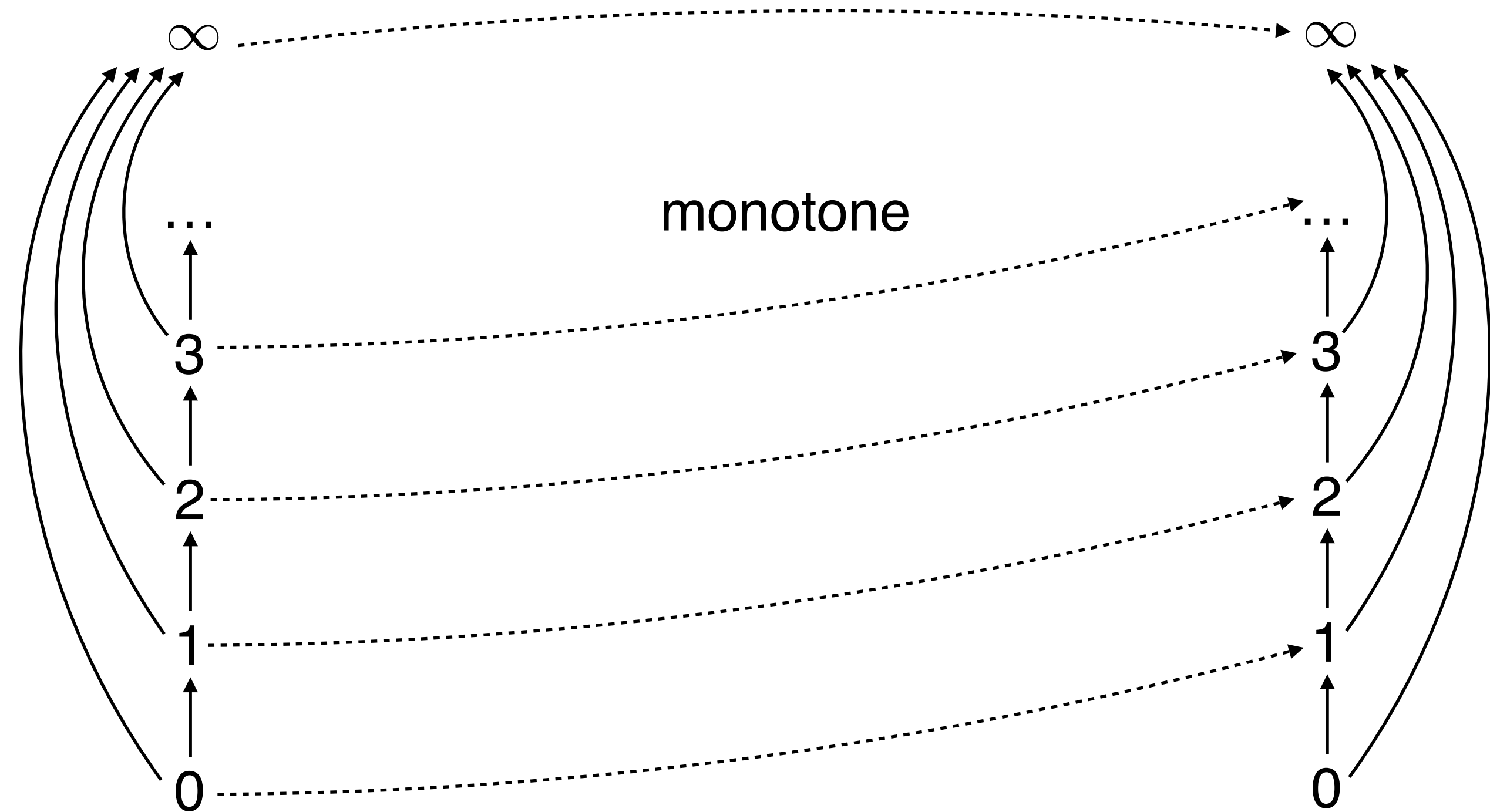


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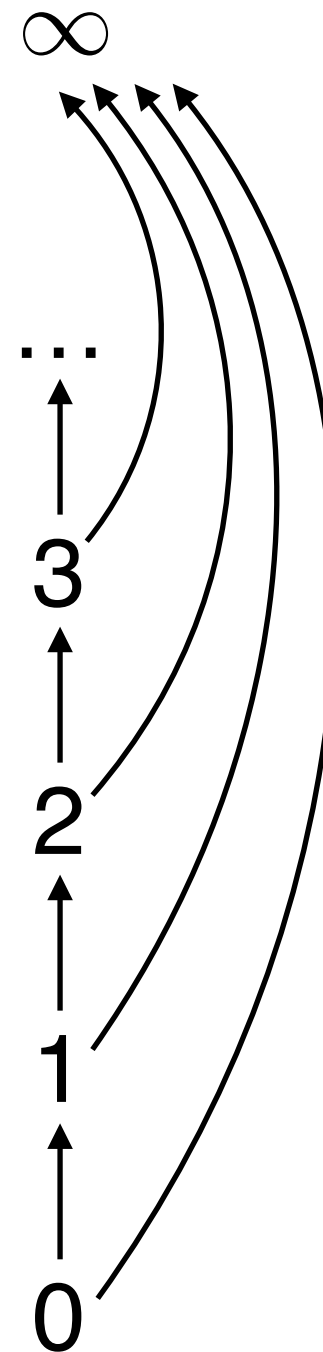
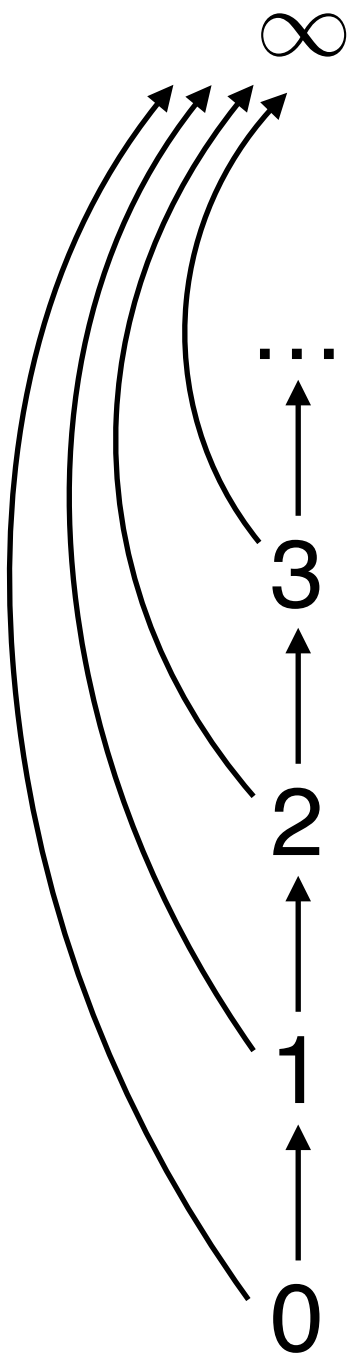
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$$f(n) = 2 \cdot n$$

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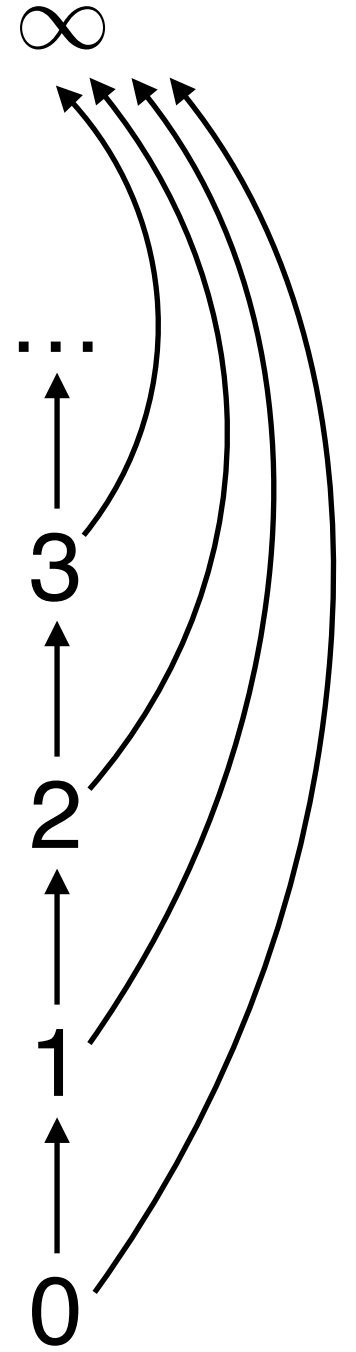
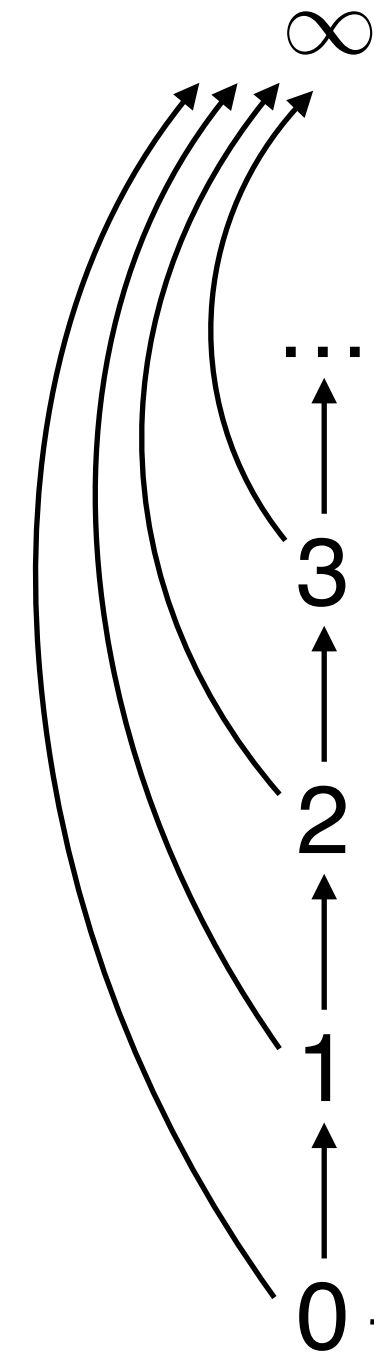


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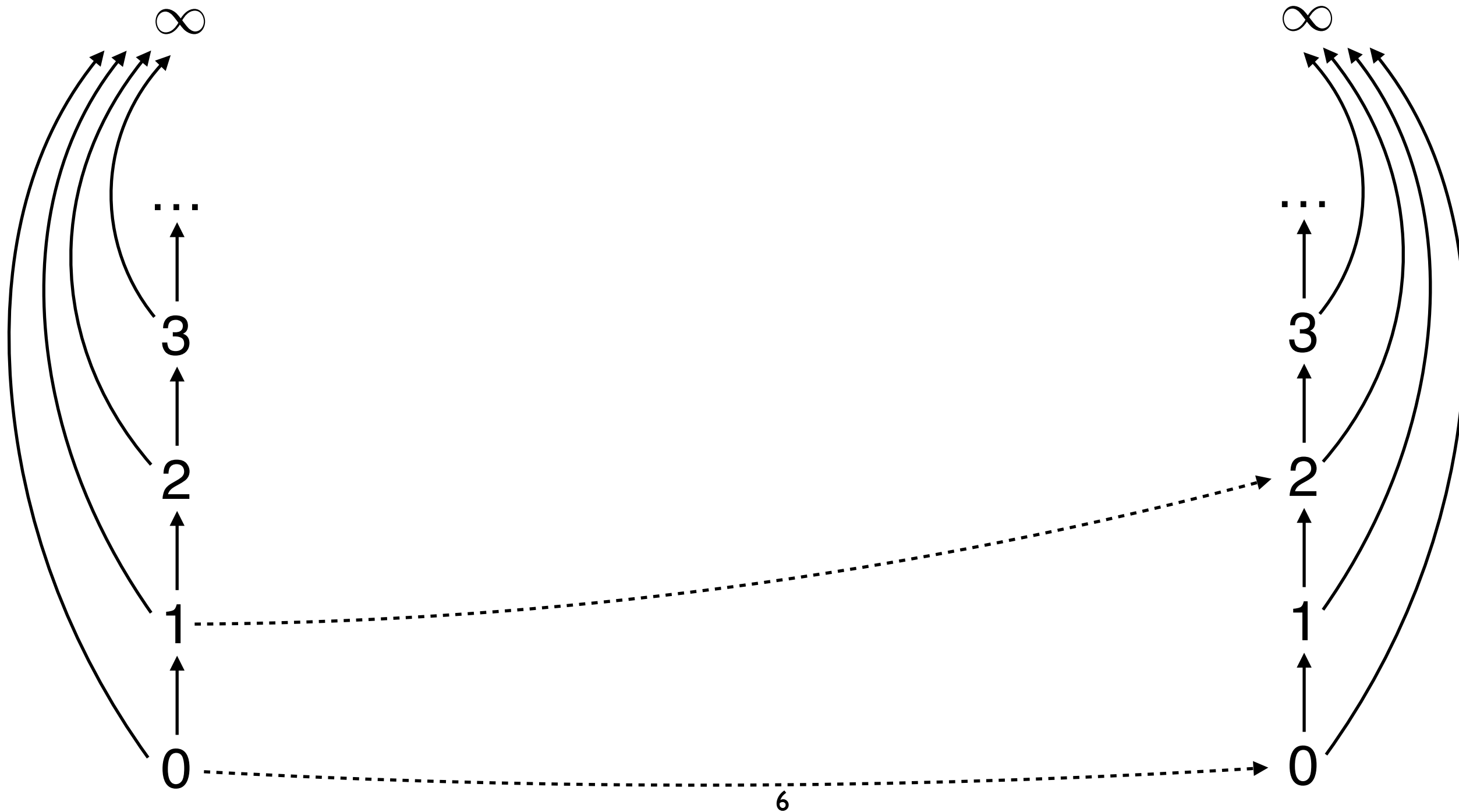


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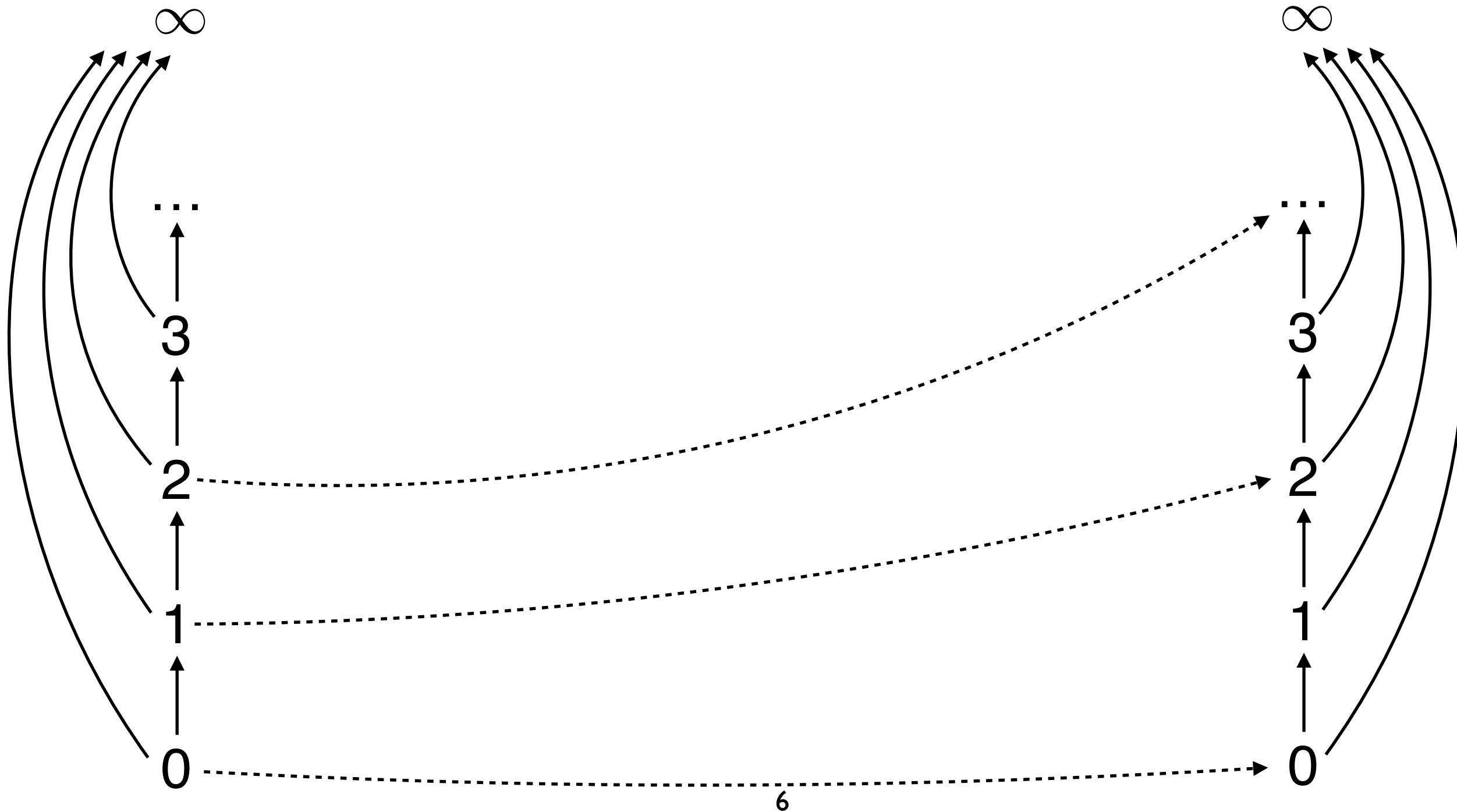


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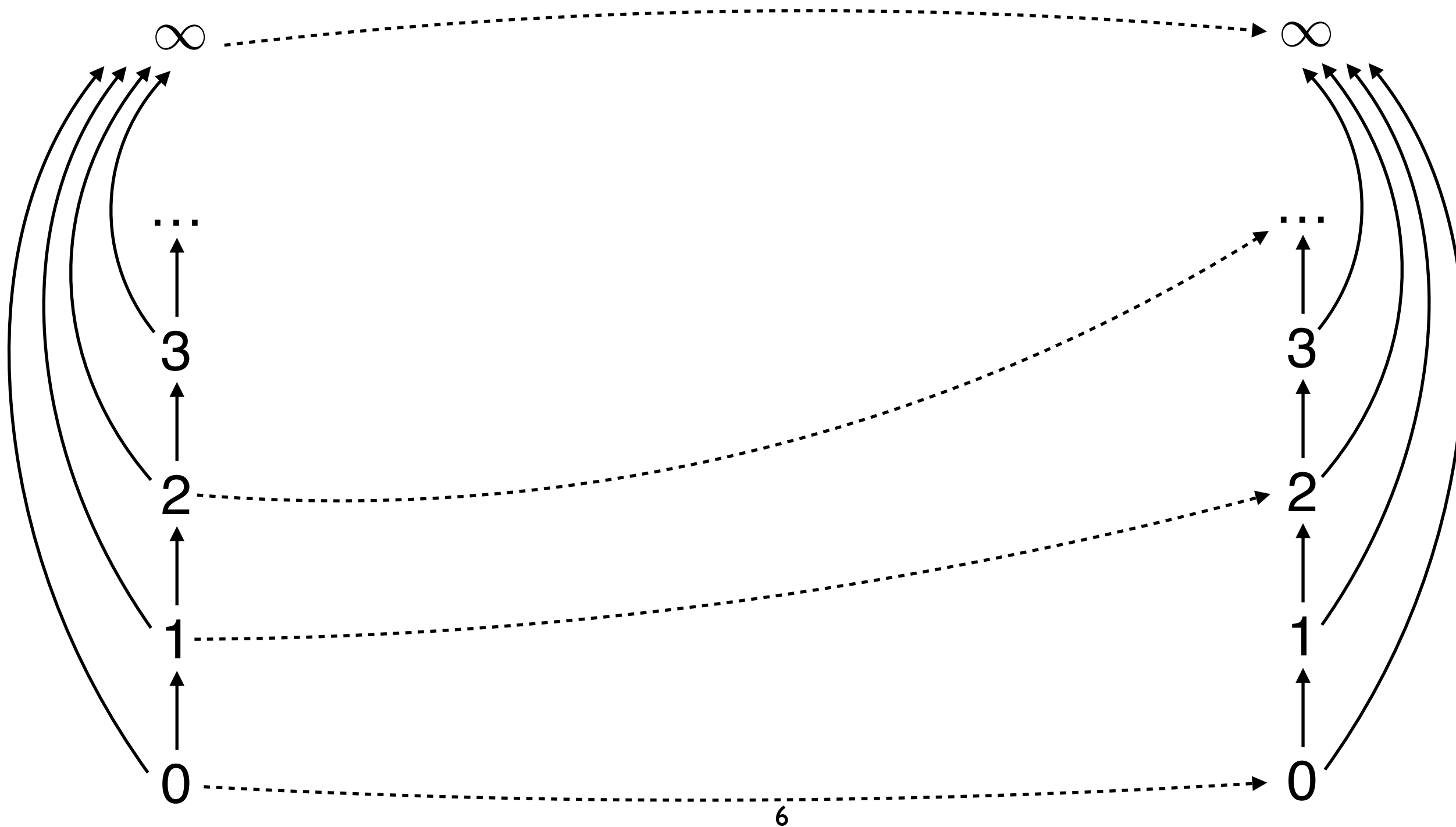


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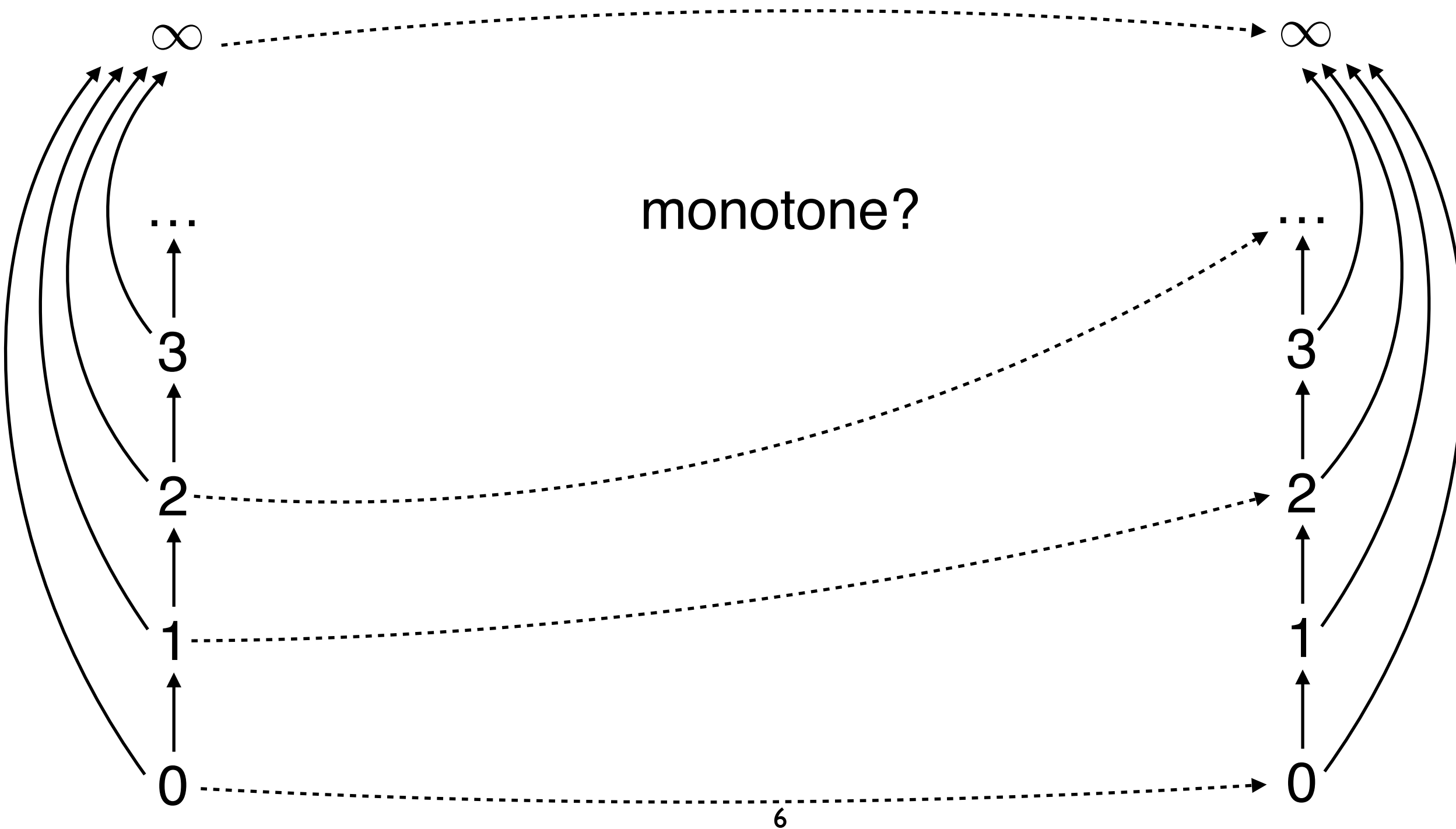


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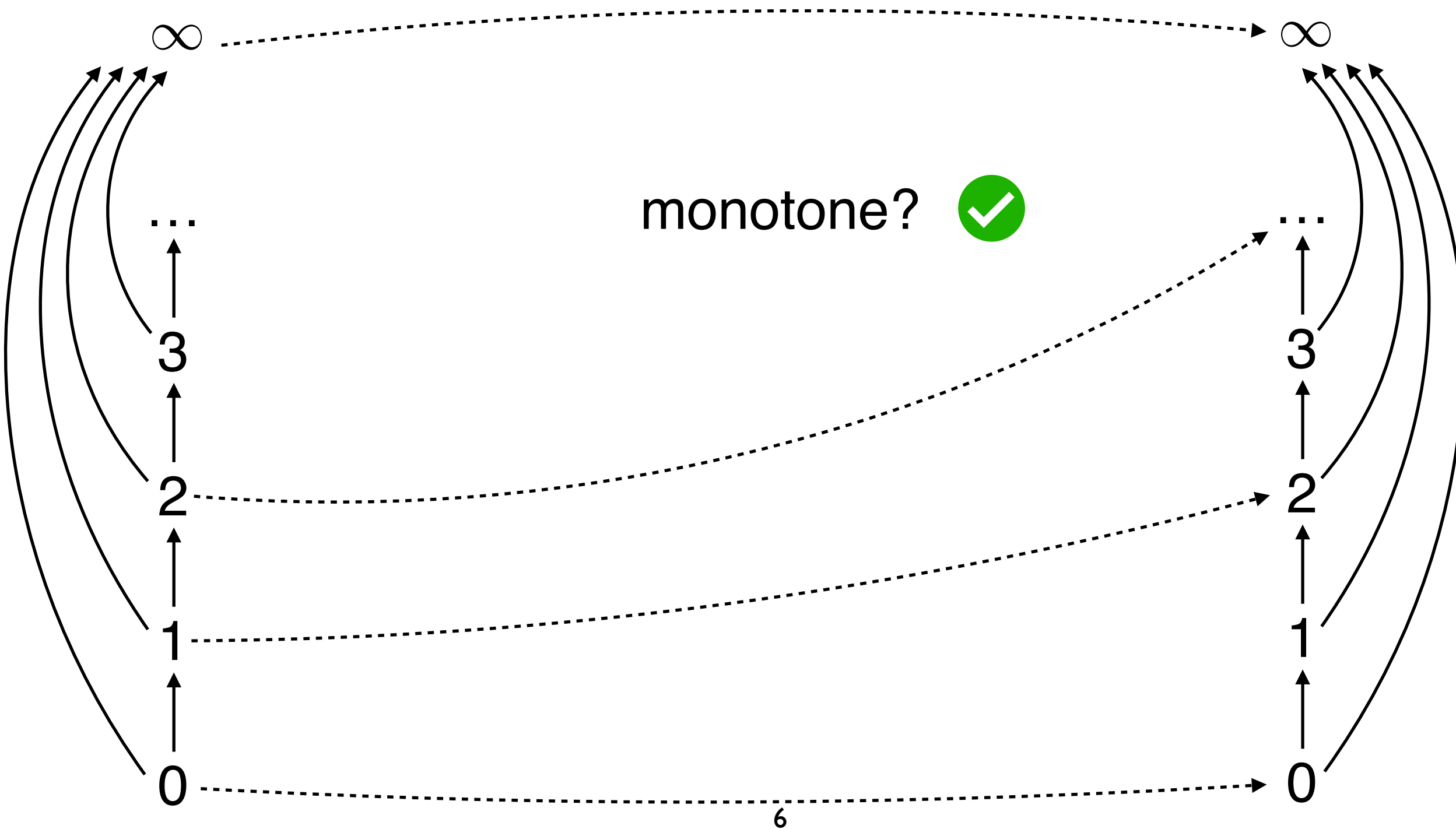


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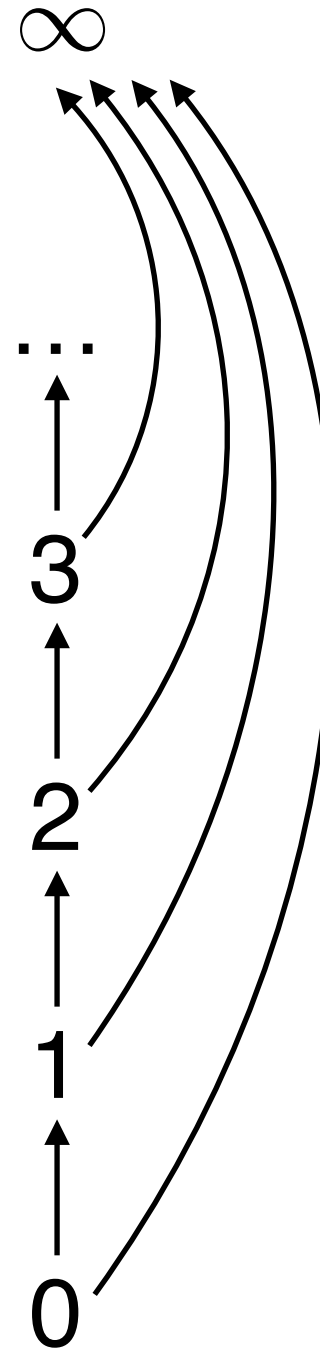
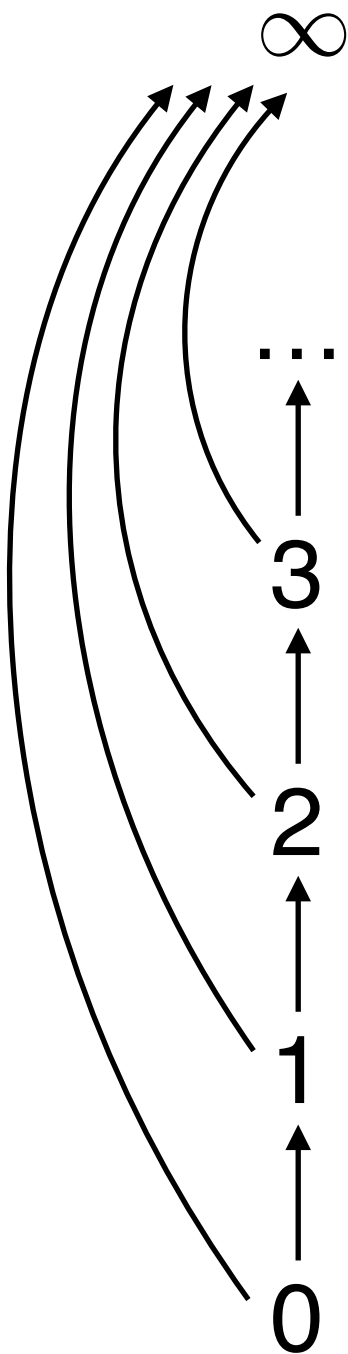
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$$f(n) = n/2$$

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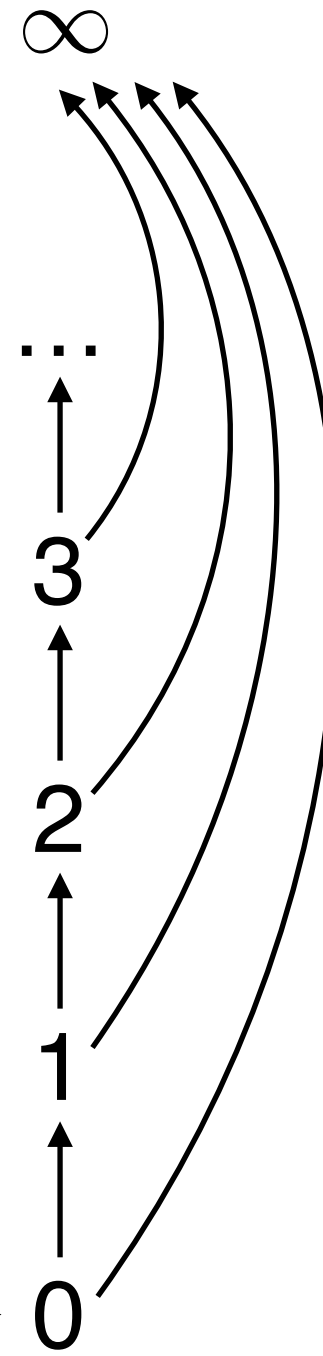
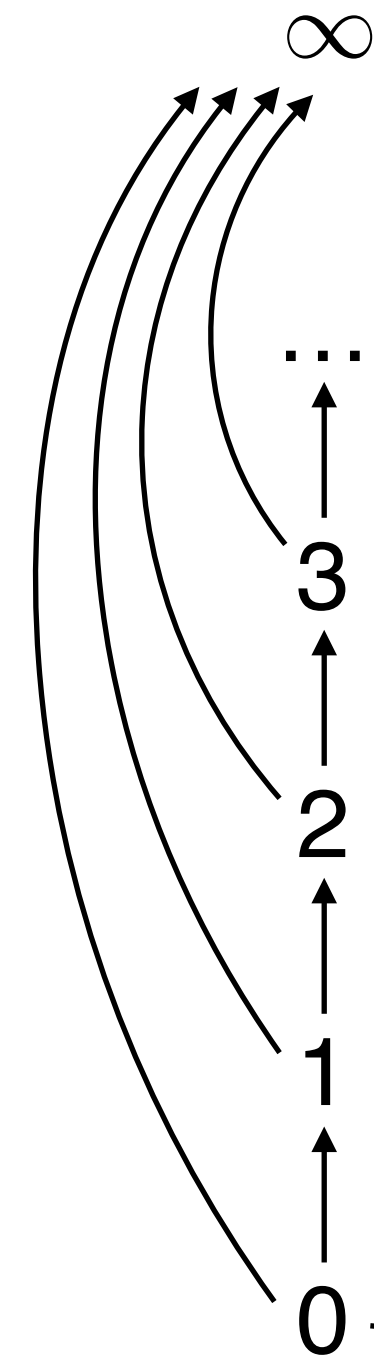


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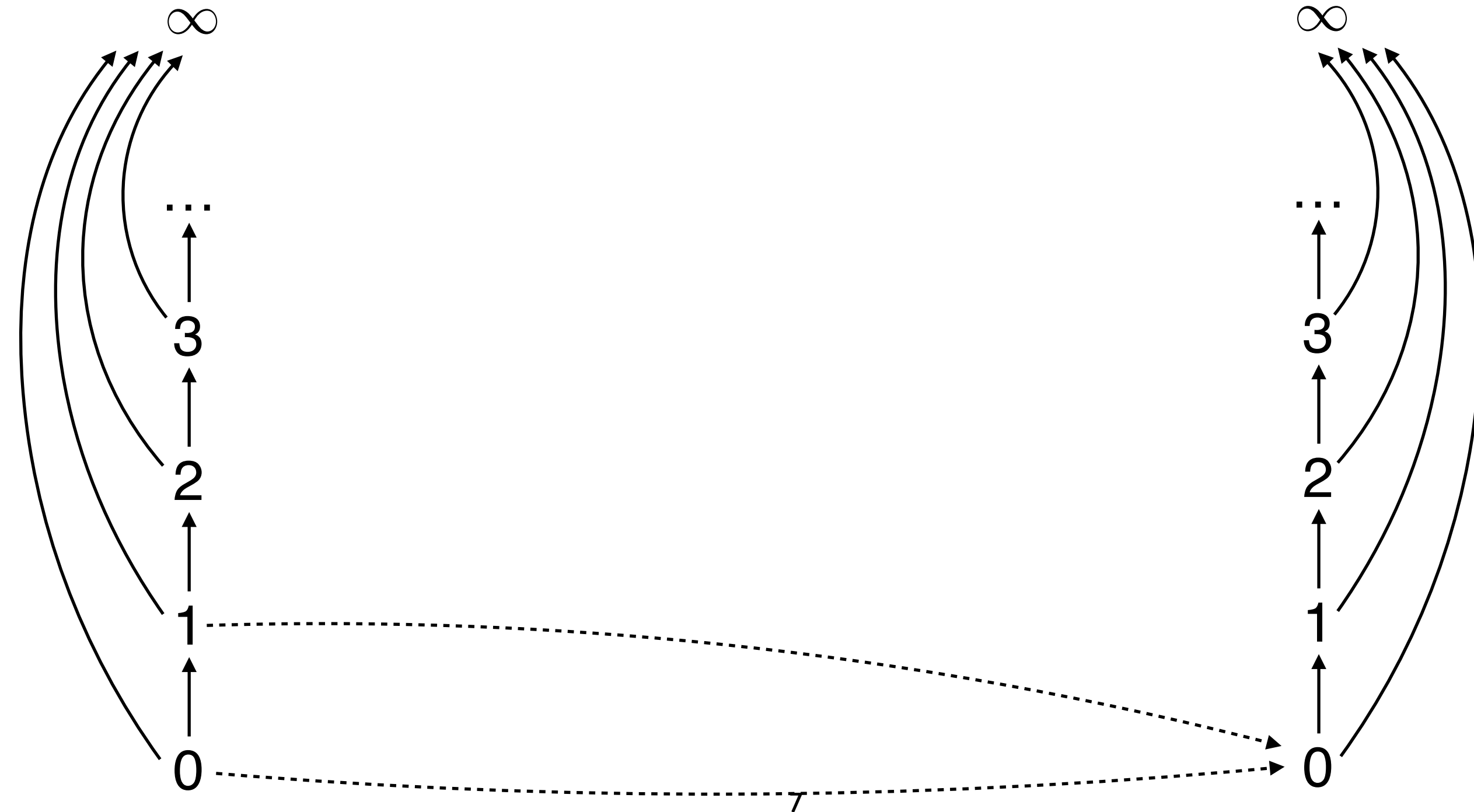


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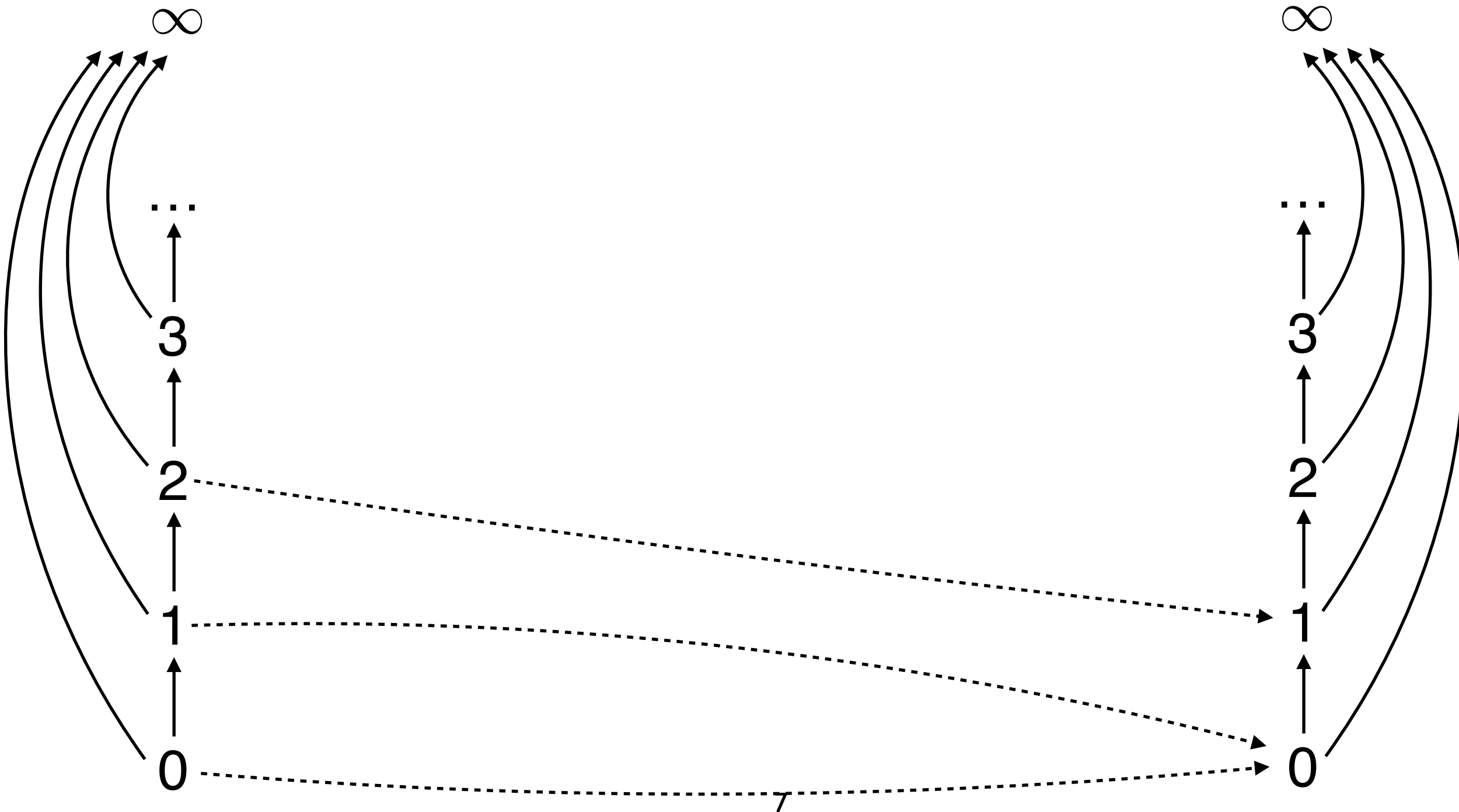


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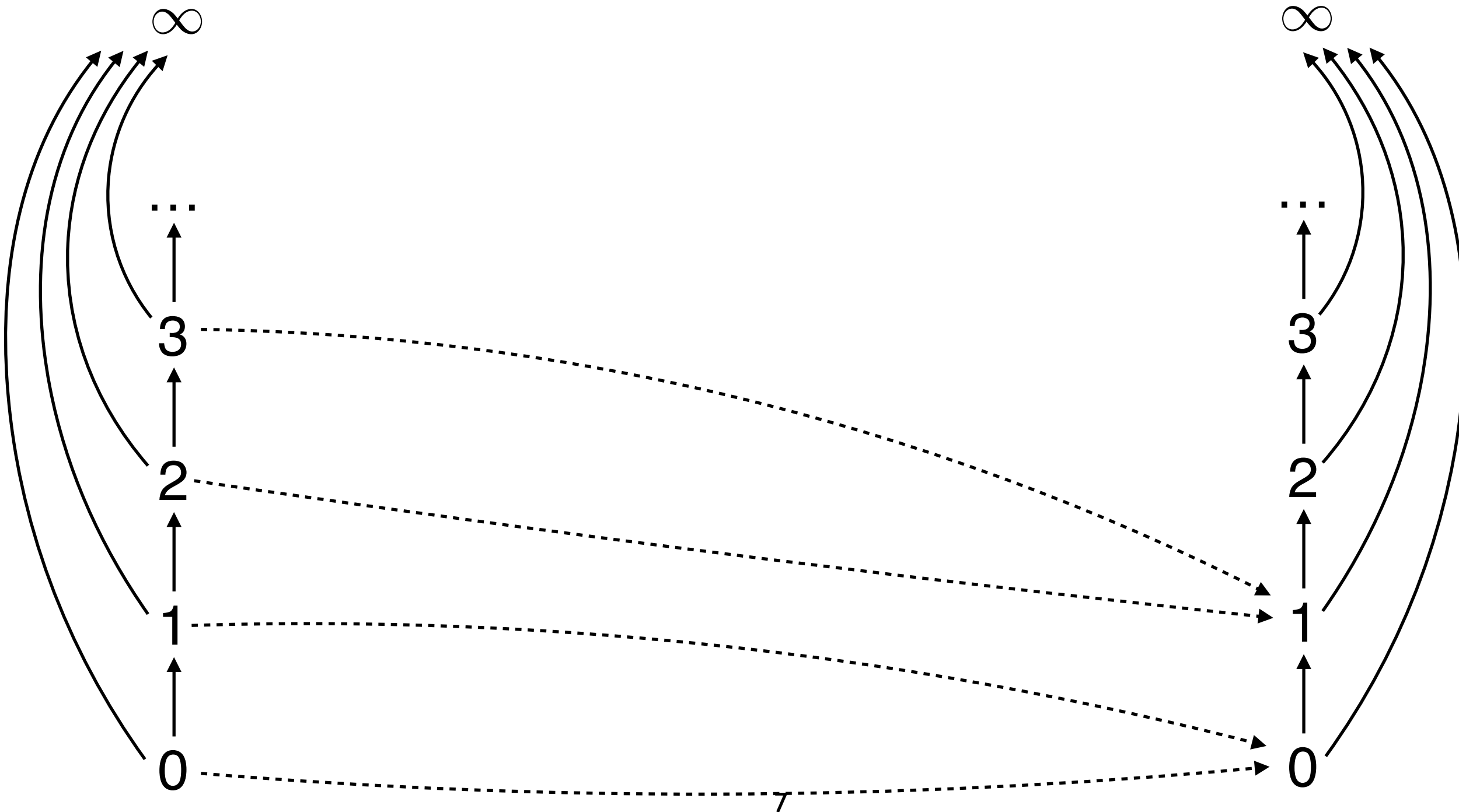


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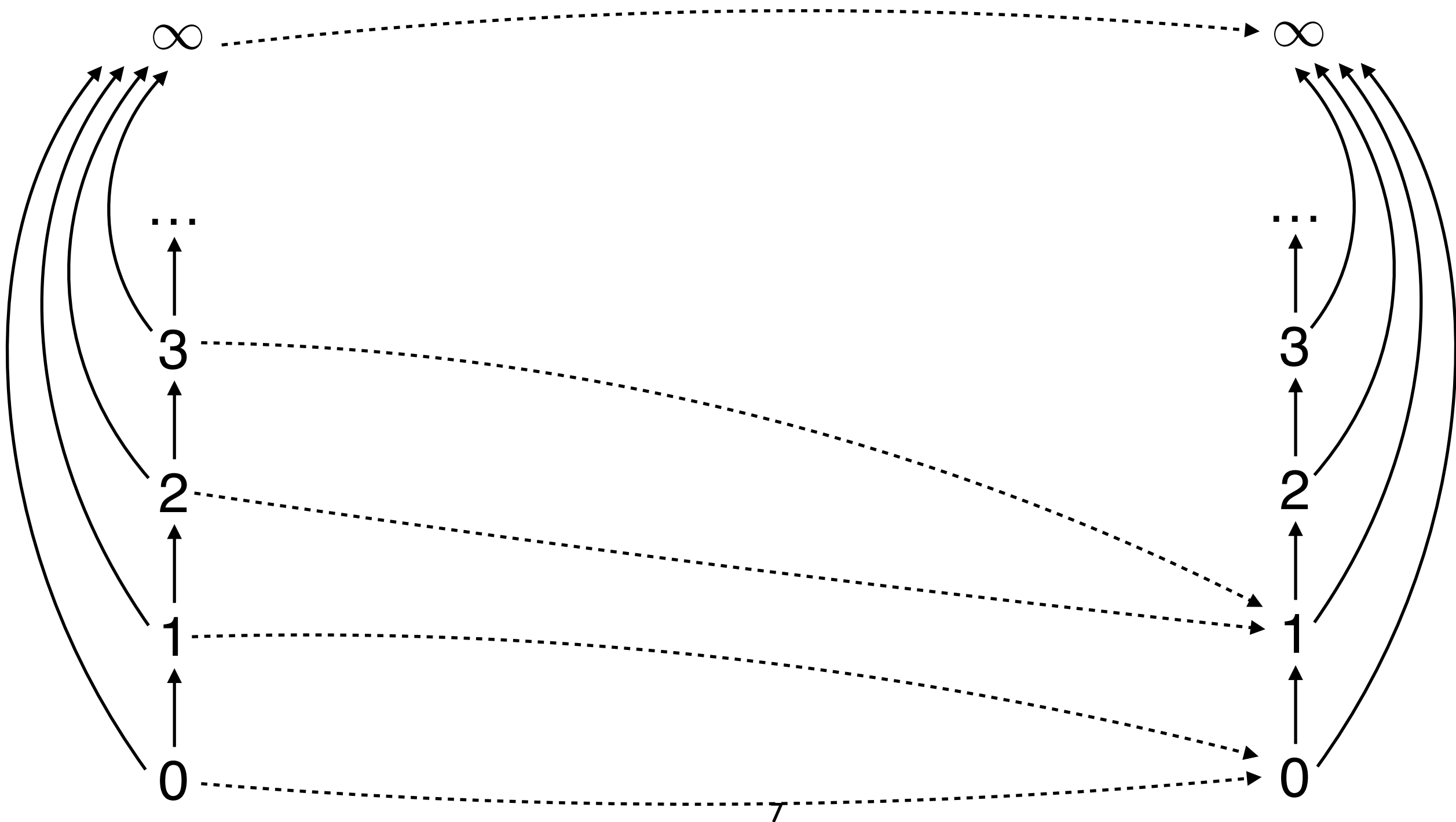


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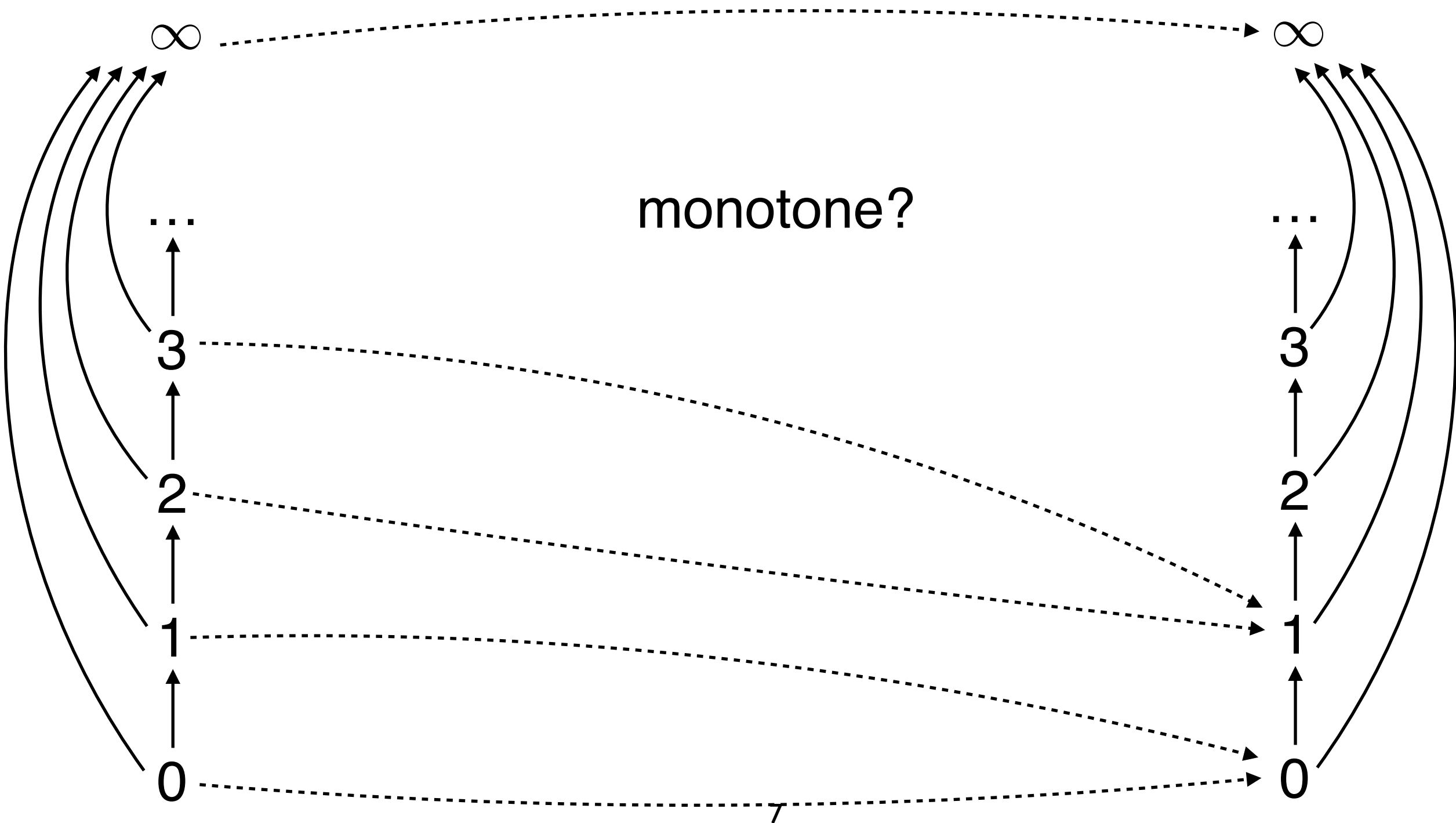


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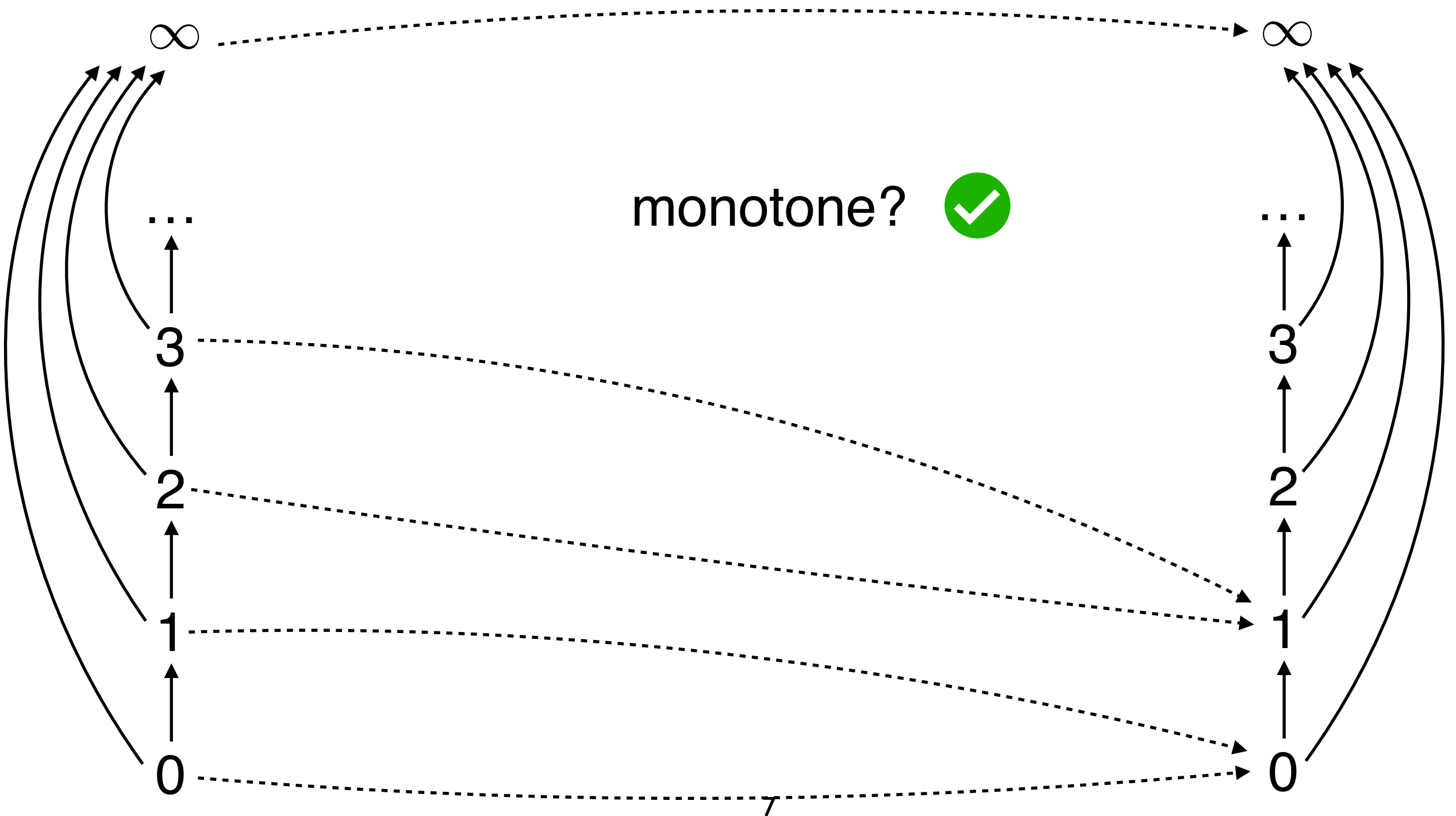


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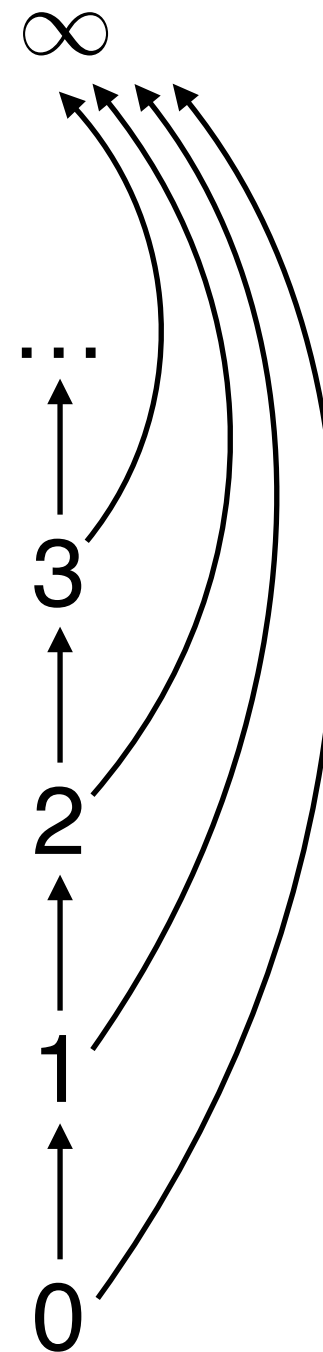
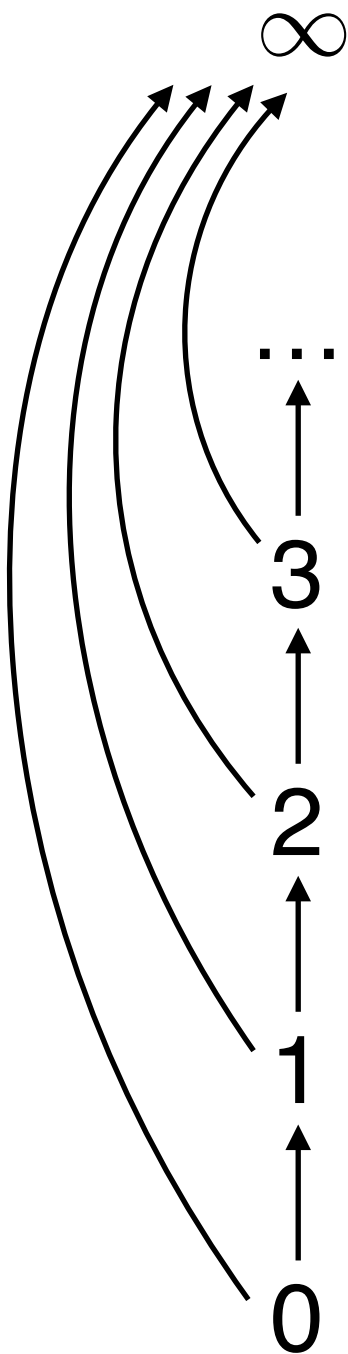


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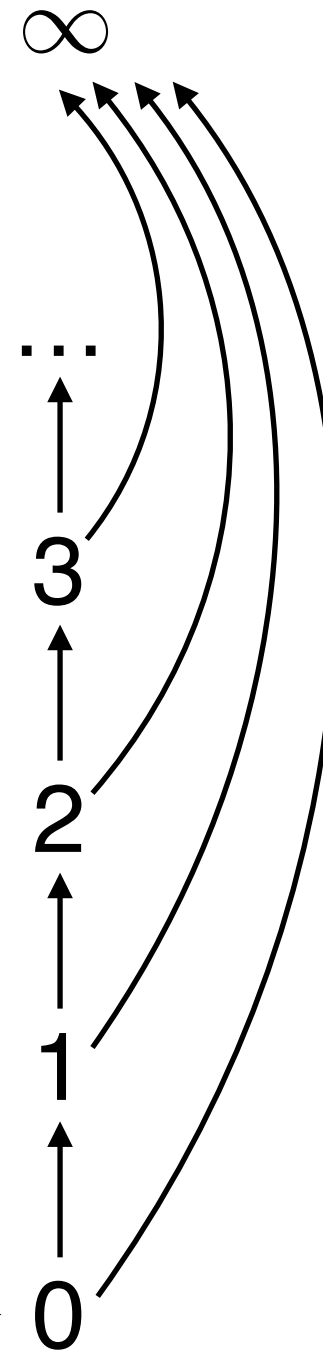
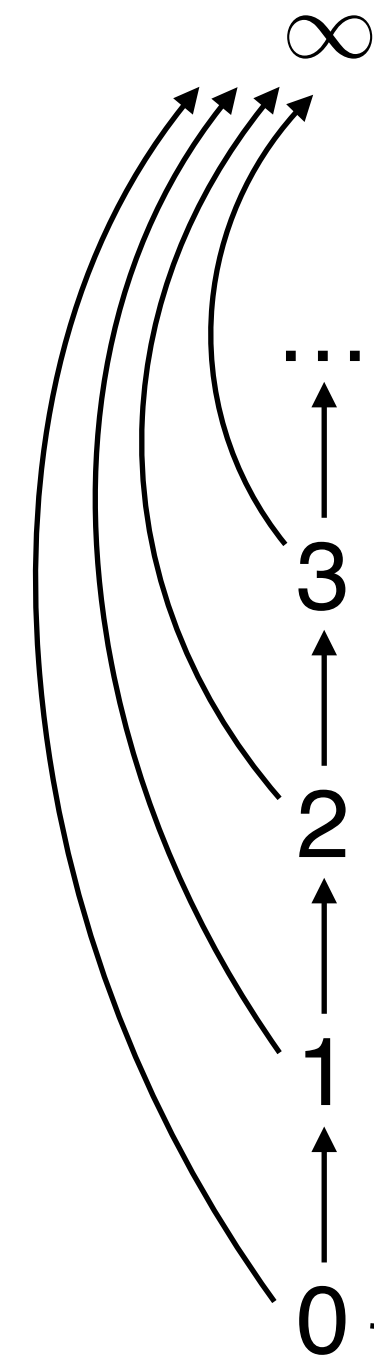


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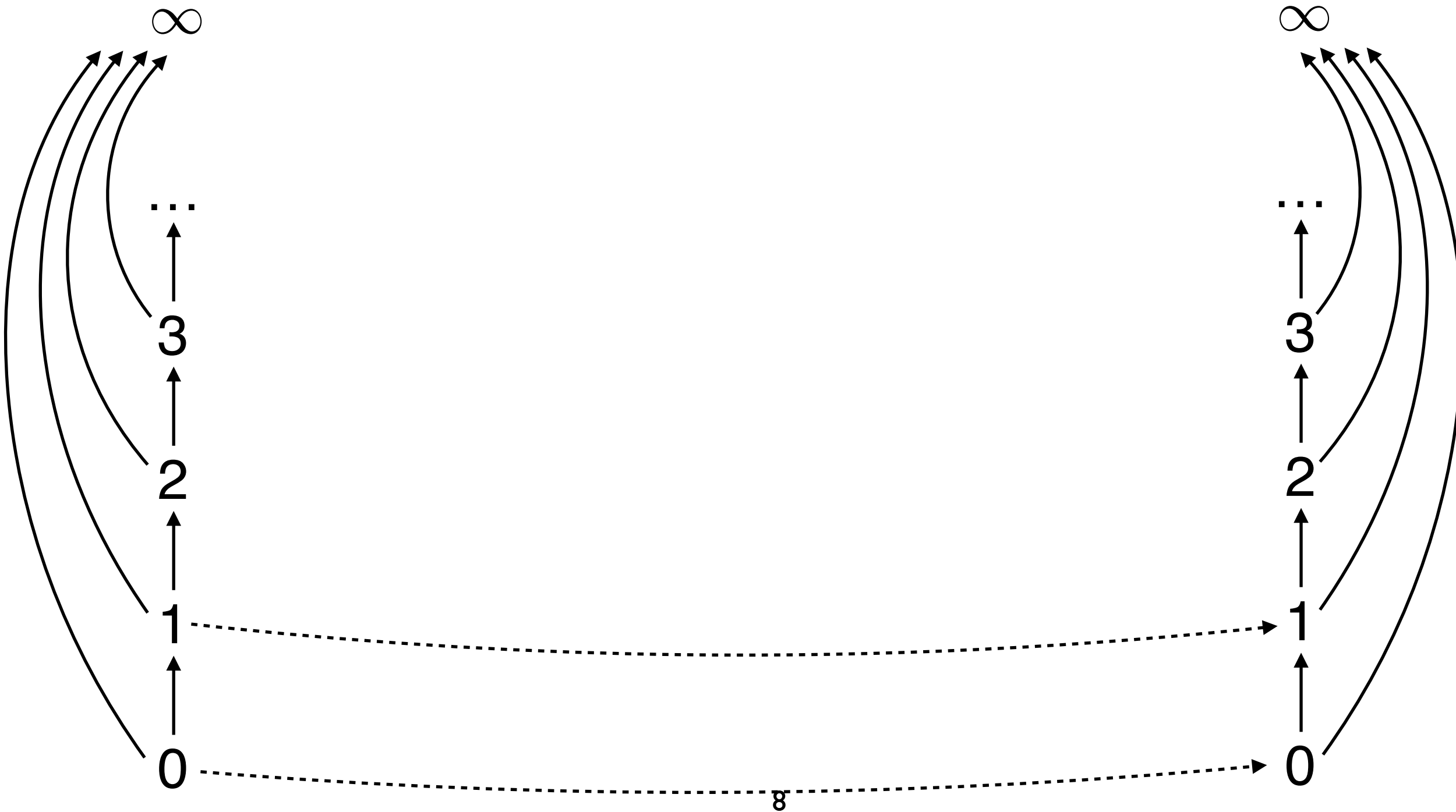


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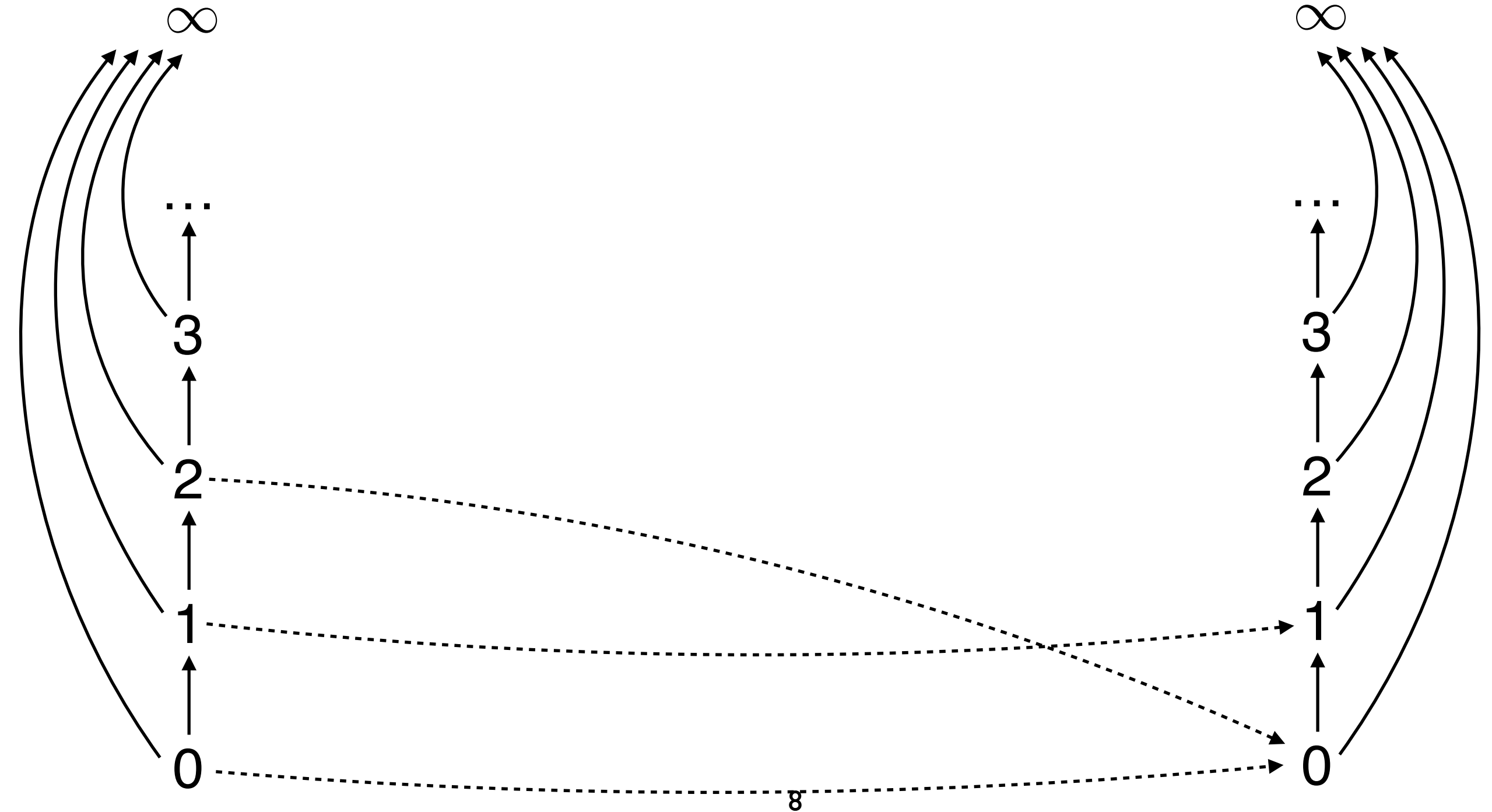


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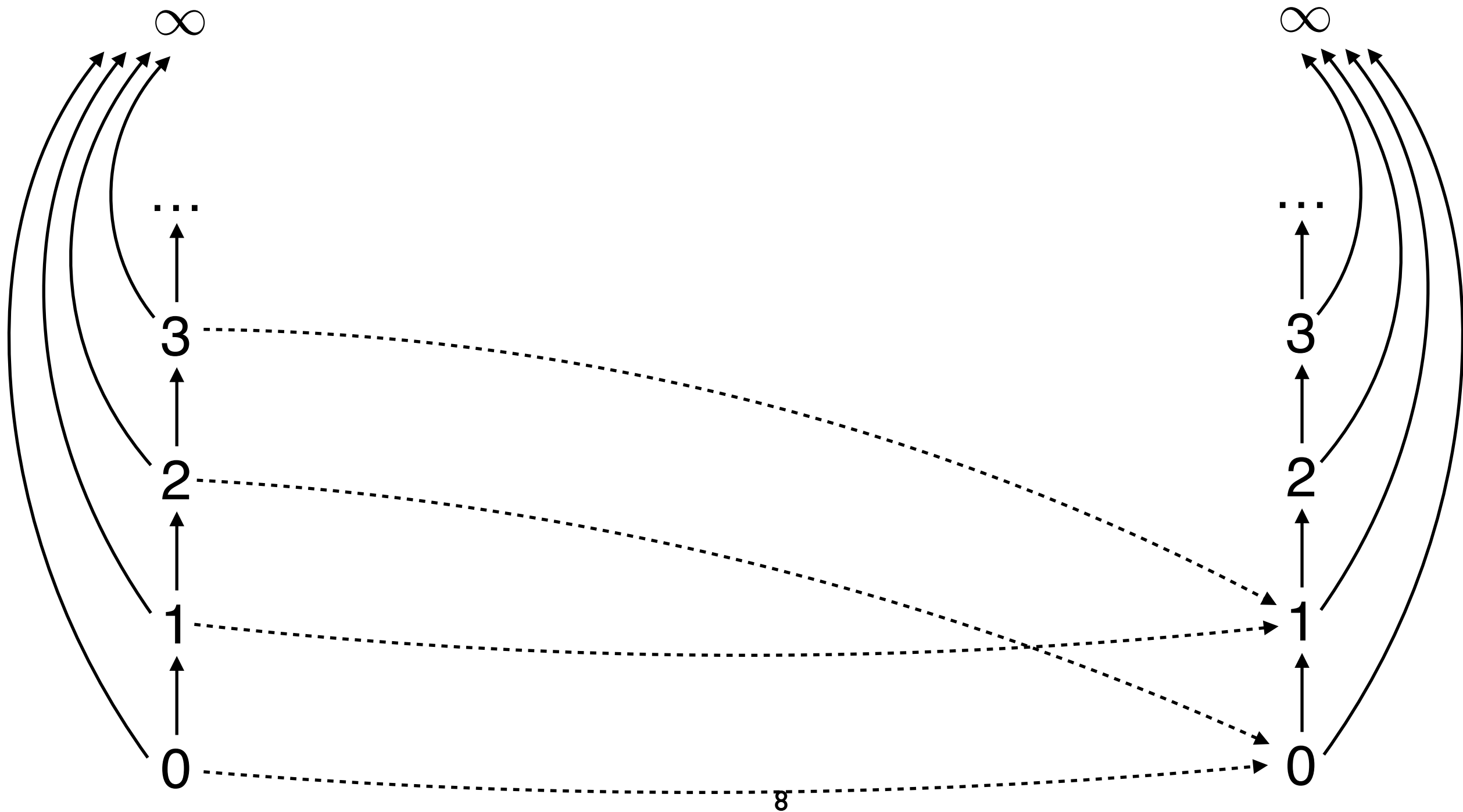


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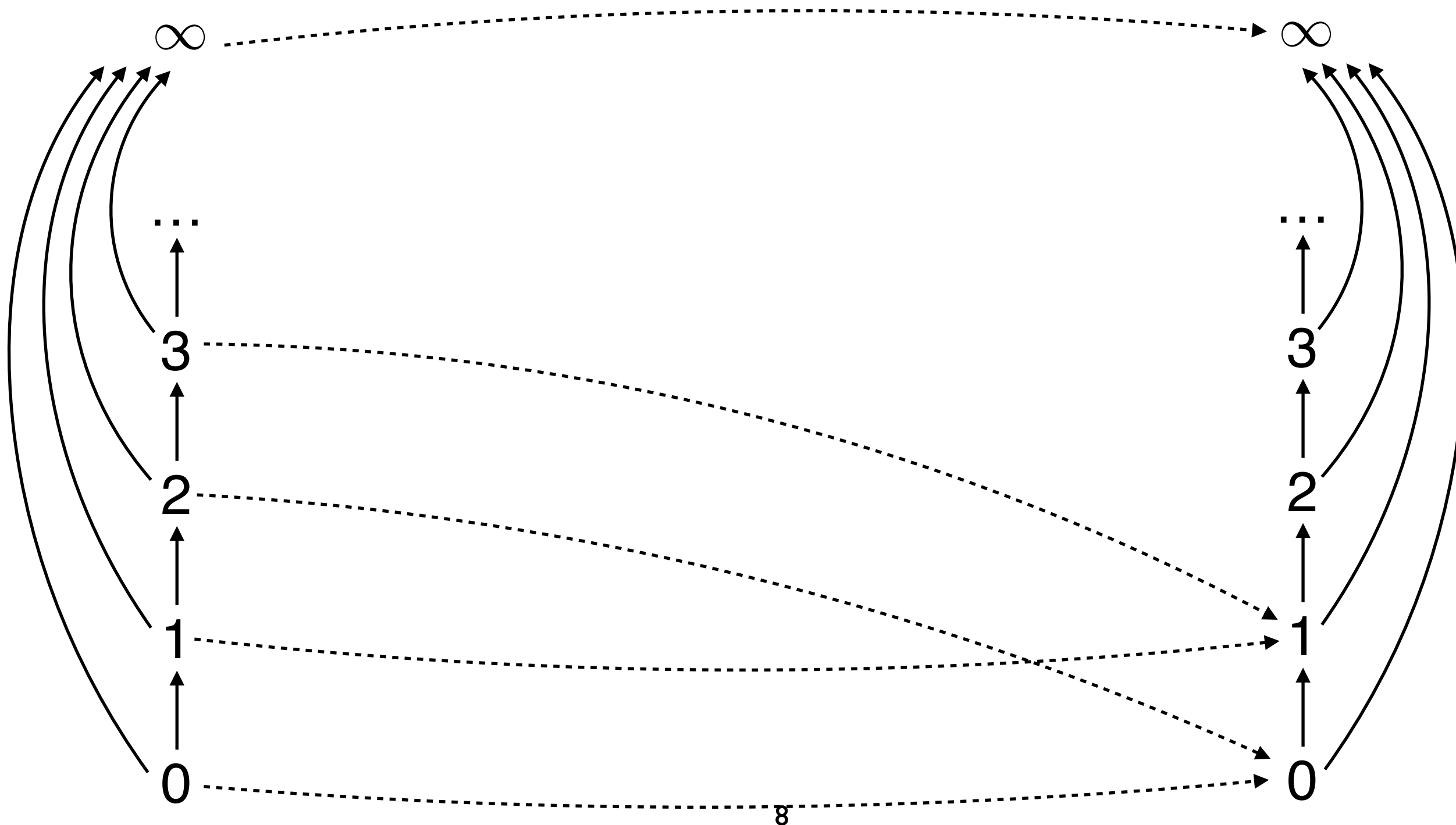


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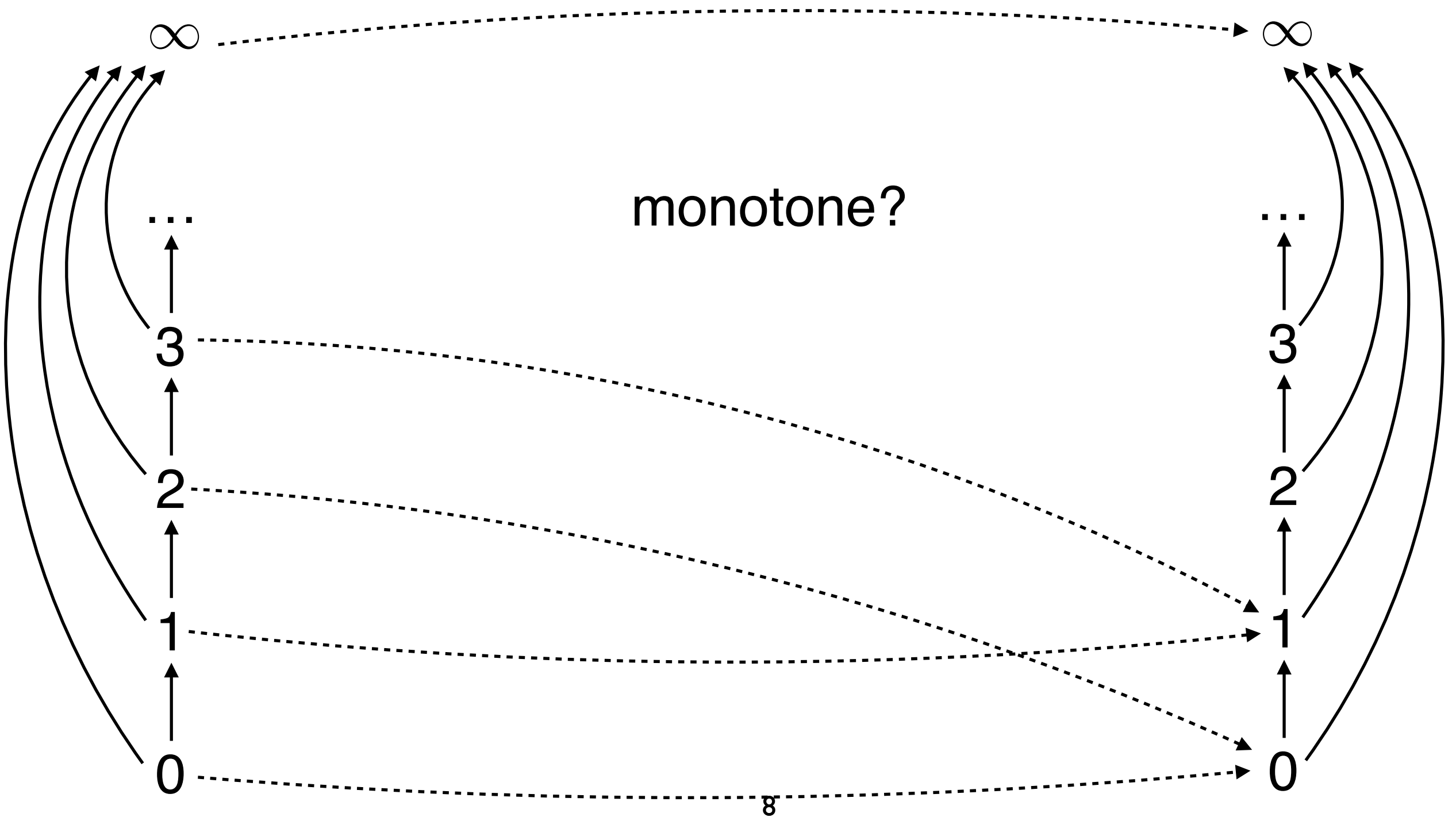


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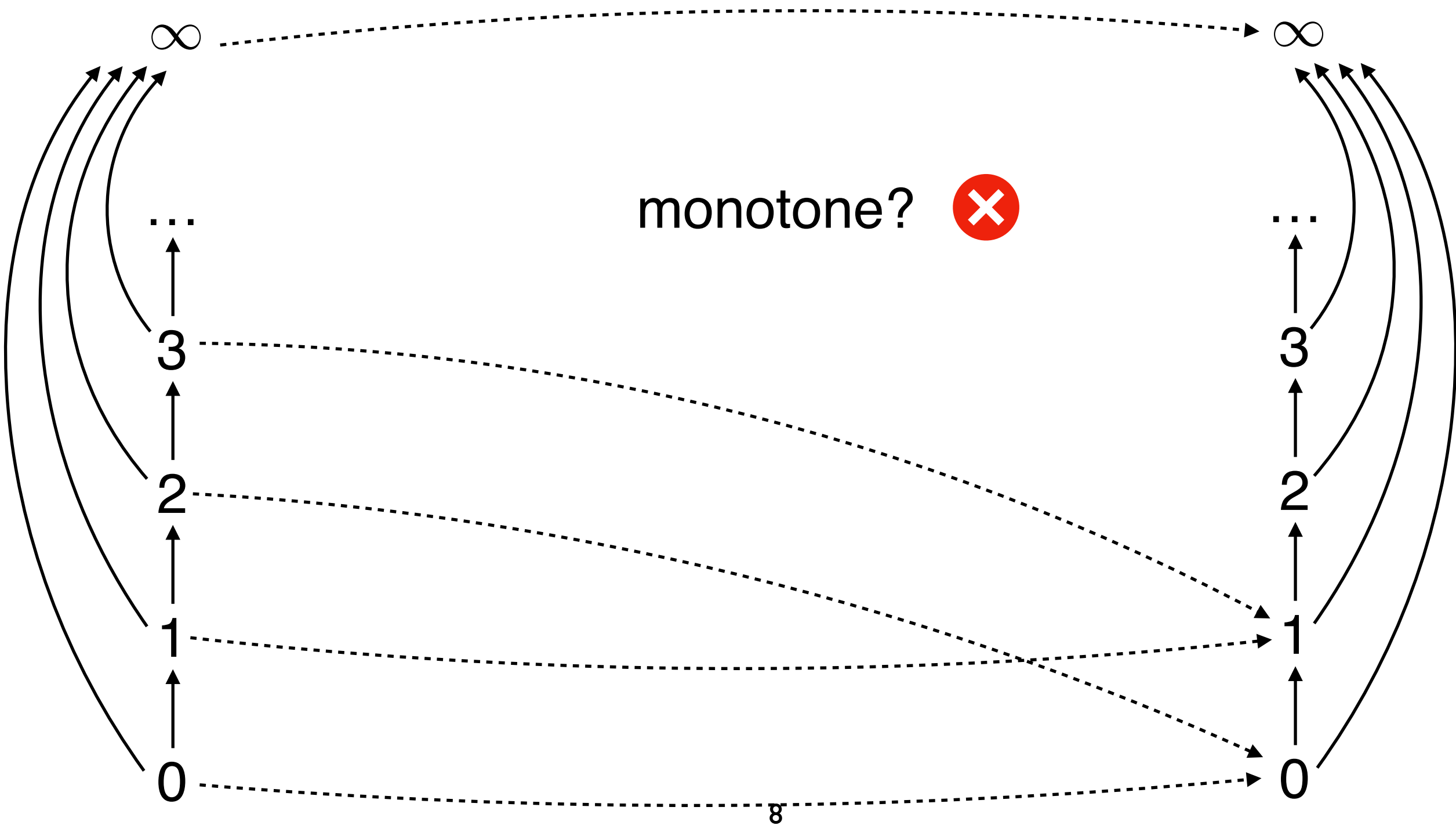


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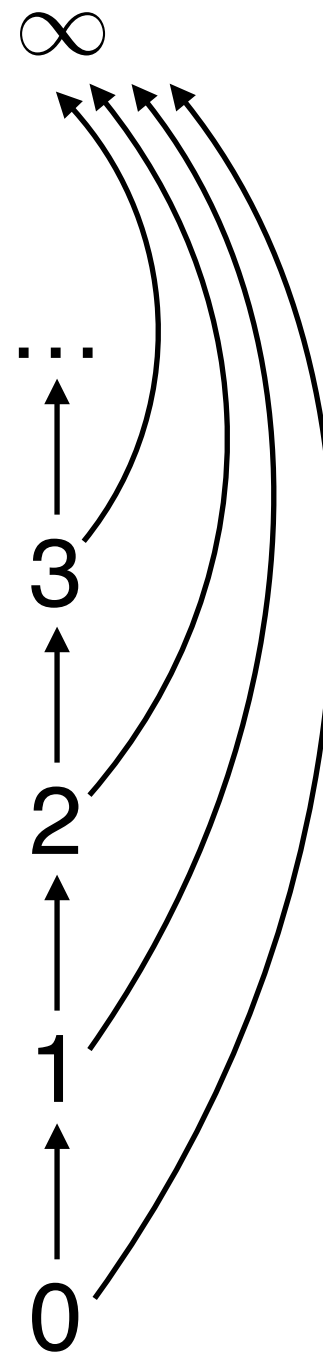
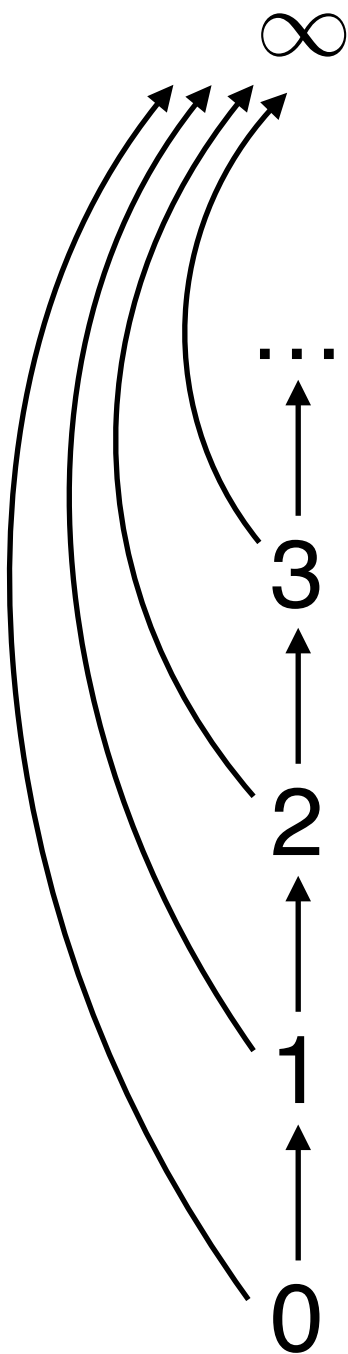


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$$(\mathbb{N} \cup \{\infty\}, \leq)$$

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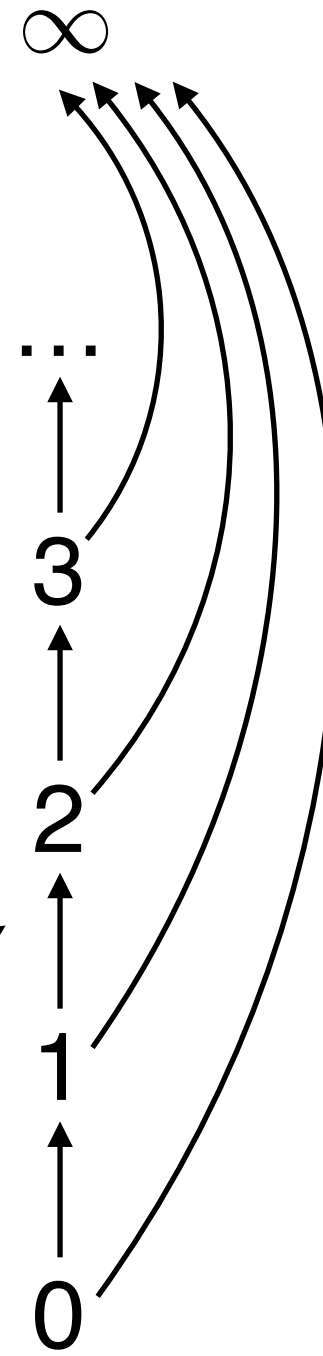
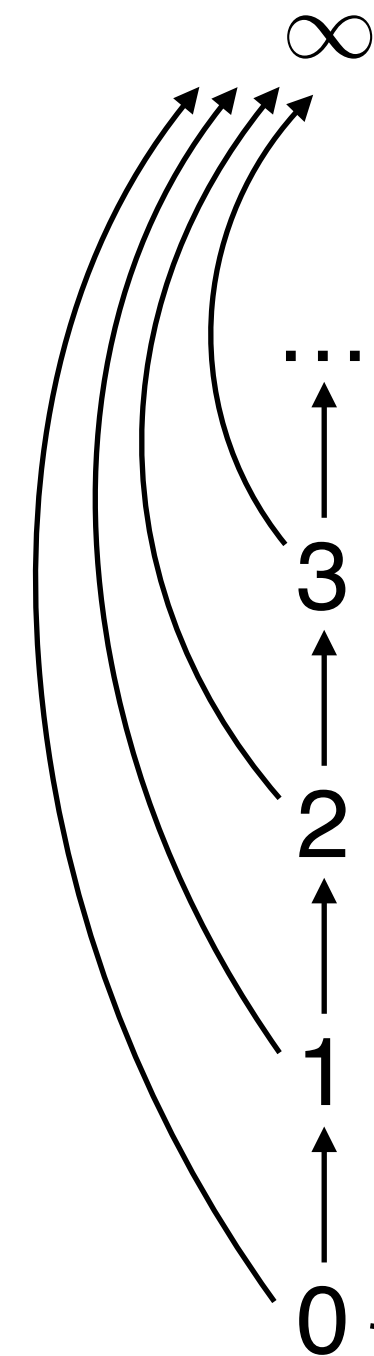


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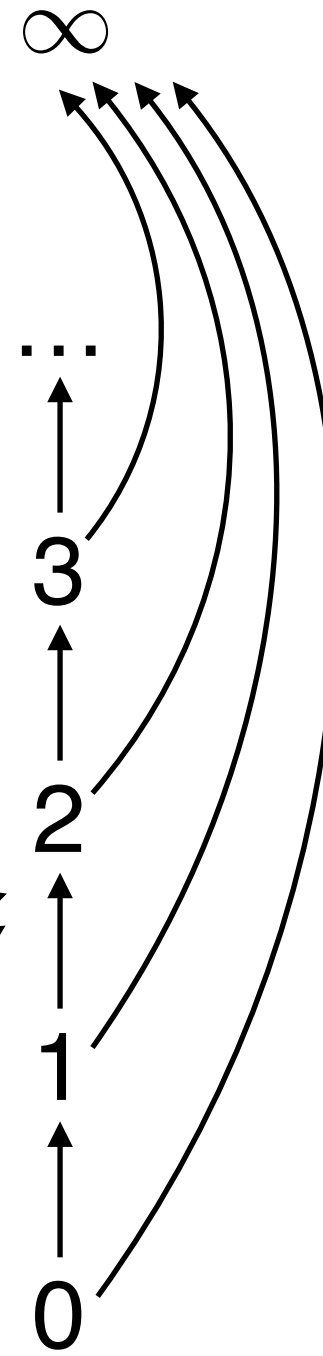
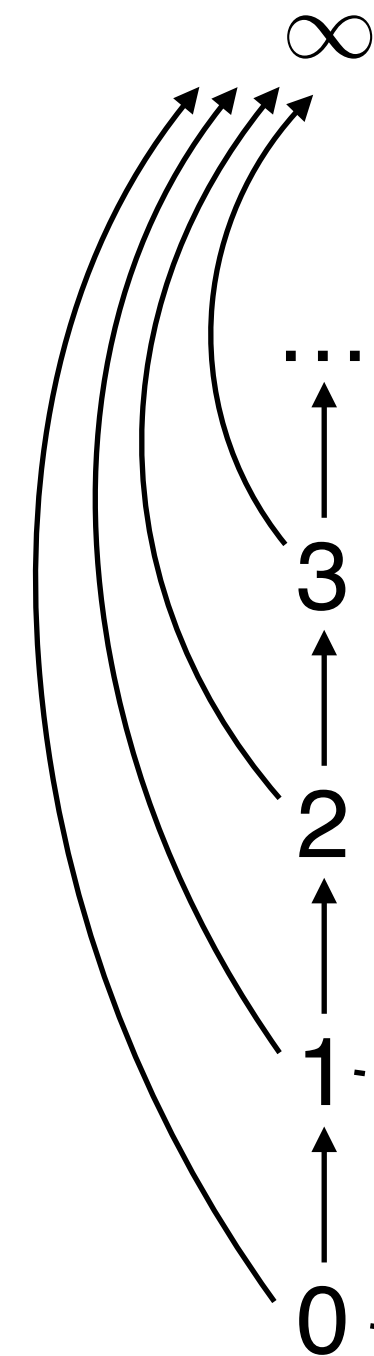


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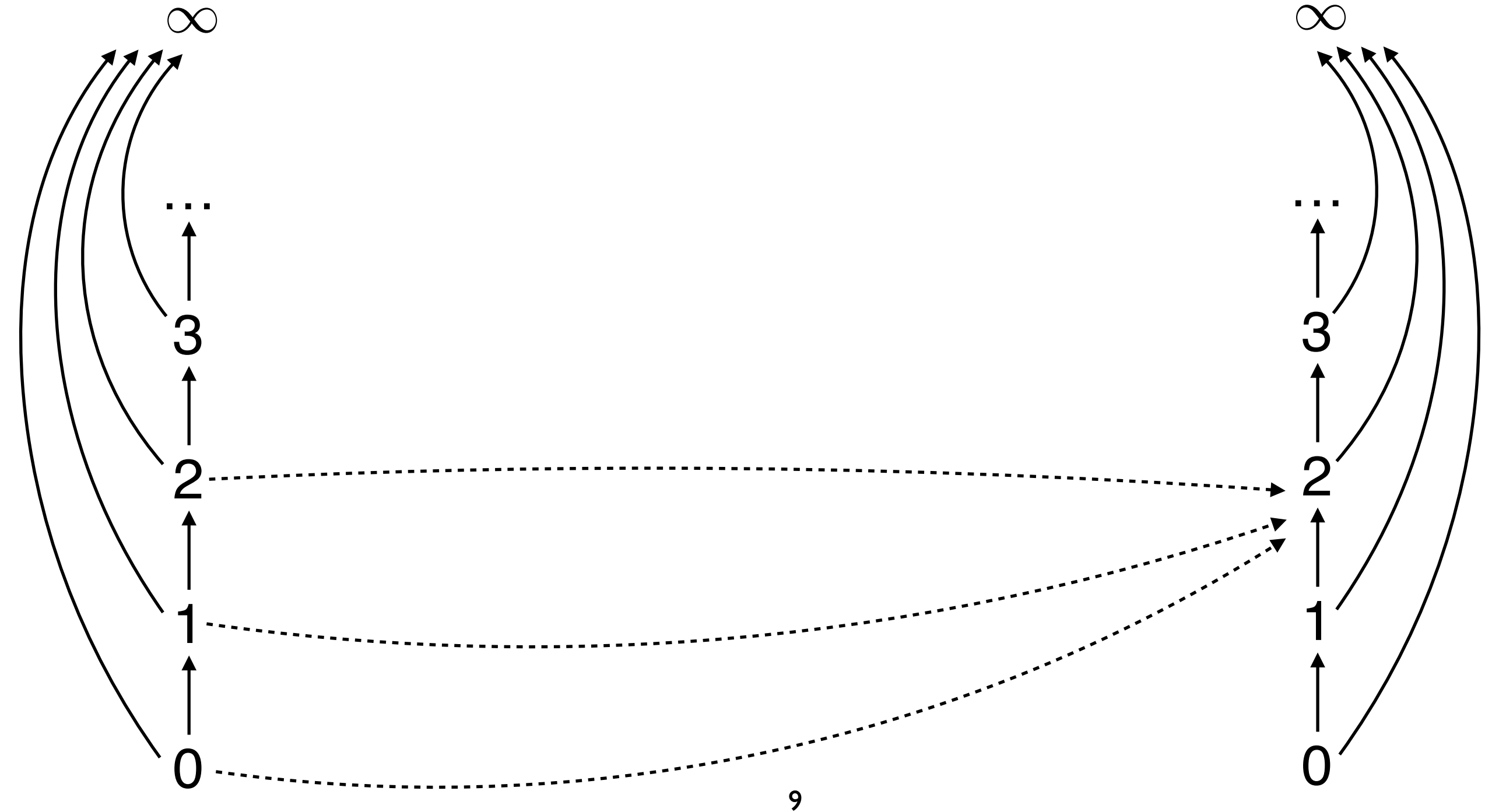


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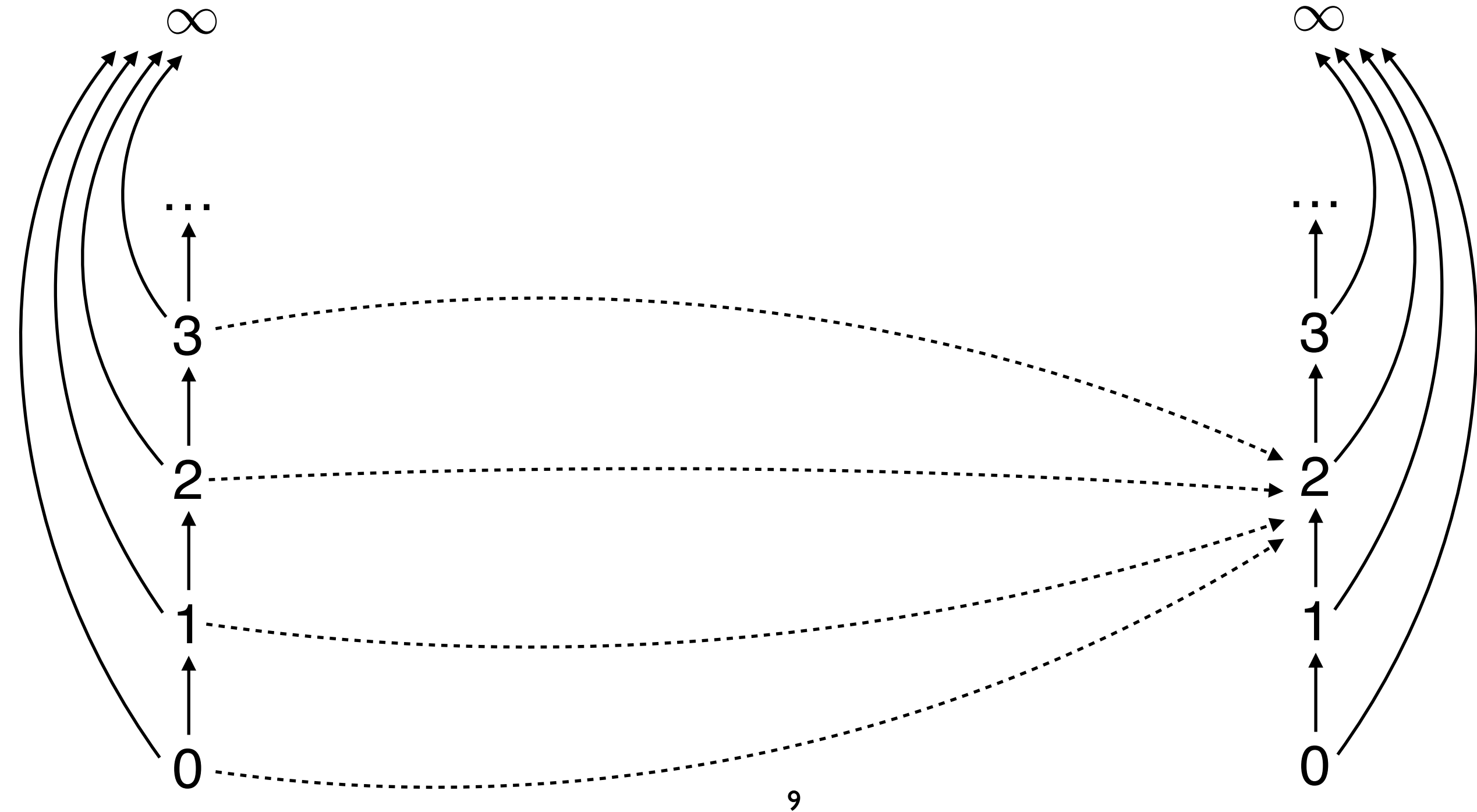


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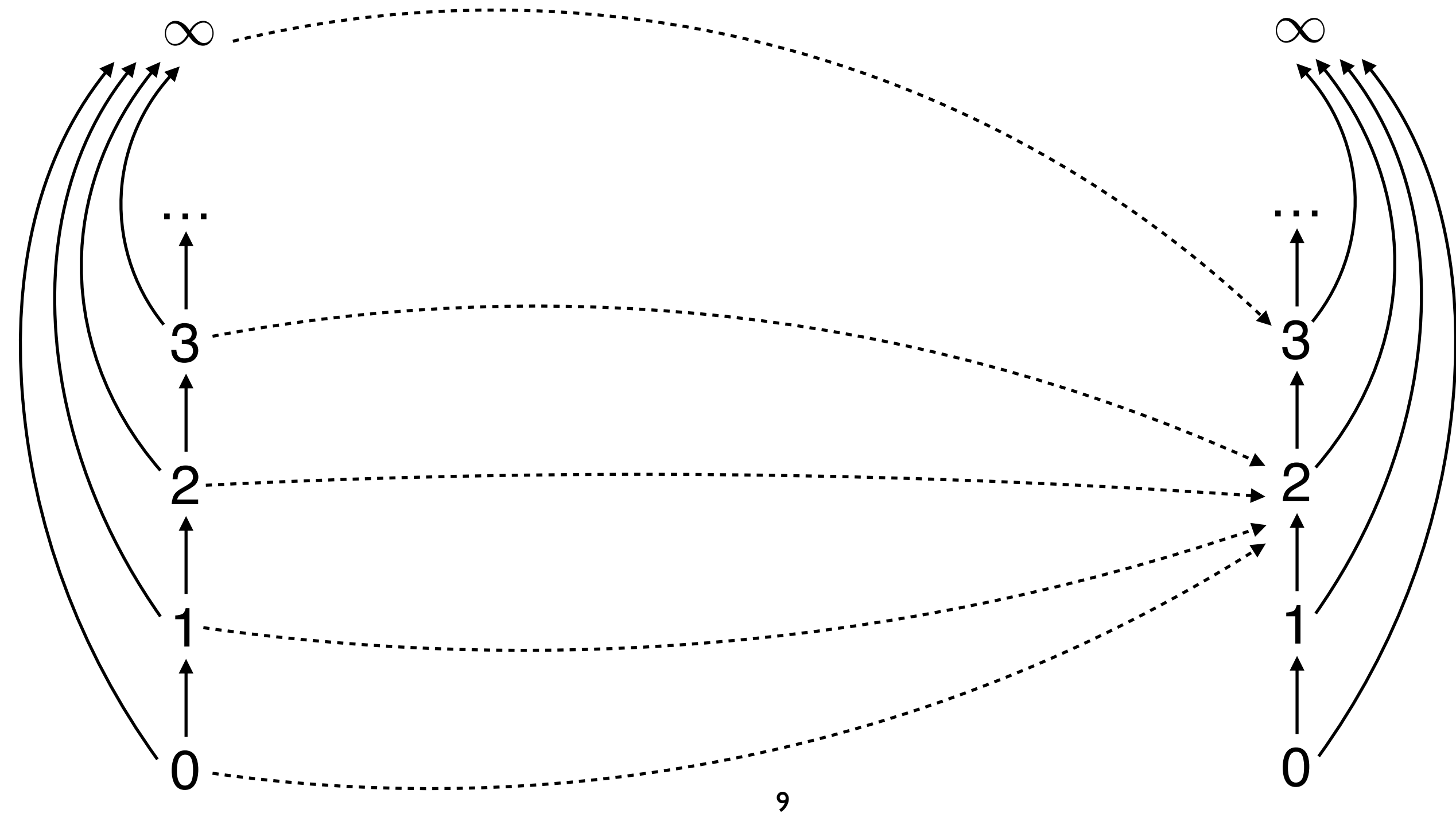


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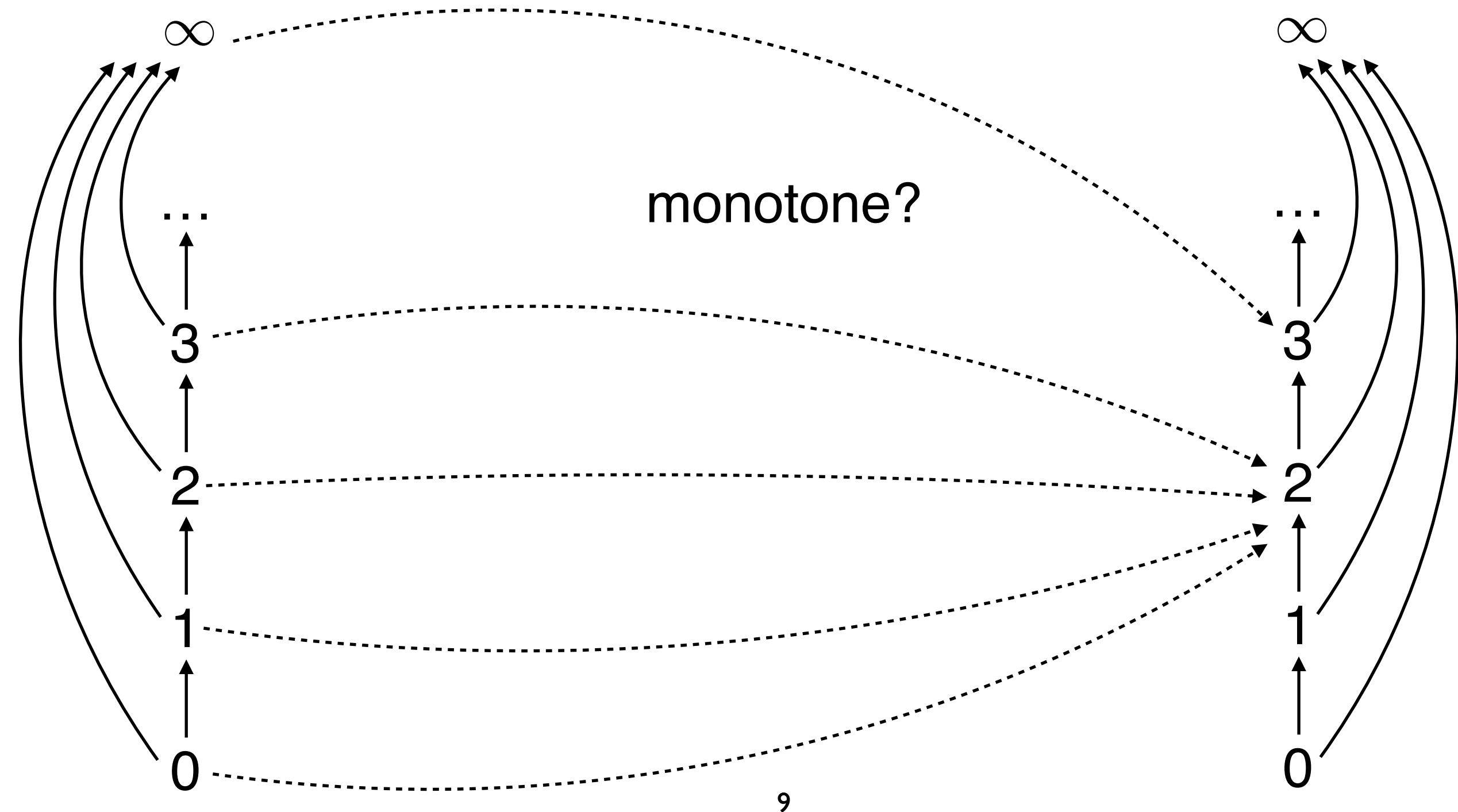


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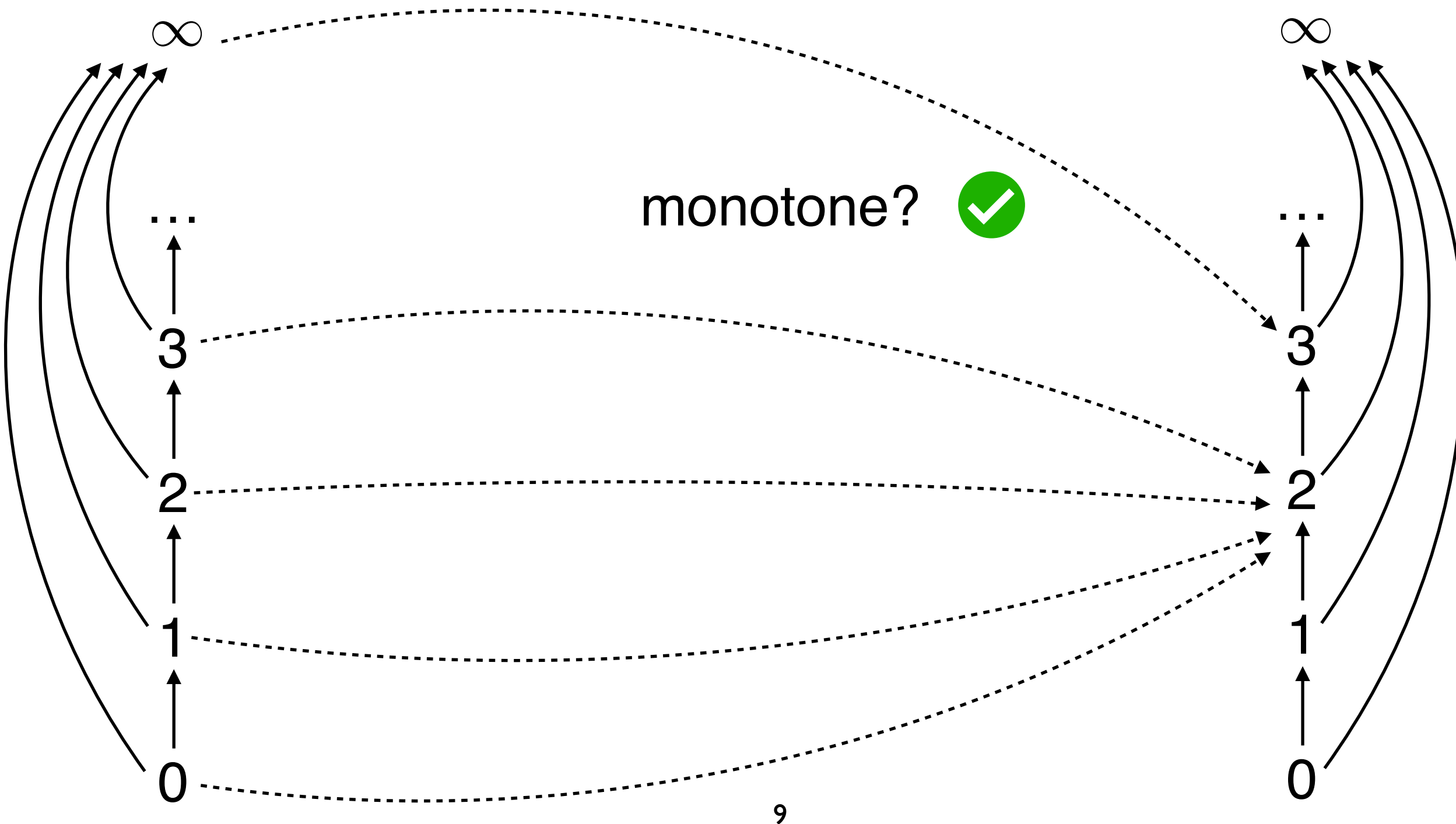


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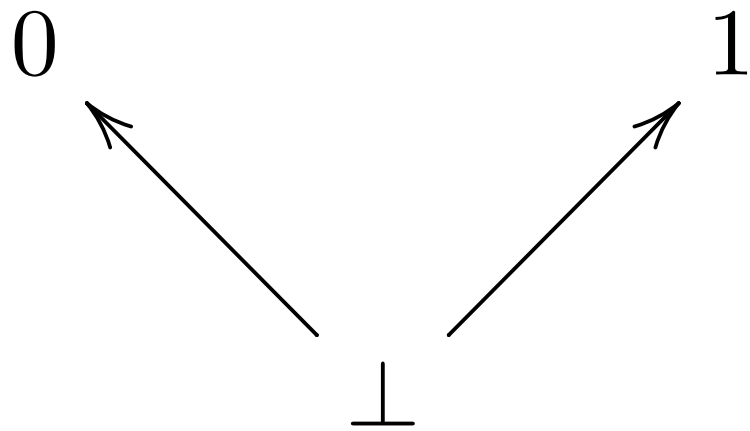
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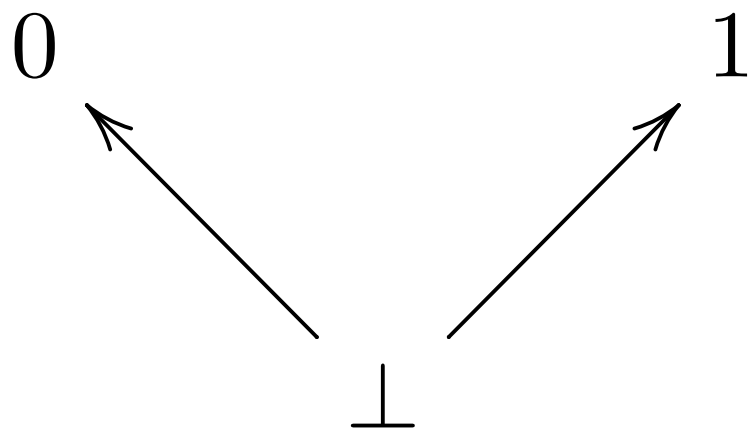
Exercise



D

$f : D \rightarrow D$

Exercise



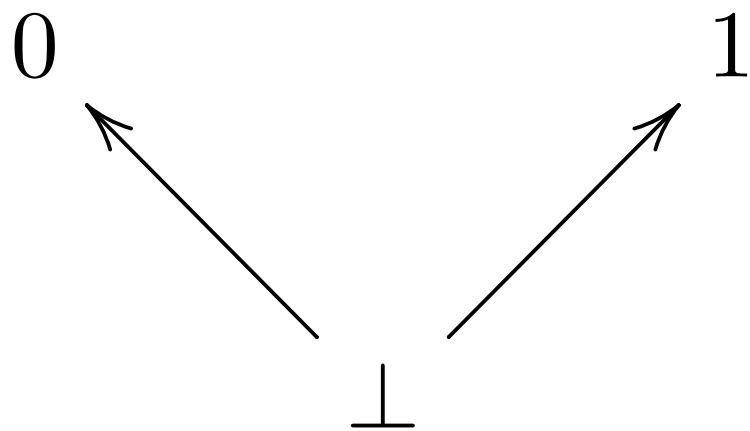
D

$$f : D \rightarrow D$$

$$f(\perp) = f(0) = 0$$

$$f(1) = 1$$

Exercise

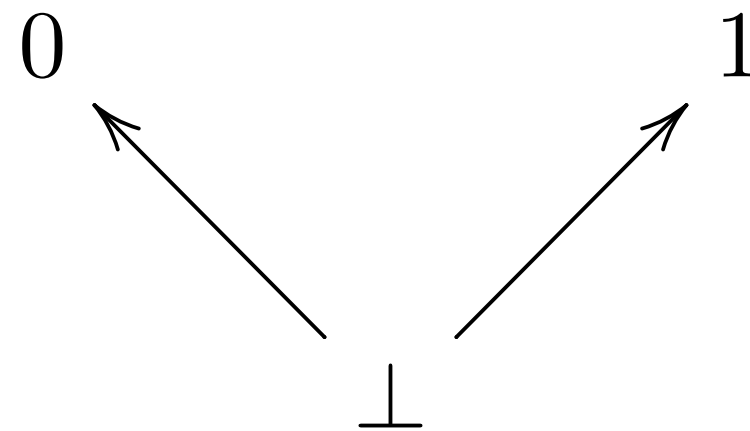


D

$$f : D \rightarrow D$$

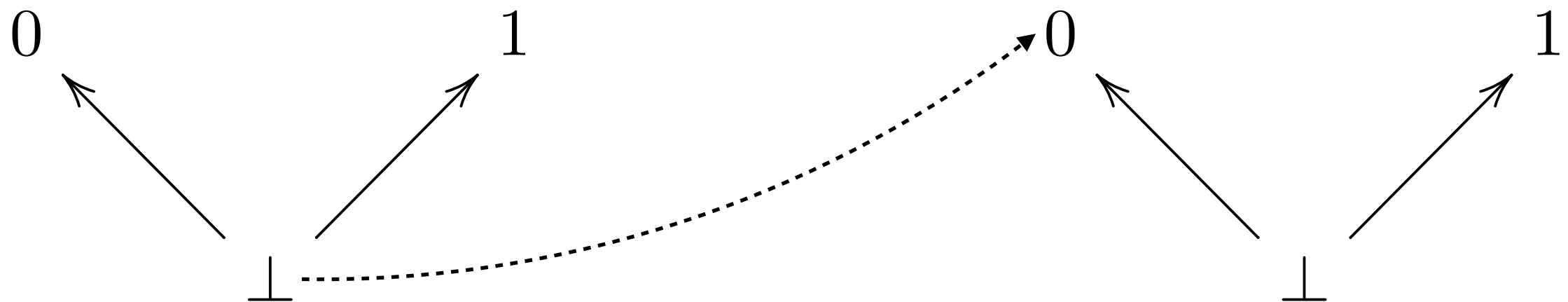
$$f(\perp) = f(0) = 0$$

$$f(1) = 1$$



D

Exercise



D

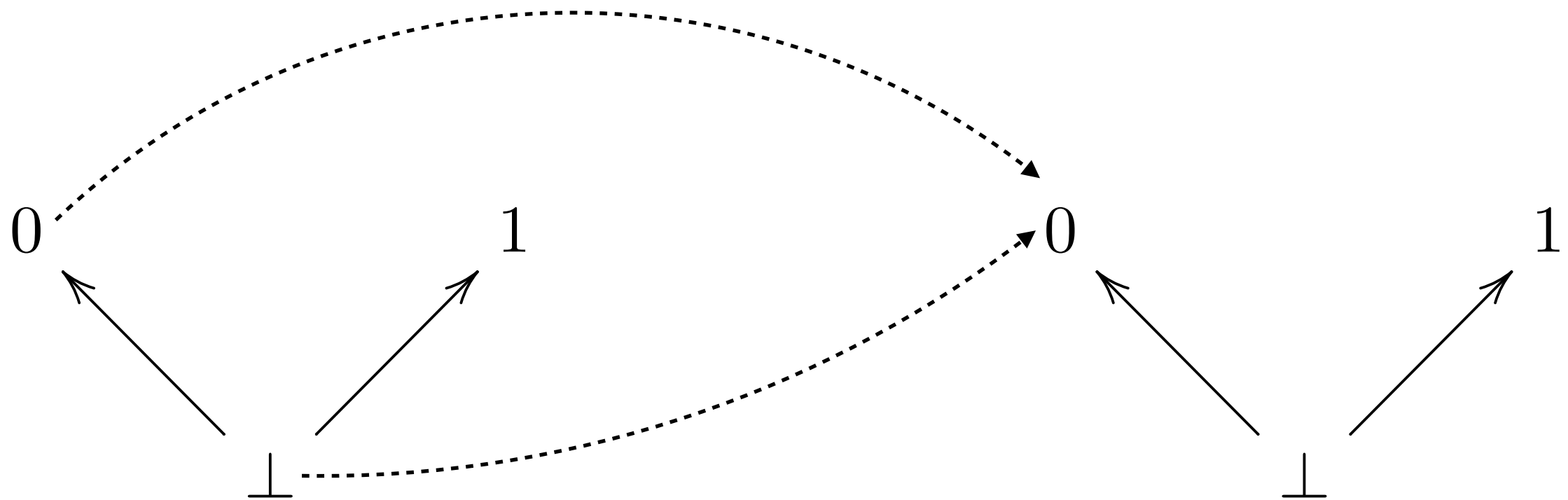
$f : D \rightarrow D$

D

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Exercise



D

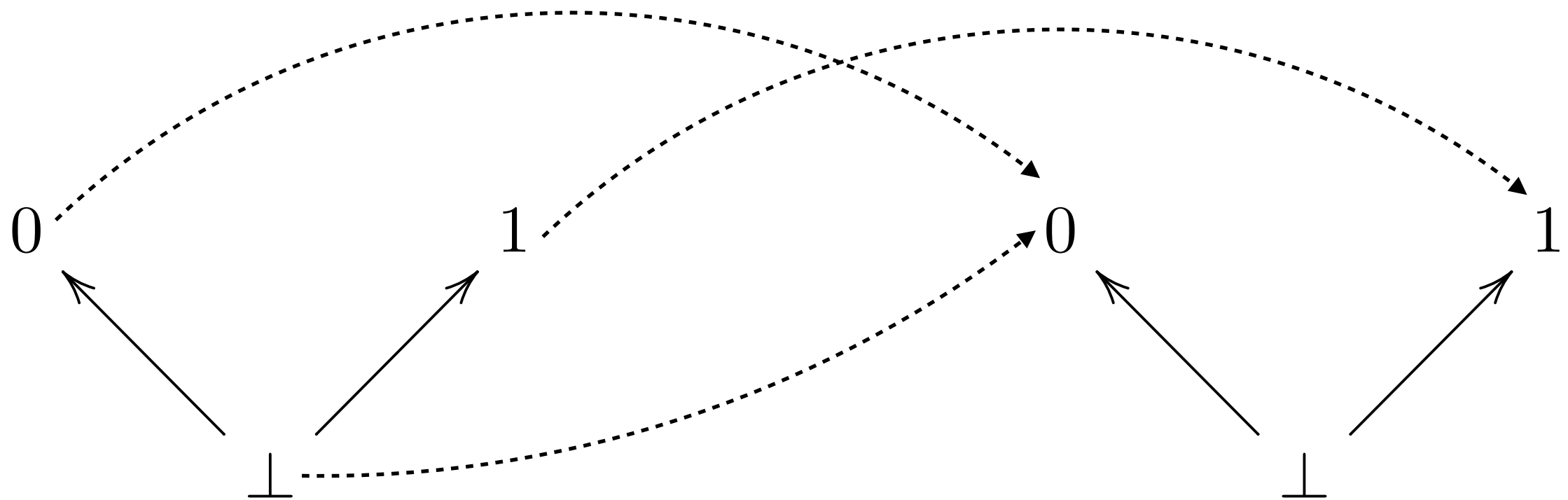
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D

Exercise



D

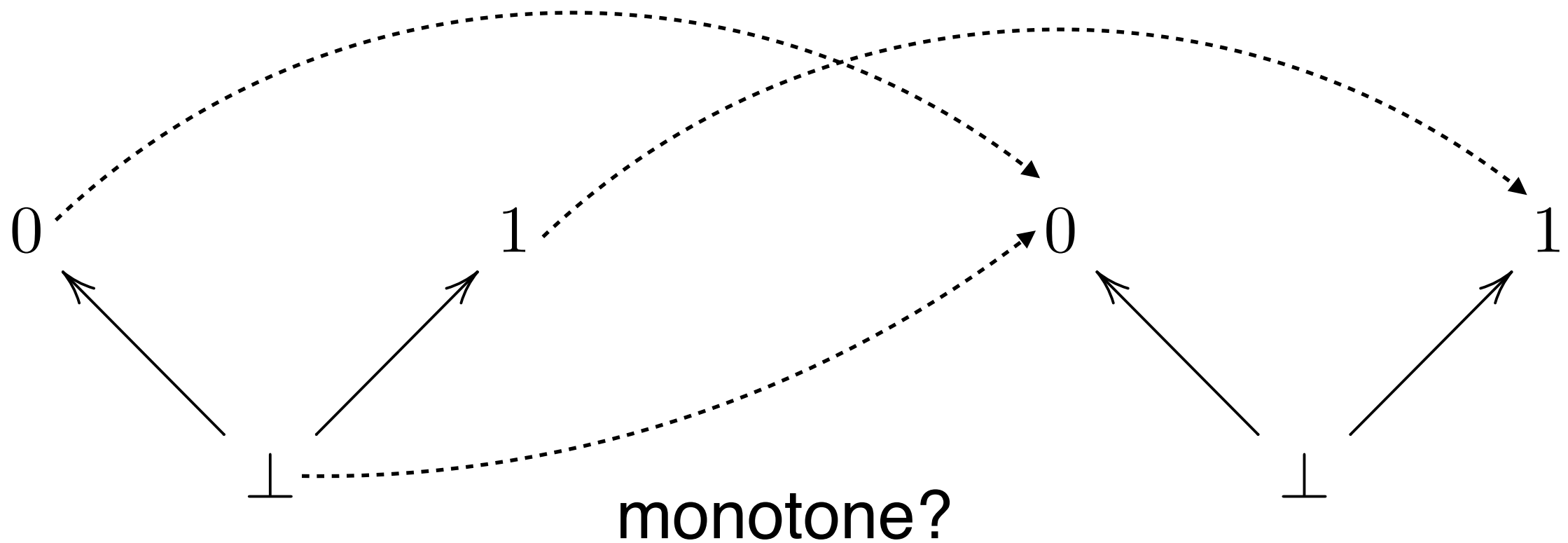
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Exercise



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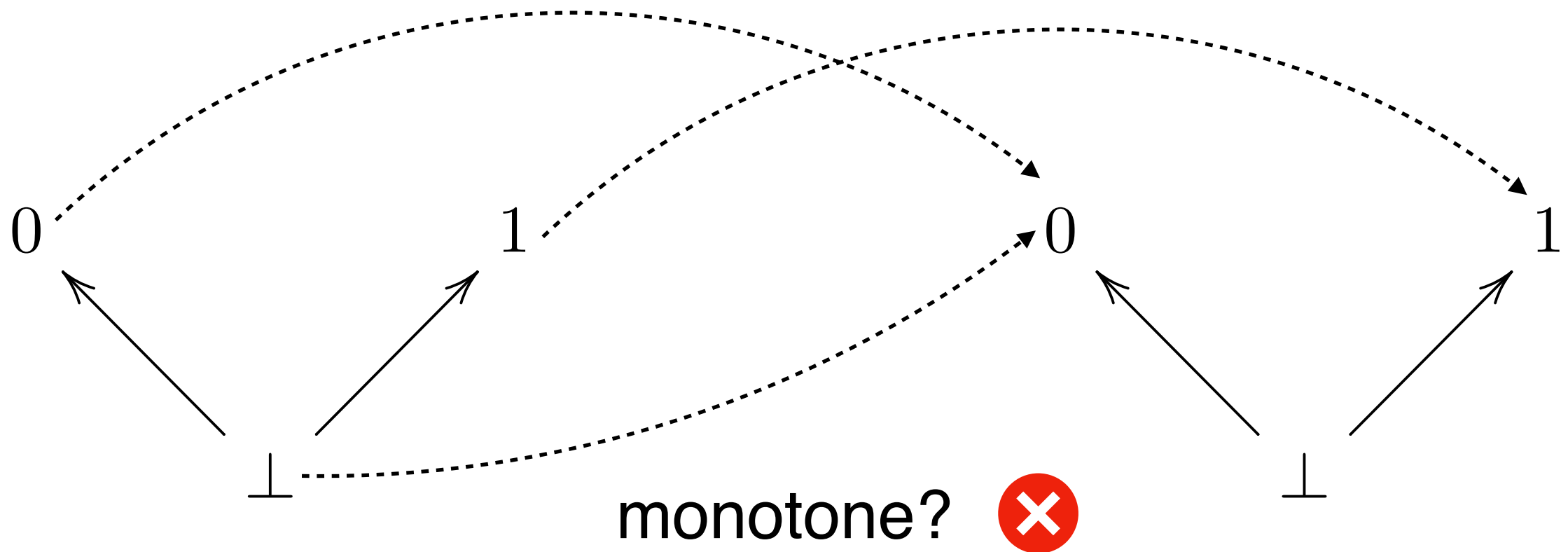
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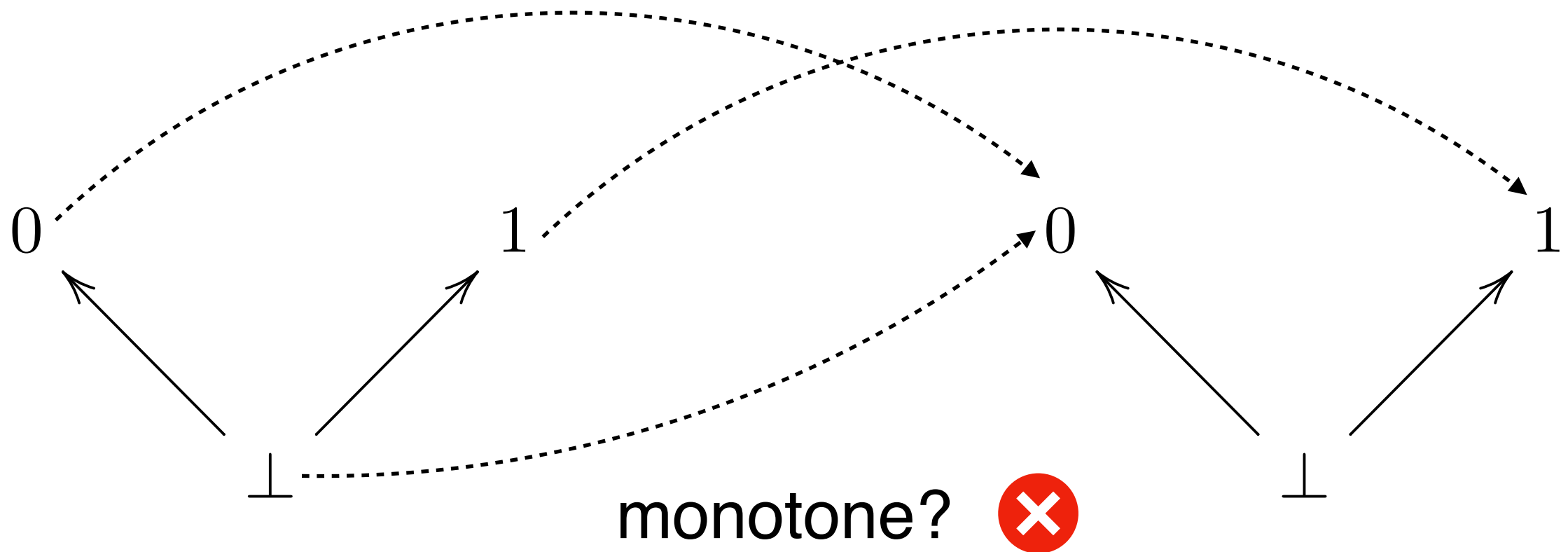
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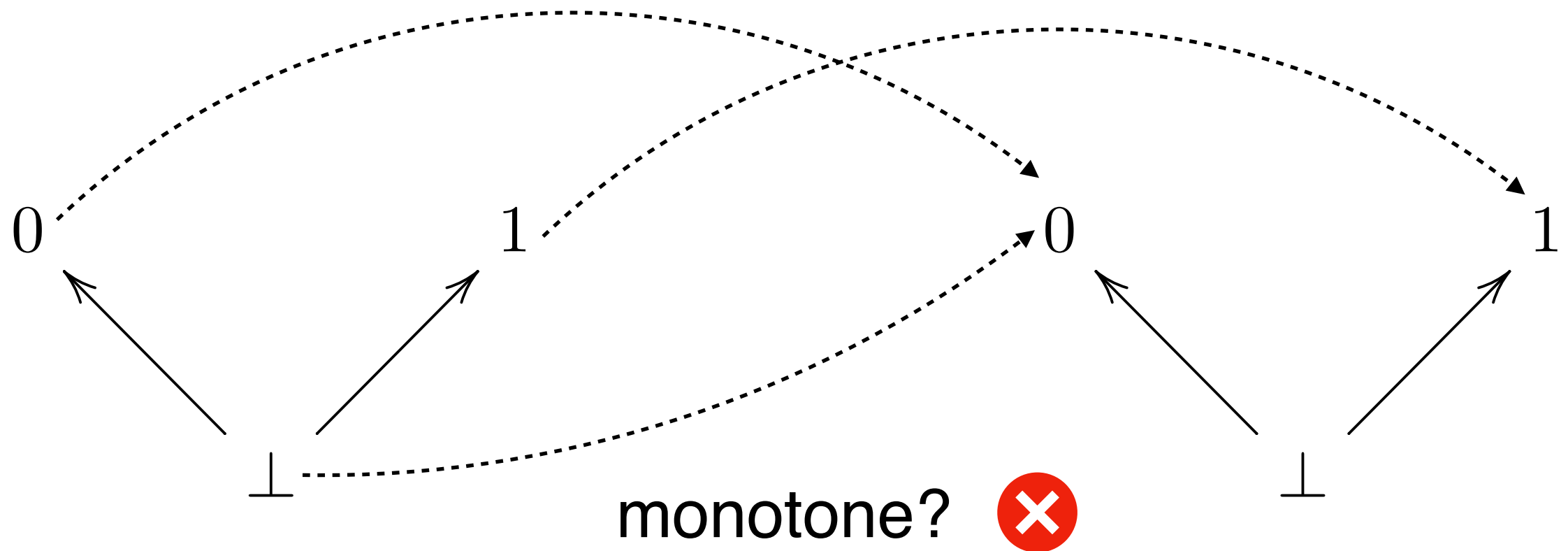
D

$$f(\perp) = f(0) = 0$$

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$$\perp \sqsubseteq 1$$

Exercise



D

$f : D \rightarrow D$

D

$$f(\perp) = f(0) = 0$$

$$f(1) = 1$$

$$\perp \sqsubseteq 1$$

$$f(\perp) = 0 \not\sqsubseteq 1 = f(1)$$

Exercise

$$(\wp(\mathbb{N}), \subseteq) \quad f(S) = \{ m \in \mathbb{N} \mid \exists n \in S, m \leq n \} \quad (\wp(\mathbb{N}), \subseteq)$$

Exercise

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monotone?

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$$(\wp(\mathbb{N}), \subseteq) \quad f(S) = \{ m \in \mathbb{N} \mid \forall n \in S, n < m \} \quad (\wp(\mathbb{N}), \subseteq)$$

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monotone? 

Composition

TH. Any composition of monotone functions is monotone

$(D, \sqsubseteq_D)$	PO	$f : D \rightarrow E$	monotone
$(E, \sqsubseteq_E)$	PO	$g : E \rightarrow F$	monotone
$(F, \sqsubseteq_F)$	PO		

Composition

TH. Any composition of monotone functions is monotone

$$\begin{array}{llll} (D, \sqsubseteq_D) & \text{PO} & f : D \rightarrow E & \text{monotone} \\ (E, \sqsubseteq_E) & \text{PO} & g : E \rightarrow F & \text{monotone} \\ (F, \sqsubseteq_F) & \text{PO} & & \end{array} \quad \Rightarrow \quad \begin{array}{l} h = g \circ f : D \rightarrow F \\ \text{monotone} \end{array}$$

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proof. we need to prove $\forall x, y \in D. x \sqsubseteq_D y \Rightarrow h(x) \sqsubseteq_F h(y)$

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take $x \sqsubseteq_D y$

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we want to prove $h(x) \sqsubseteq_F h(y)$

then $f(x) \sqsubseteq_E f(y)$ because f is monotone

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$$\begin{array}{c} = \\ h(x) \end{array}$$

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$$\begin{array}{ccc} = & & = \\ h(x) & & h(y) \end{array}$$

Continuous functions

Continuous function

$(D, \sqsubseteq_D)$ CPO $(E, \sqsubseteq_E)$ CPO $f : D \rightarrow E$ monotone

f is continuous if

Continuous function

$(D, \sqsubseteq_D)$ CPO $(E, \sqsubseteq_E)$ CPO $f : D \rightarrow E$ monotone

f is **continuous** if $\forall \{d_i\}_{i \in \mathbb{N}}. f \left(\bigsqcup_{i \in \mathbb{N}} d_i \right) = \bigsqcup_{i \in \mathbb{N}} f(d_i)$

Continuous function

$(D, \sqsubseteq_D)$ CPO $(E, \sqsubseteq_E)$ CPO $f : D \rightarrow E$ monotone

f is **continuous** if $\forall \{d_i\}_{i \in \mathbb{N}} \text{ chain} \cdot f \left(\bigsqcup_{i \in \mathbb{N}} d_i \right) = \bigsqcup_{i \in \mathbb{N}} f(d_i)$

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limit in D

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$\text{limit in } D$ $\text{limit in } E$

Continuous function

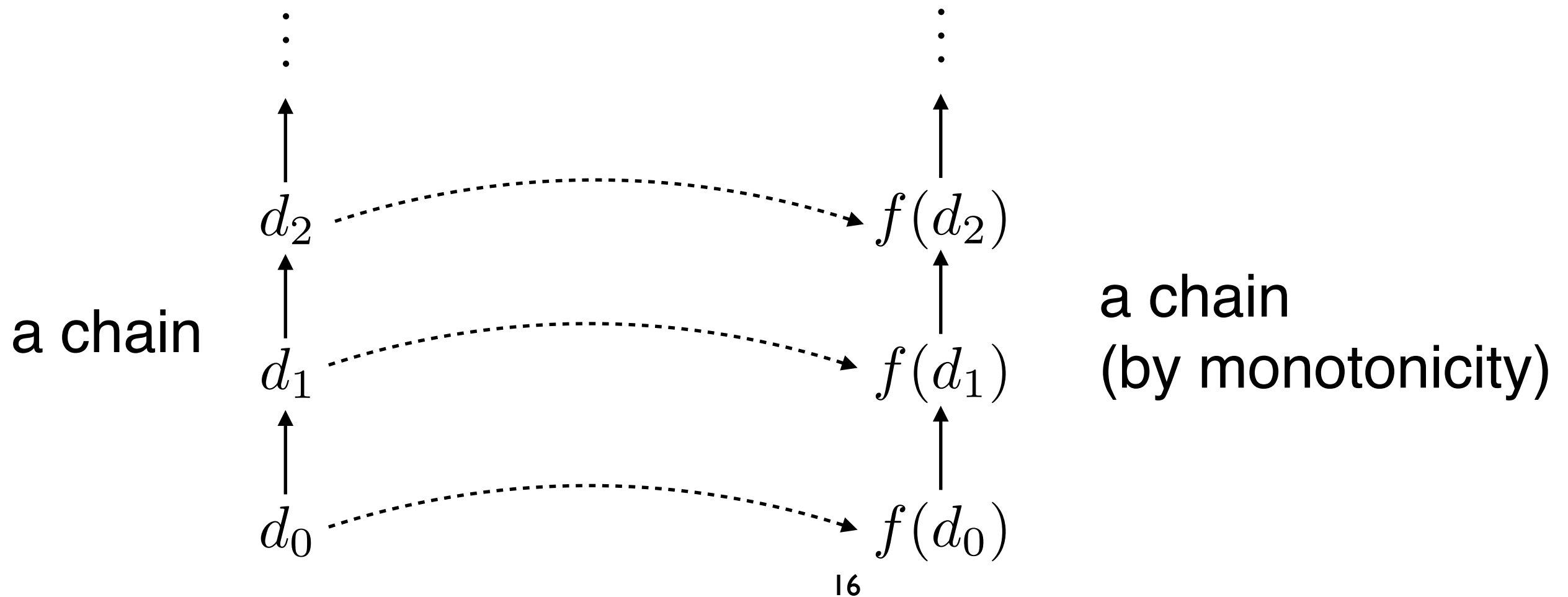
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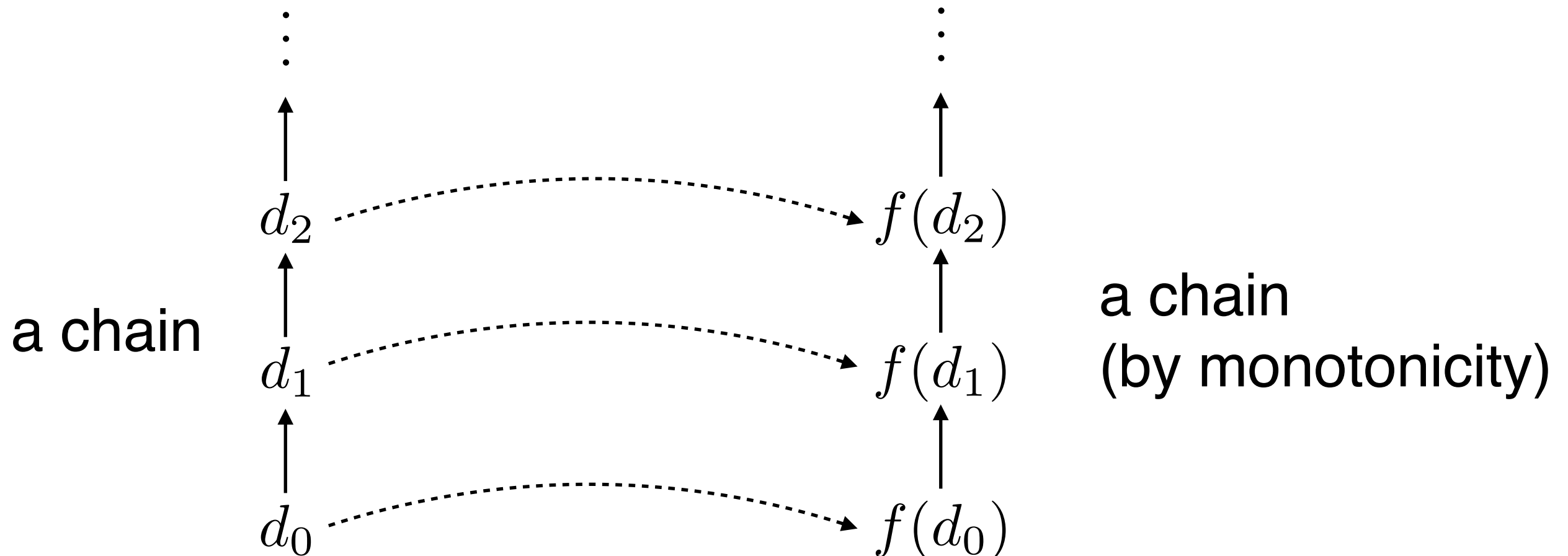
Continuous = limit preserving

Continuity illustrated



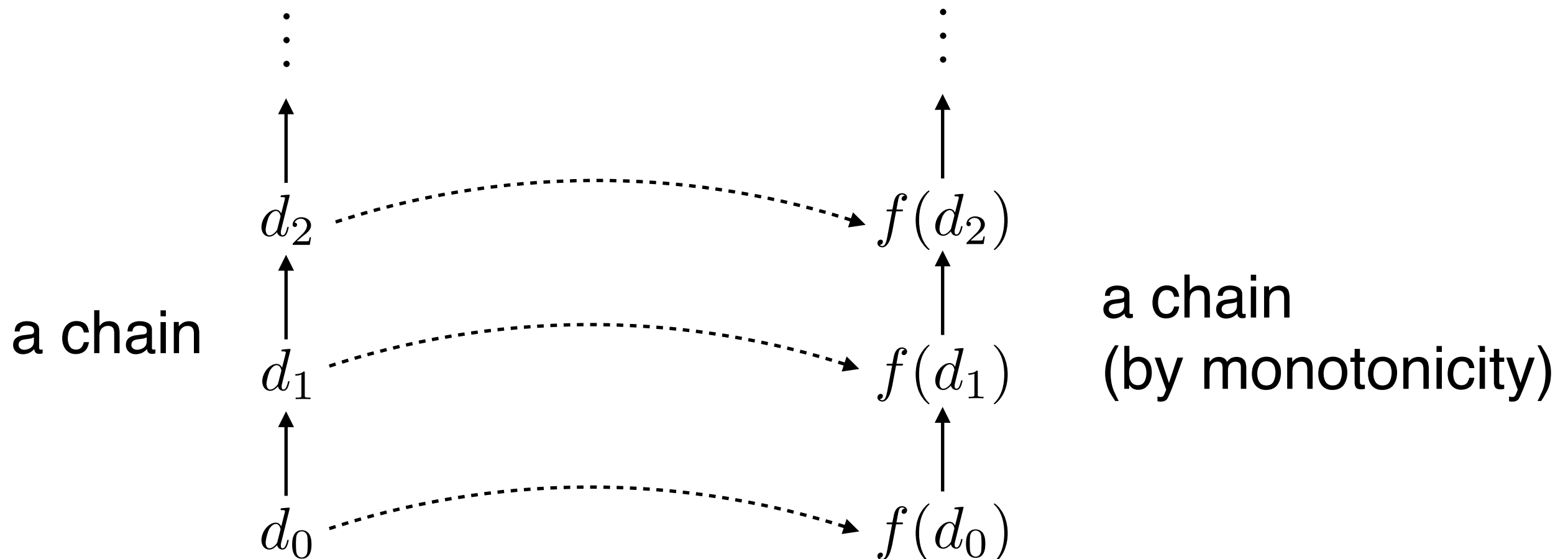
Continuity illustrated

$$\text{lub } \bigsqcup_{i \in \mathbb{N}} d_i$$



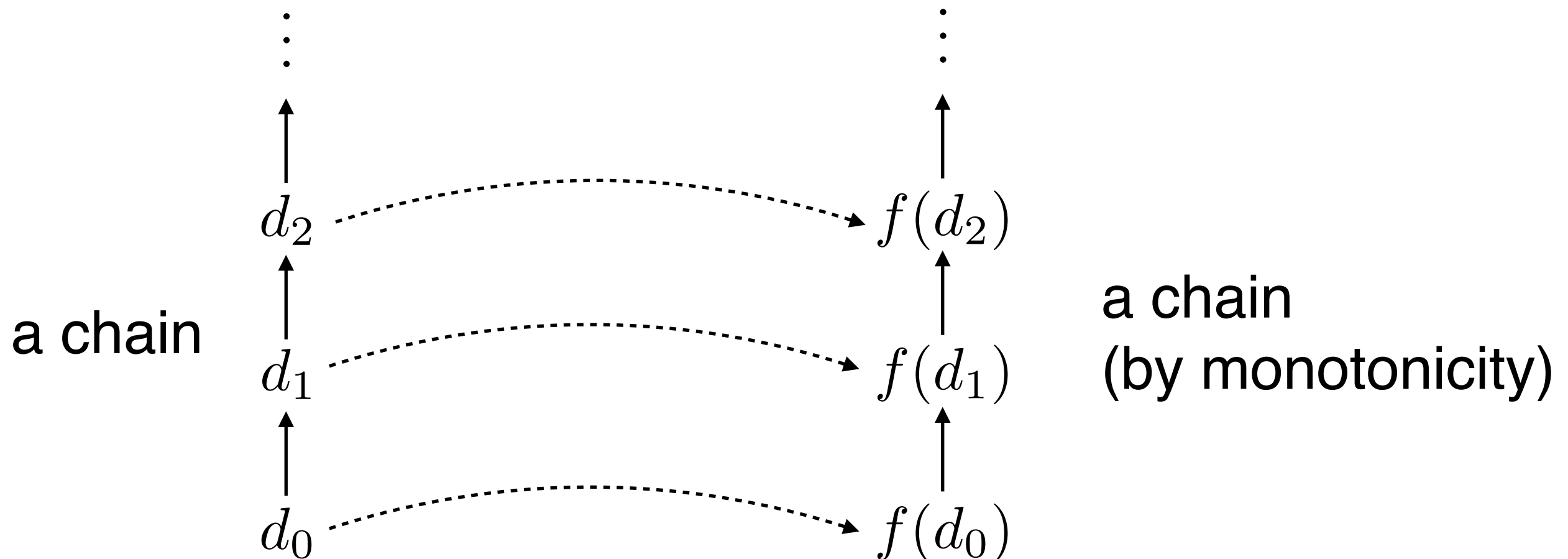
Continuity illustrated

lub $\bigsqcup_{i \in \mathbb{N}} d_i \xrightarrow{\quad} f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right)$



Continuity illustrated

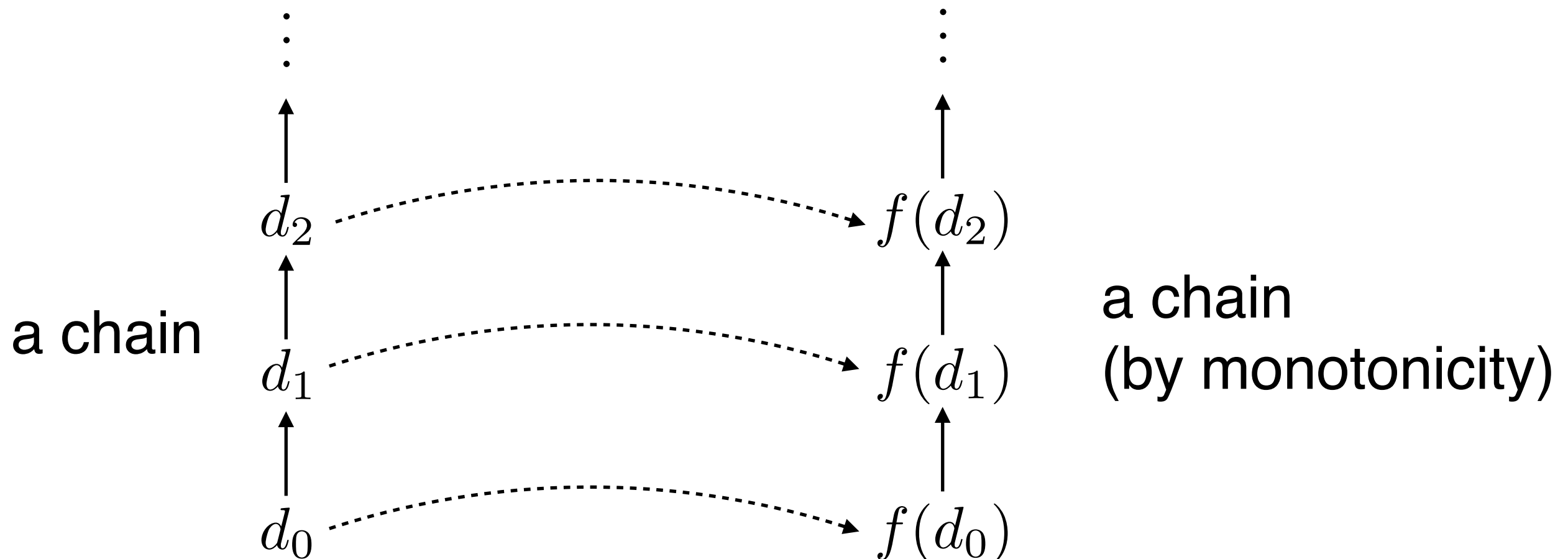
lub $\bigsqcup_{i \in \mathbb{N}} d_i \xrightarrow{\quad} f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right)$ u.b.
(by monotonicity)



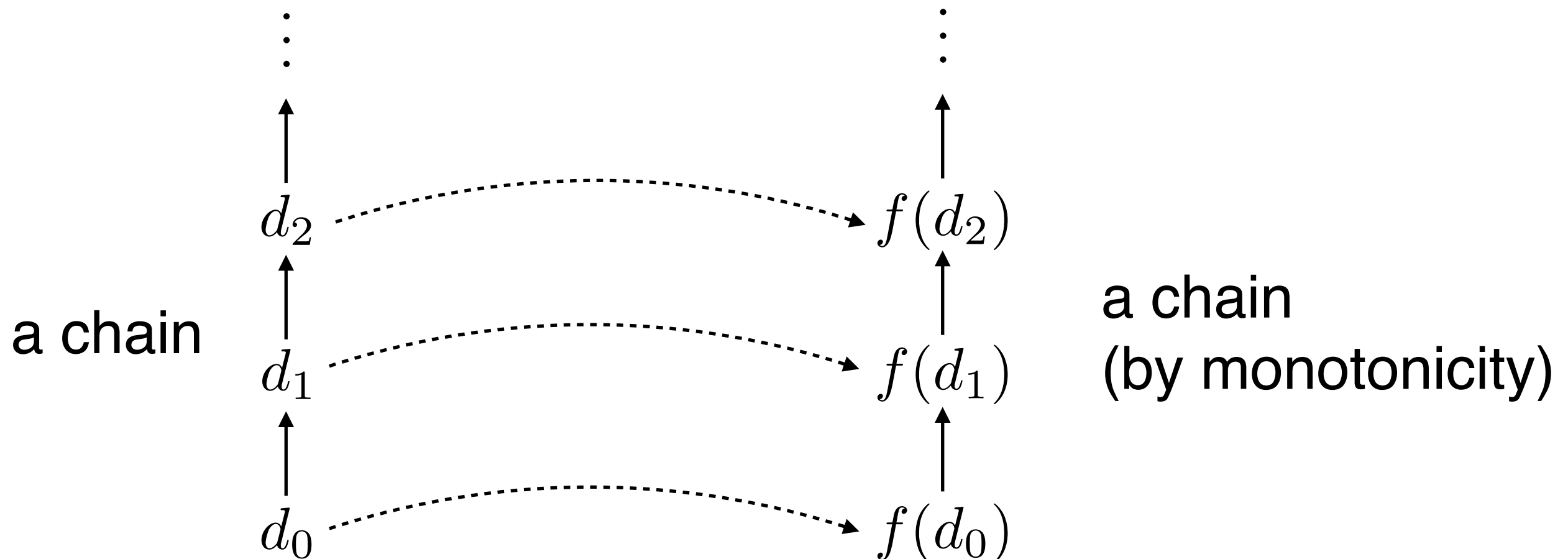
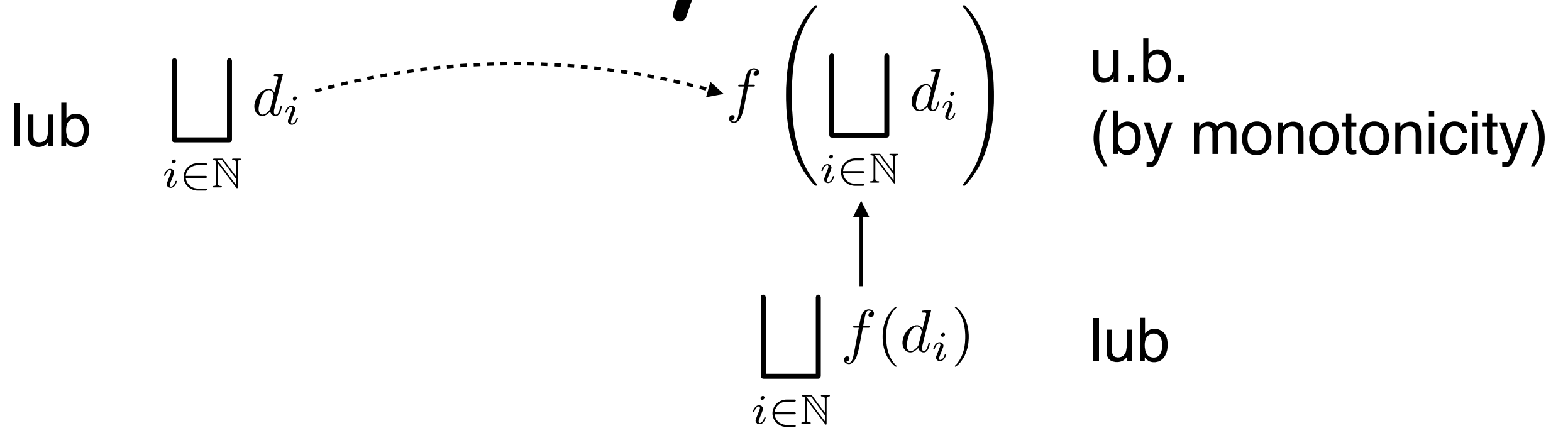
Continuity illustrated

$$\text{lub} \quad \bigsqcup_{i \in \mathbb{N}} d_i \quad \xrightarrow{\quad} \quad f \left(\bigsqcup_{i \in \mathbb{N}} d_i \right) \quad \text{u.b.} \\ \text{(by monotonicity)}$$

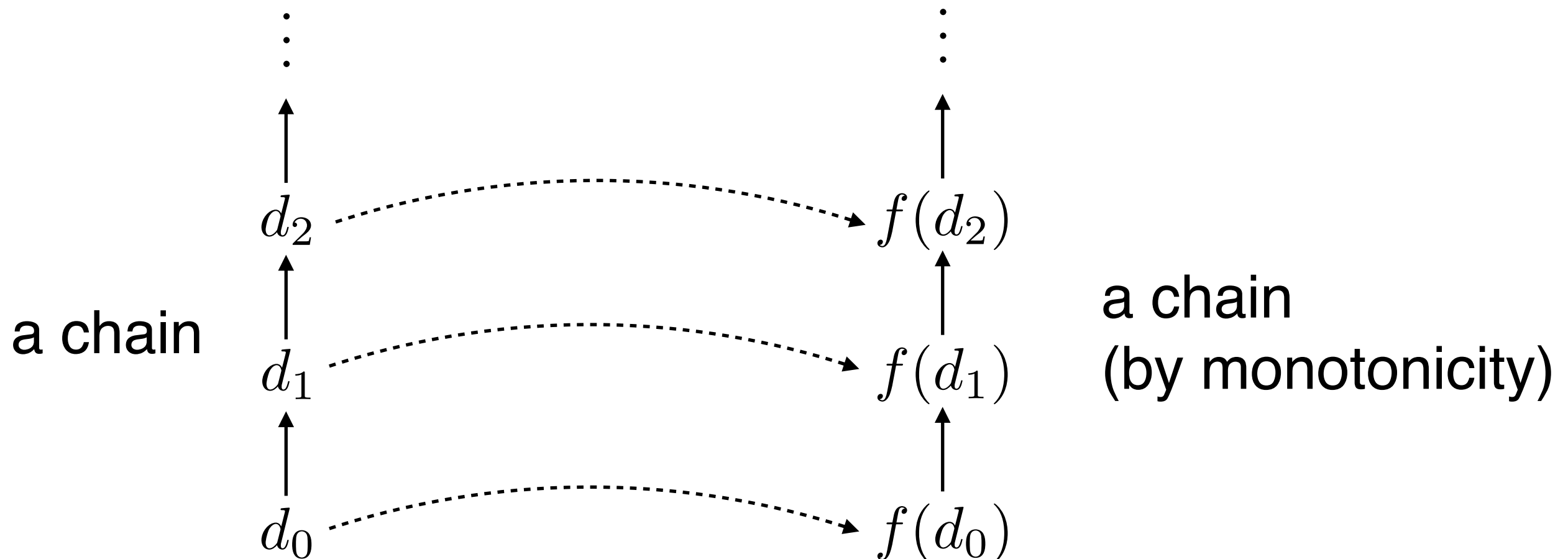
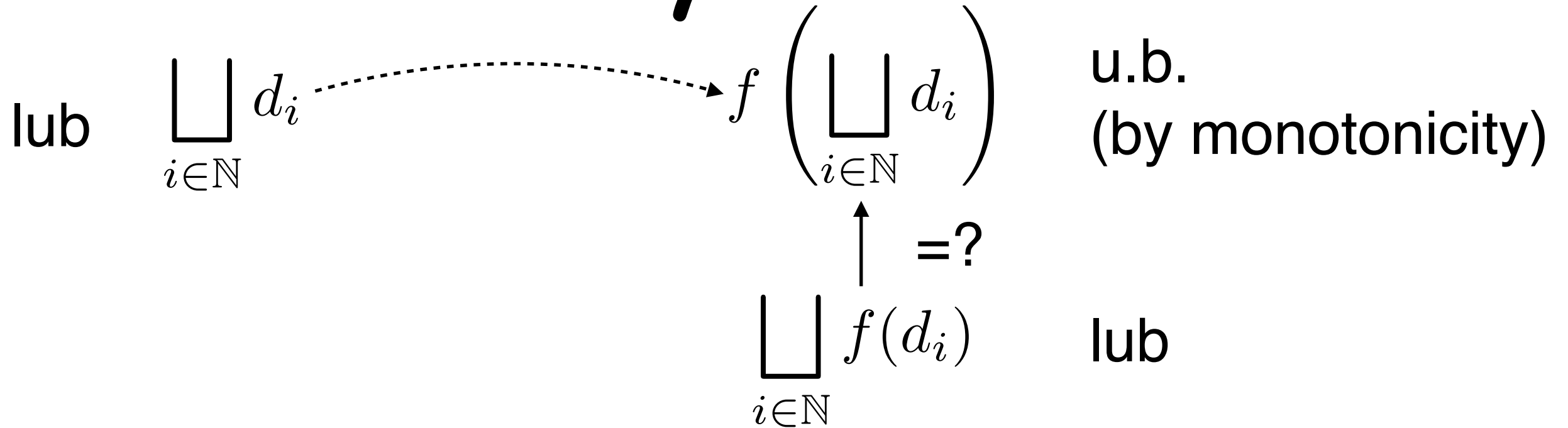
$$\bigsqcup_{i \in \mathbb{N}} f(d_i) \quad \text{lub}$$



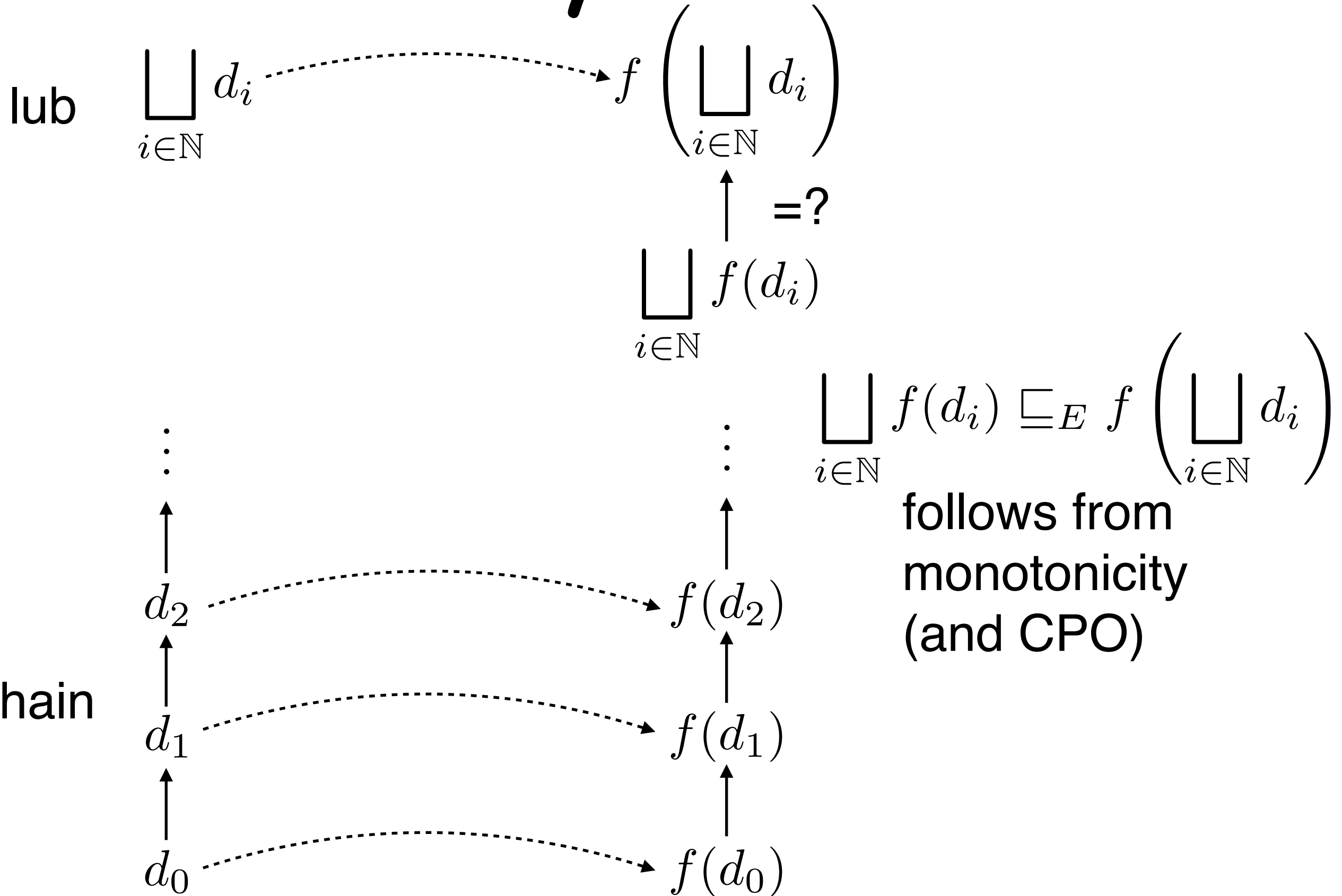
Continuity illustrated



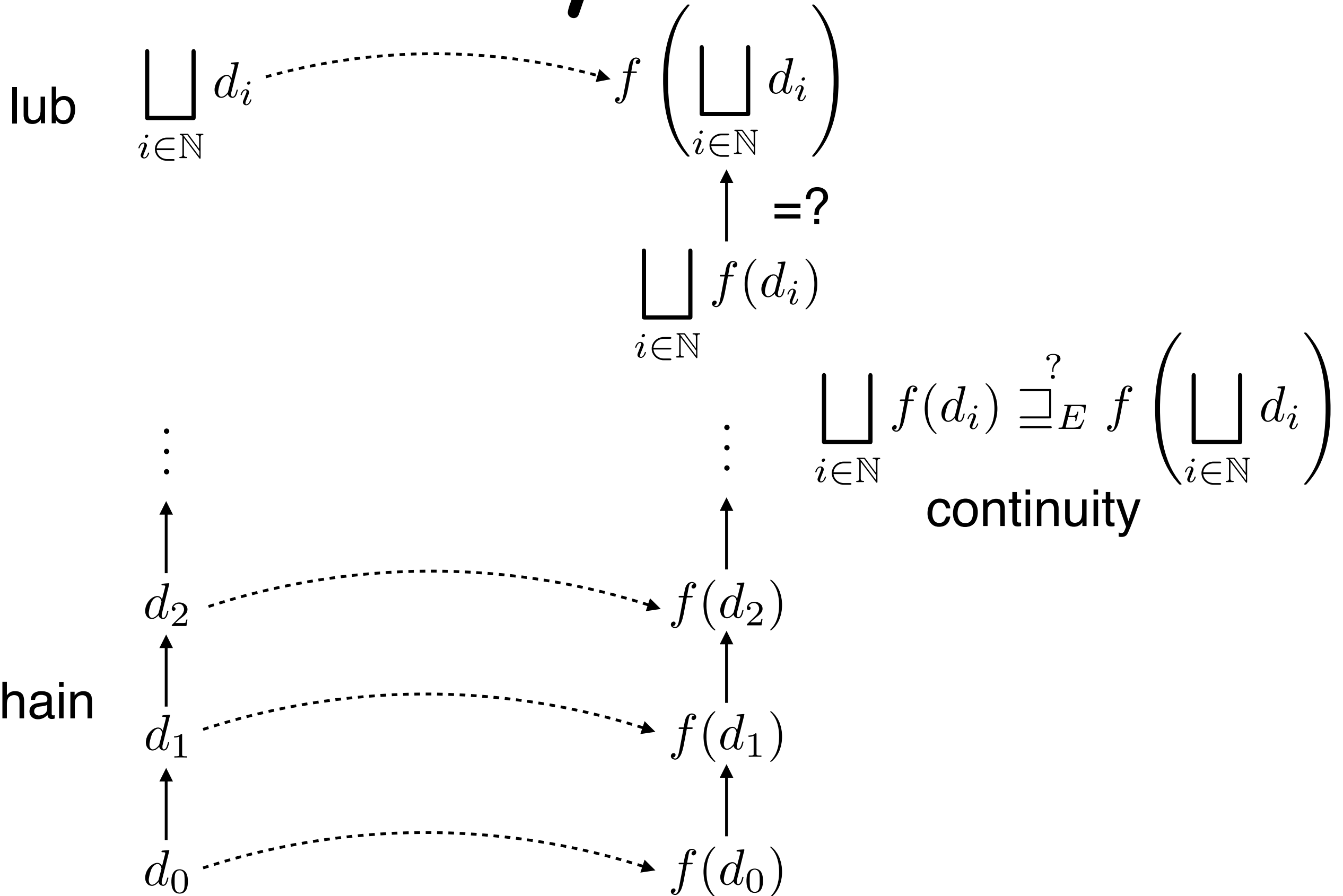
Continuity illustrated



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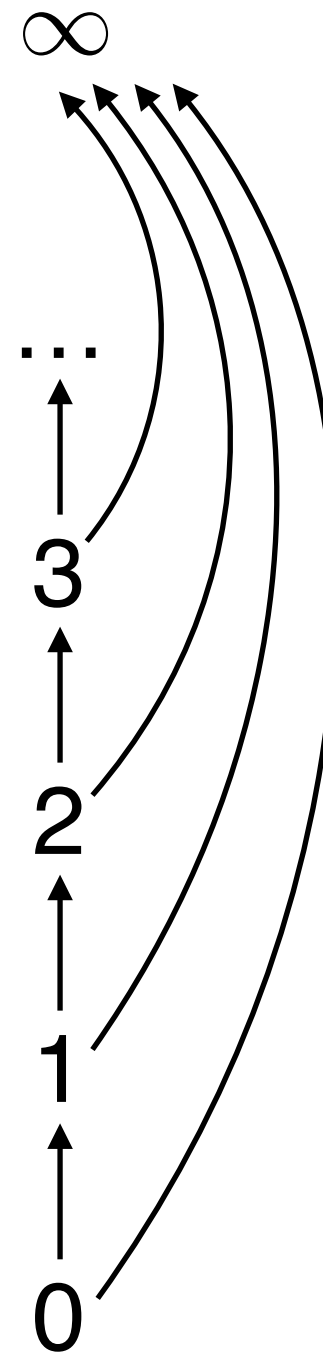
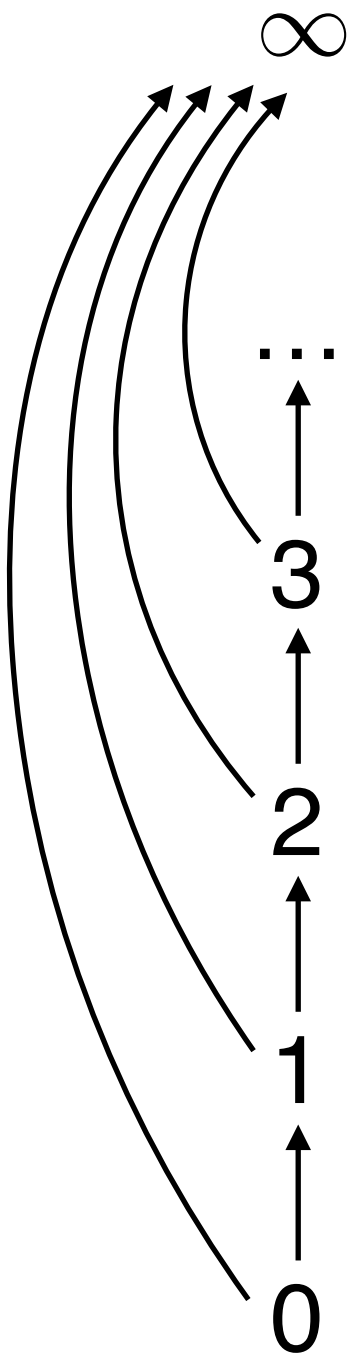
Example

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = n + 1$$

$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$

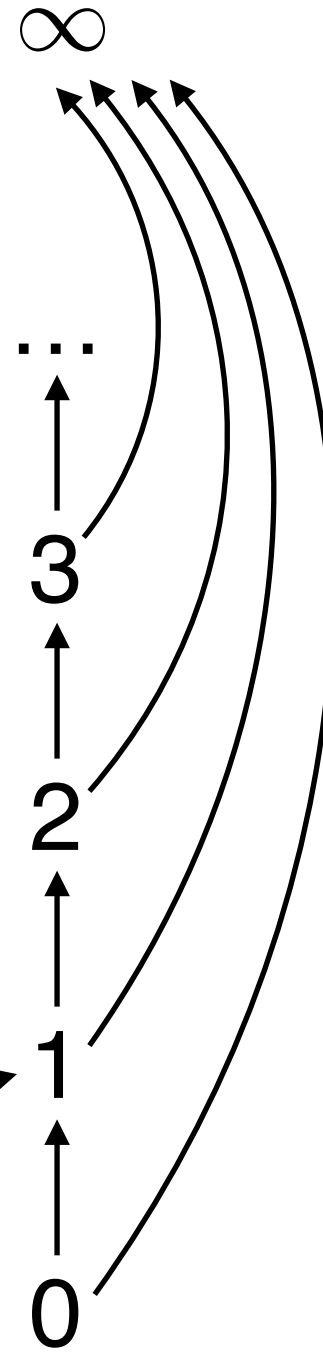
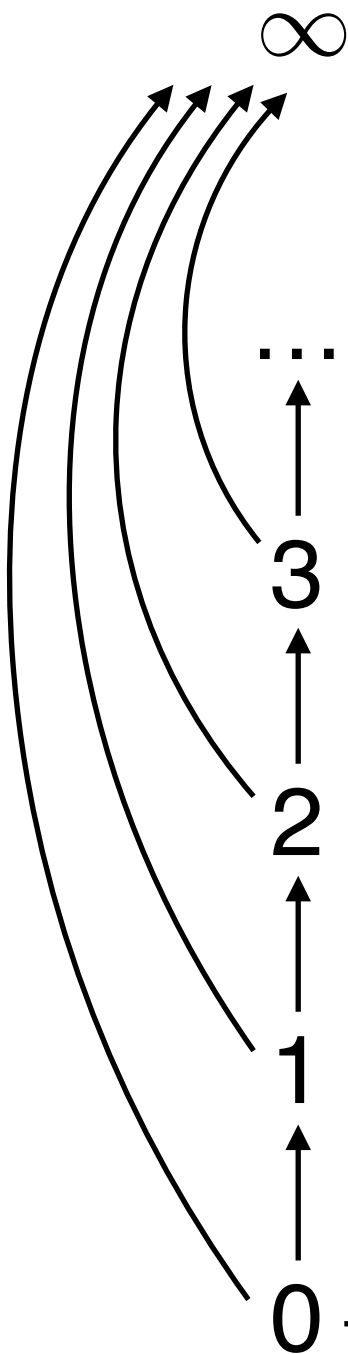


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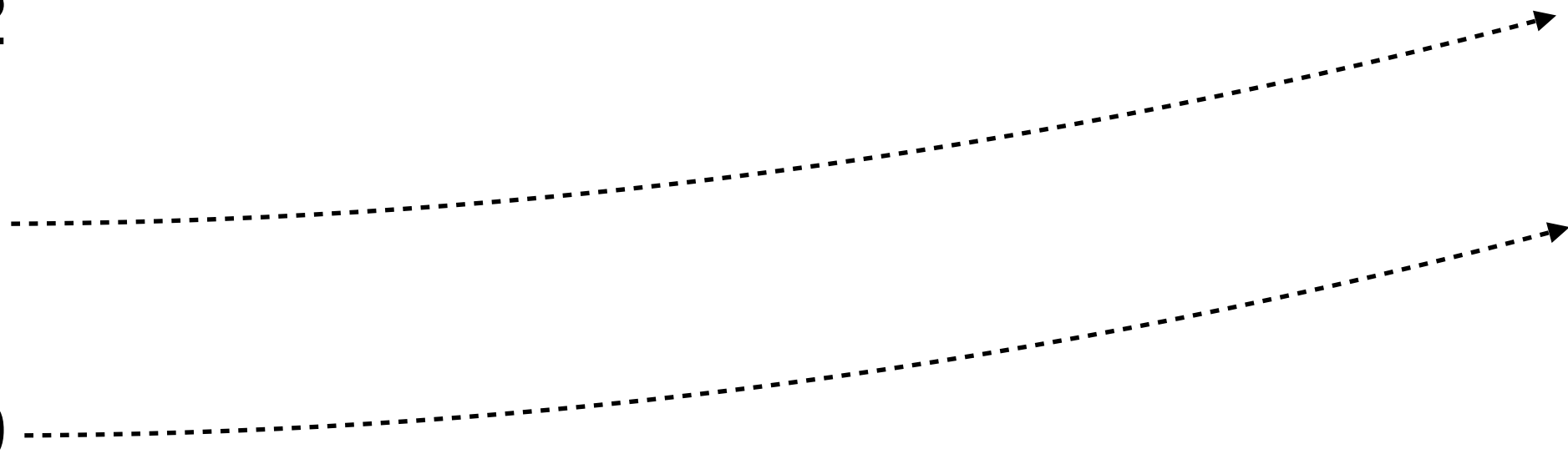
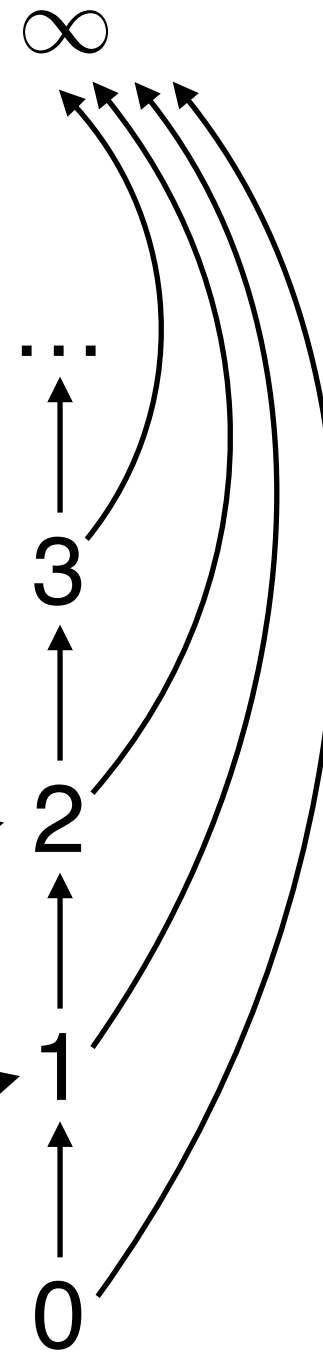
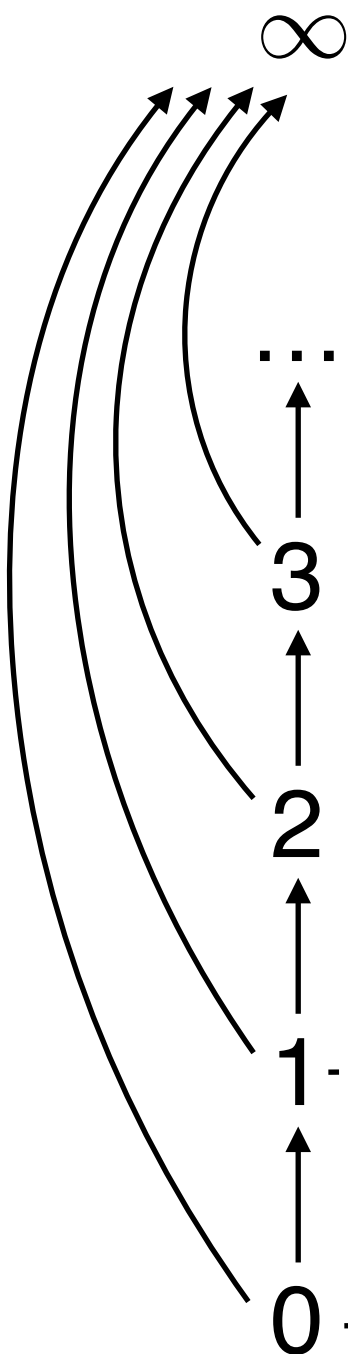


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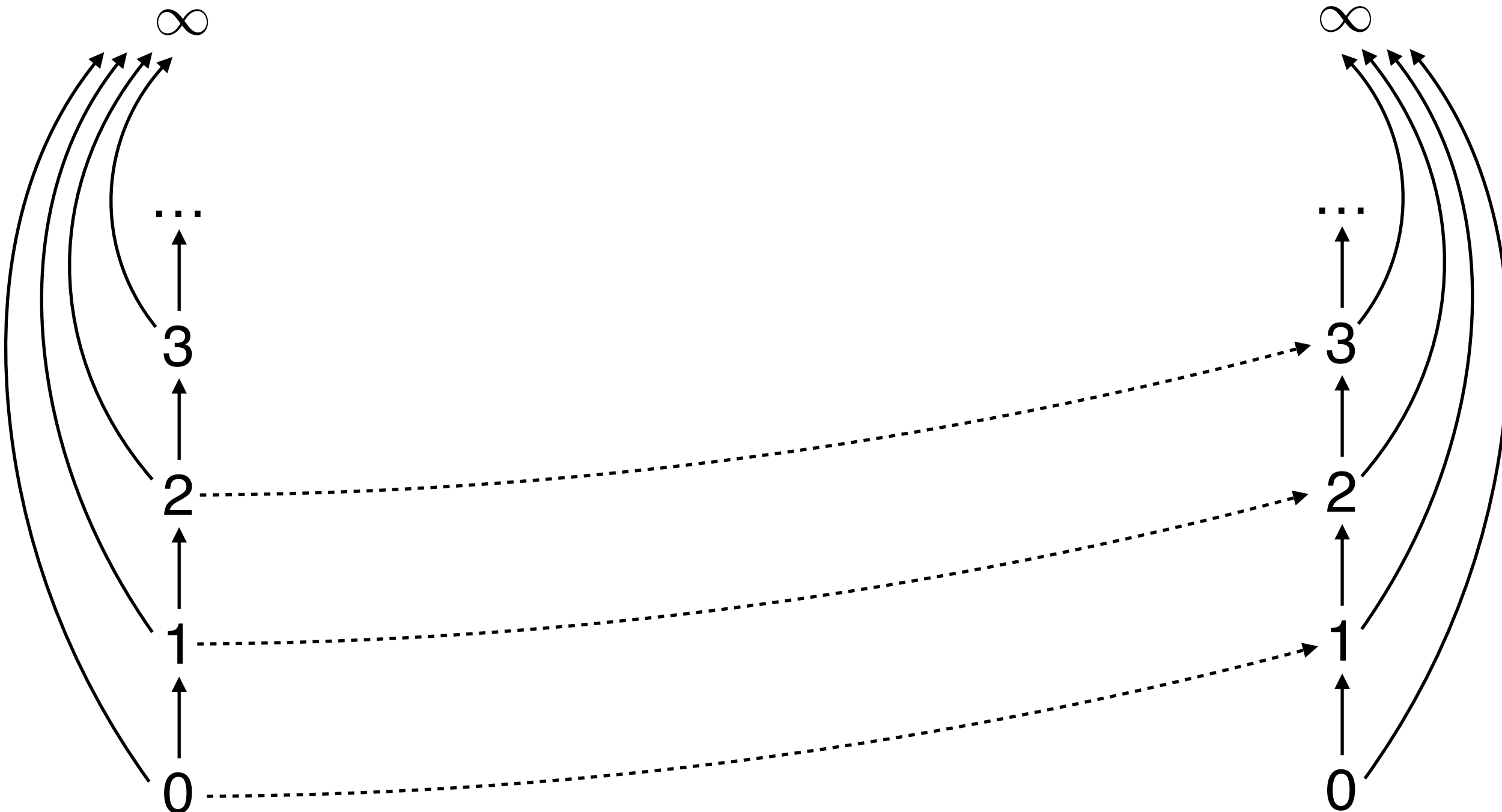


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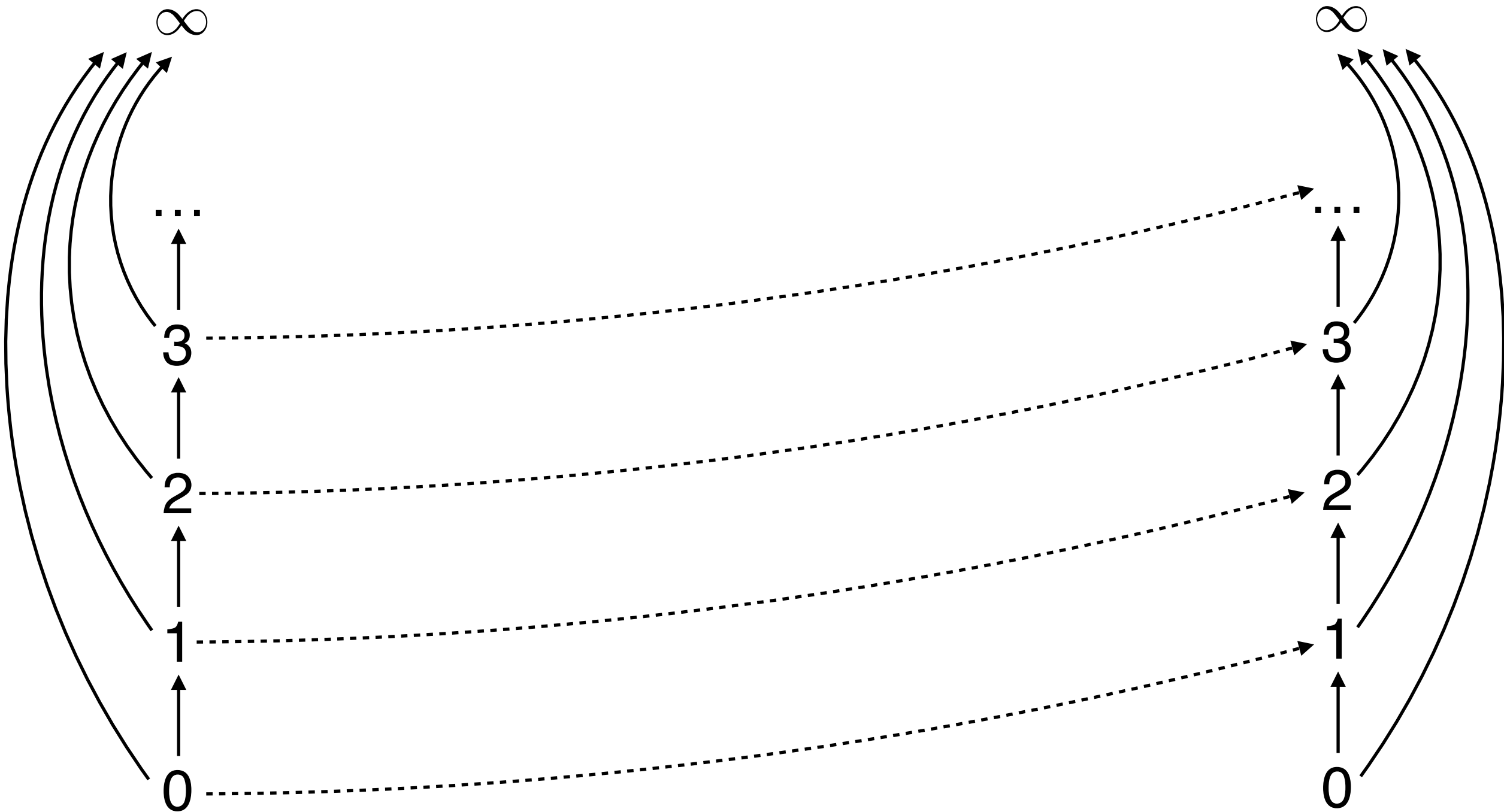


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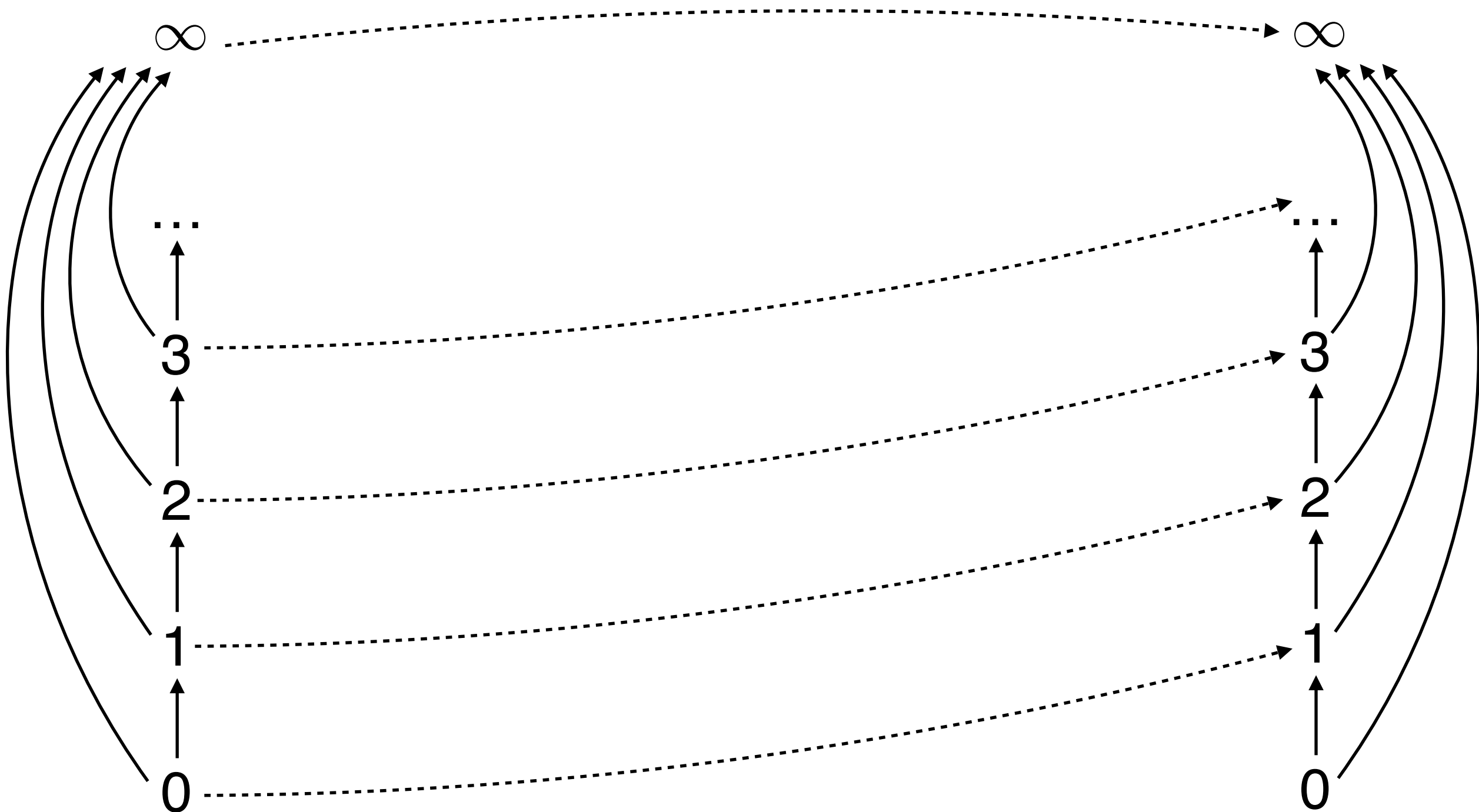


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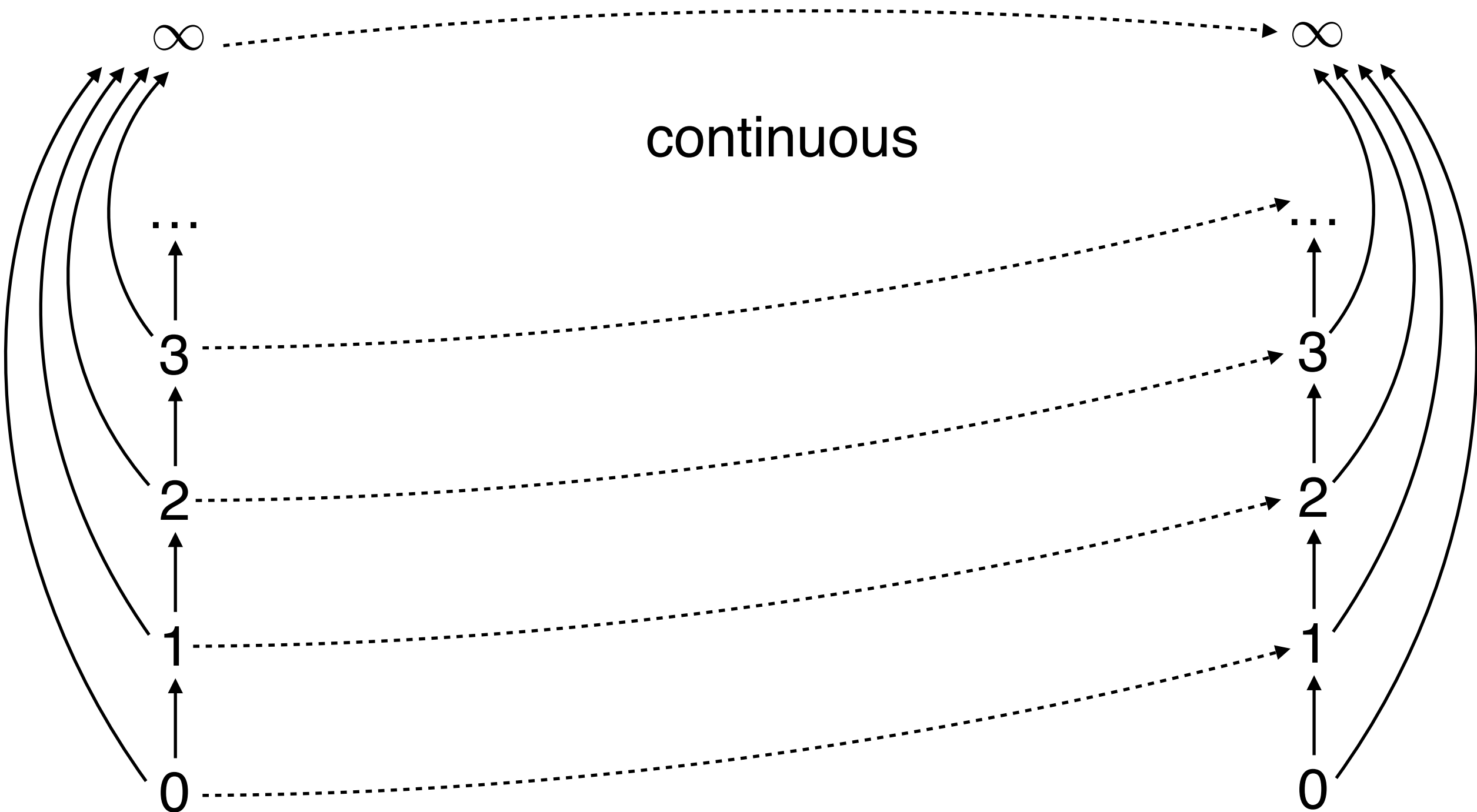


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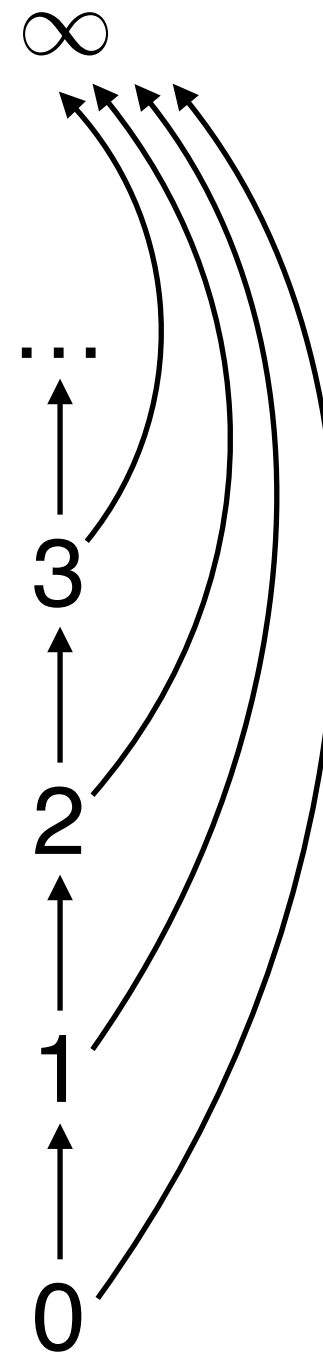
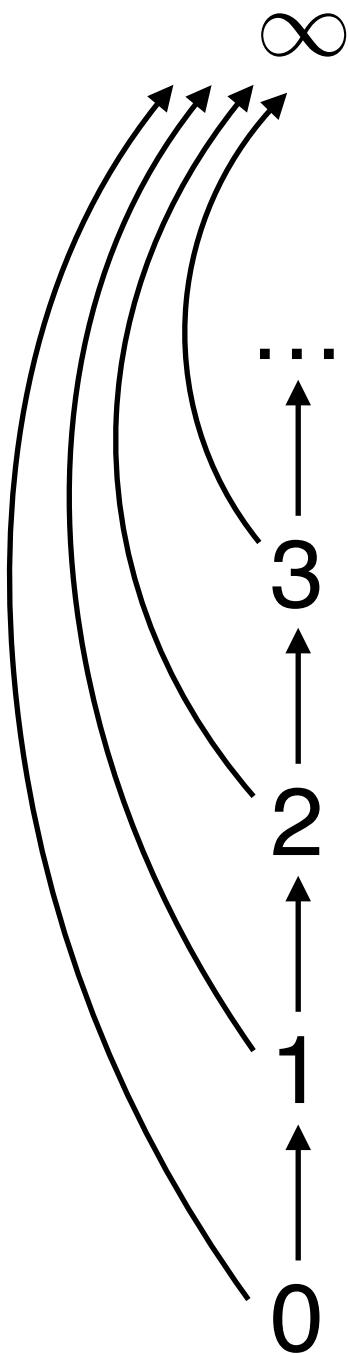
Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = 2 \cdot n$$

$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$

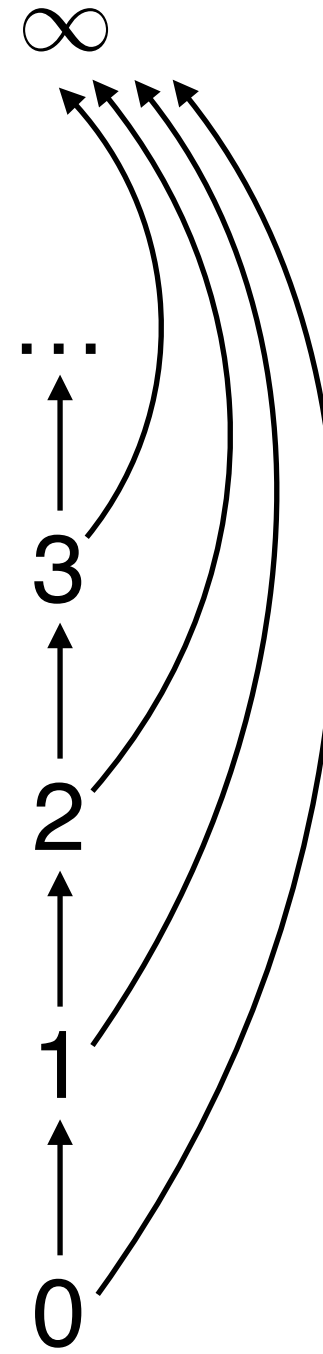
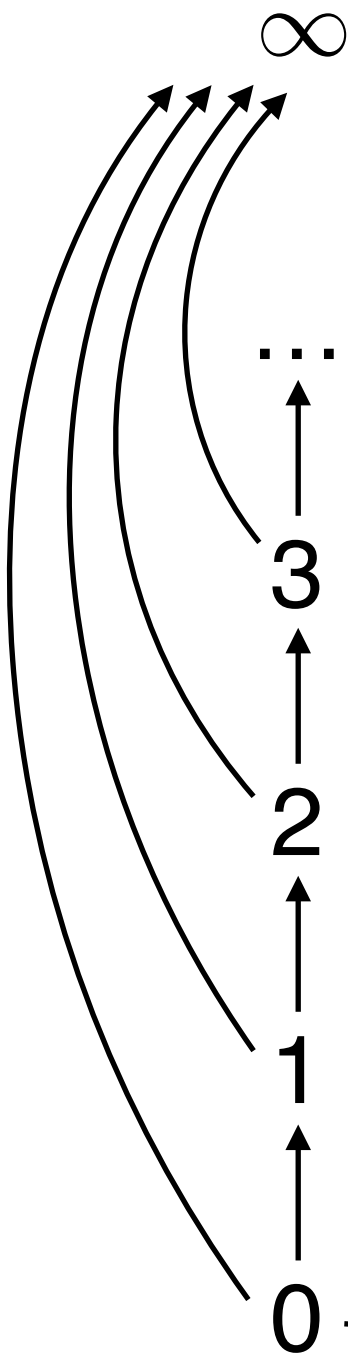


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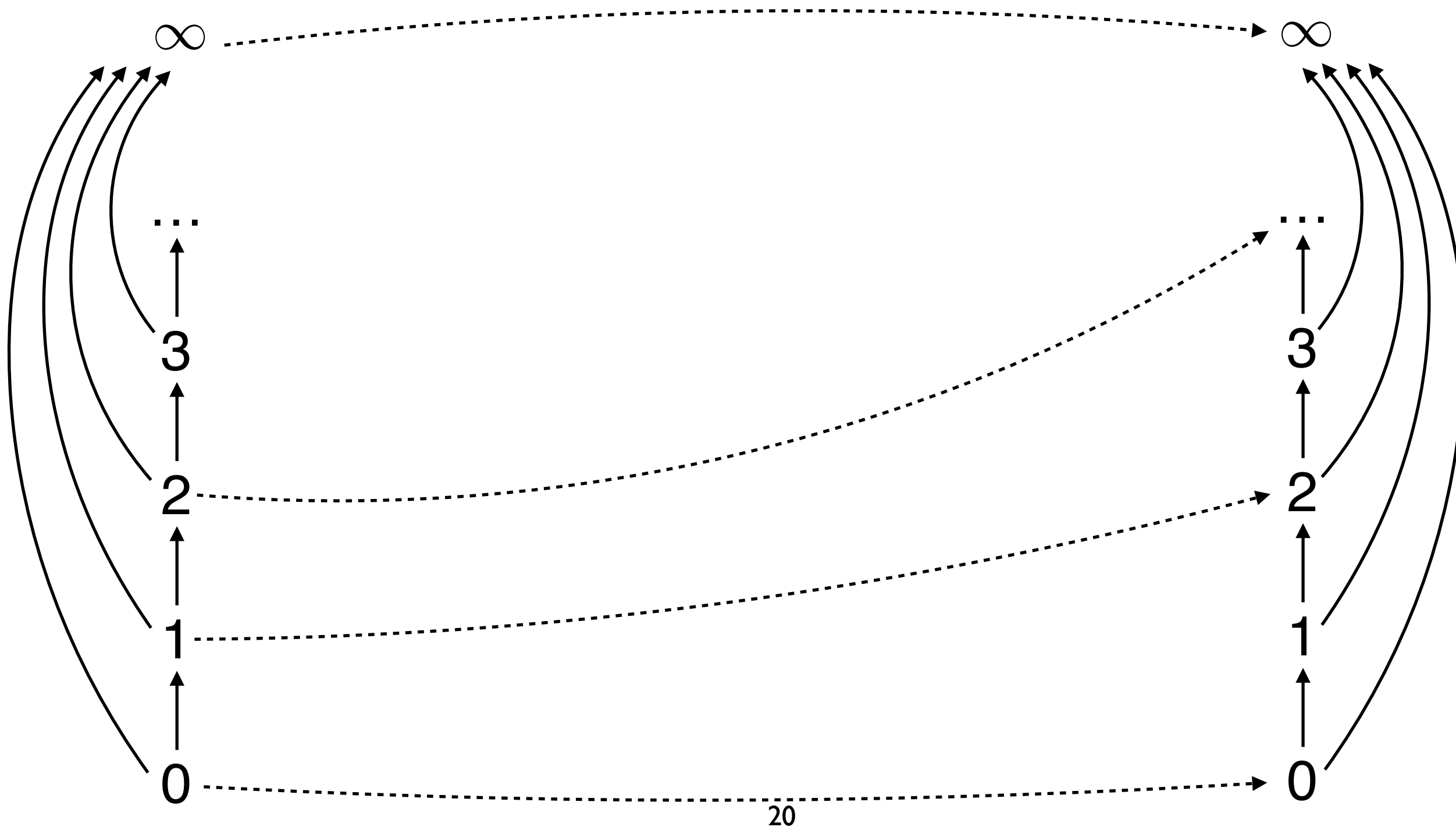


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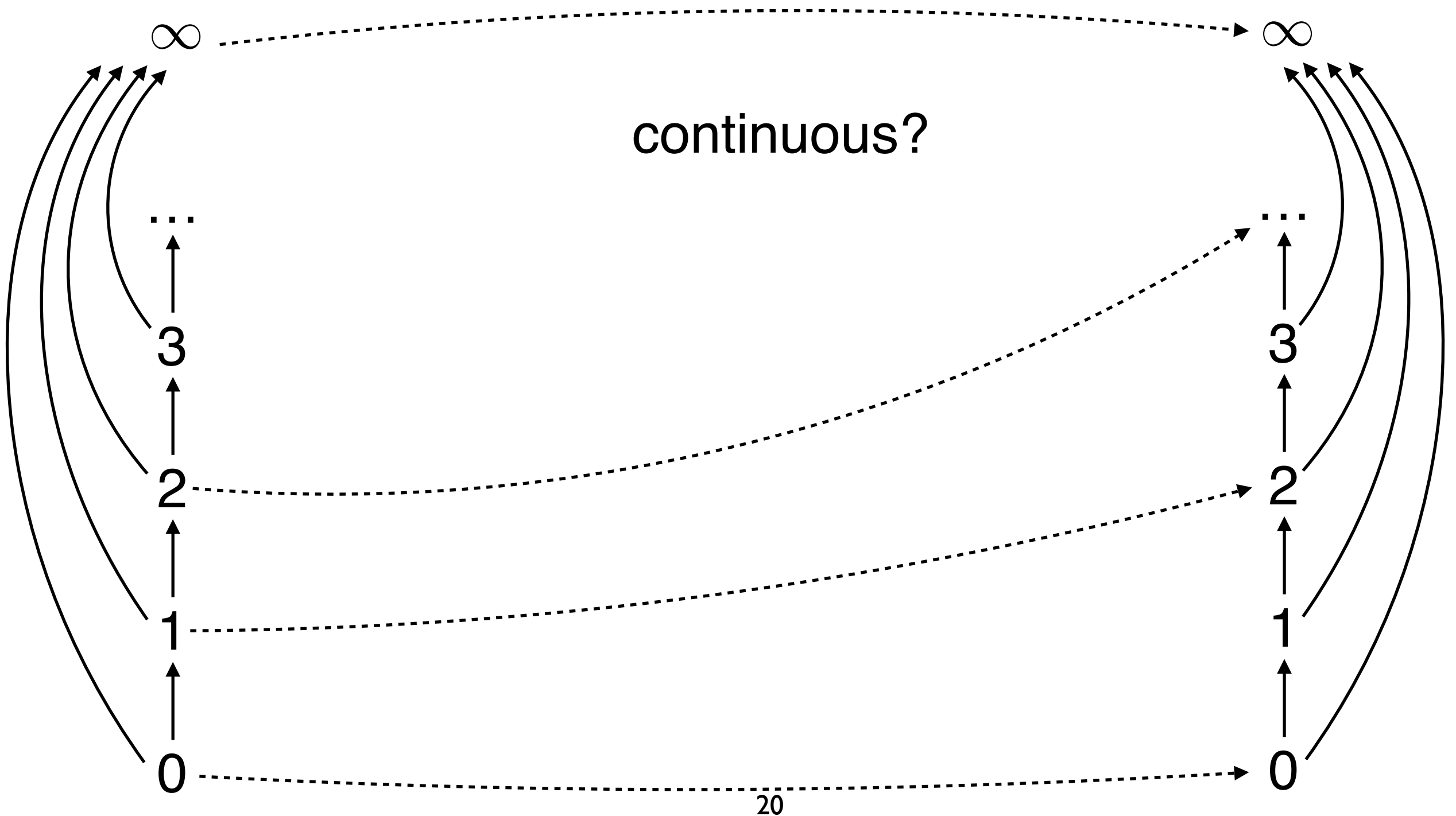
Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

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$(\mathbb{N} \cup \{\infty\}, \leq)$

continuous?



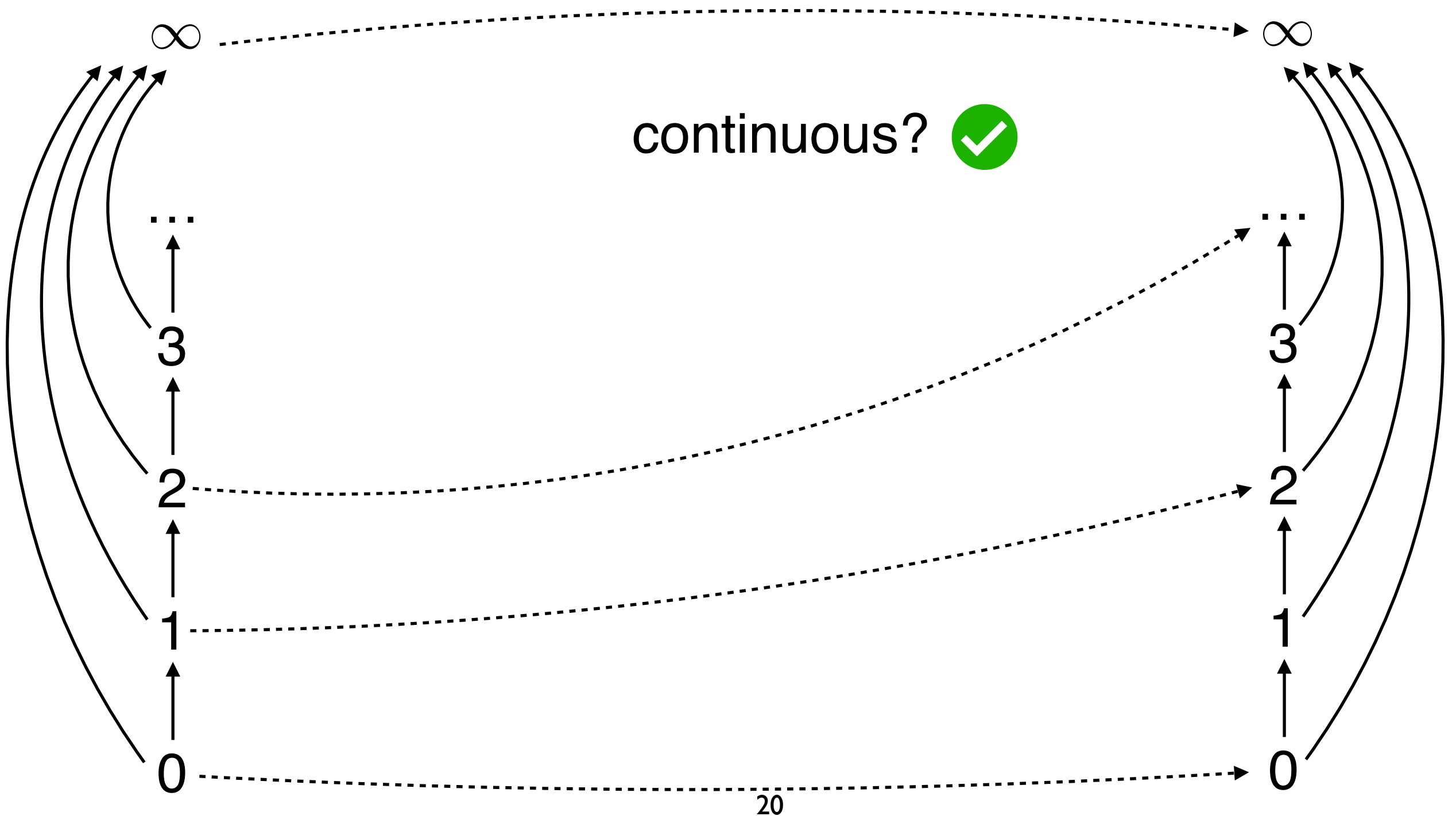
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$(\mathbb{N} \cup \{\infty\}, \leq)$

continuous? 



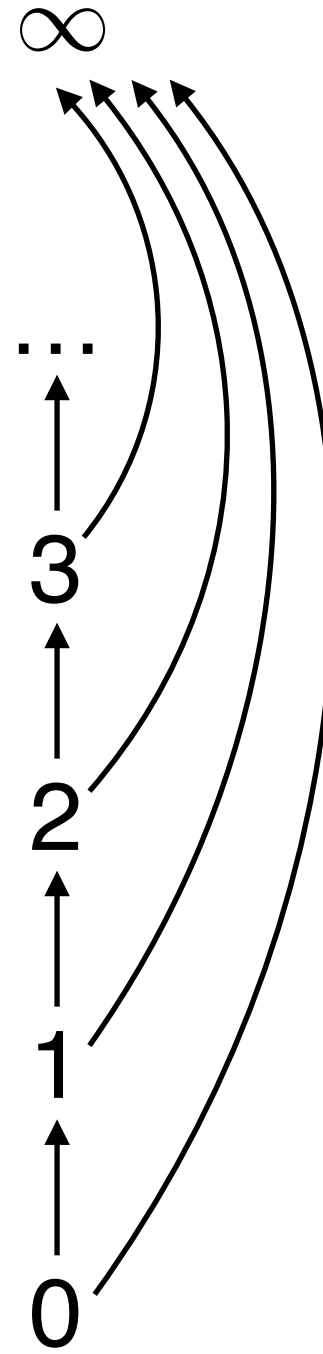
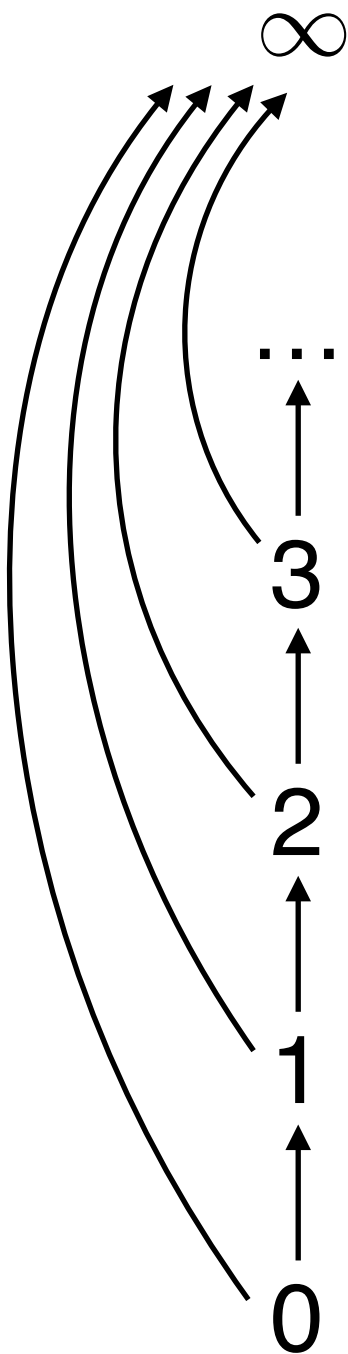
Example

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = n/2$$

$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$

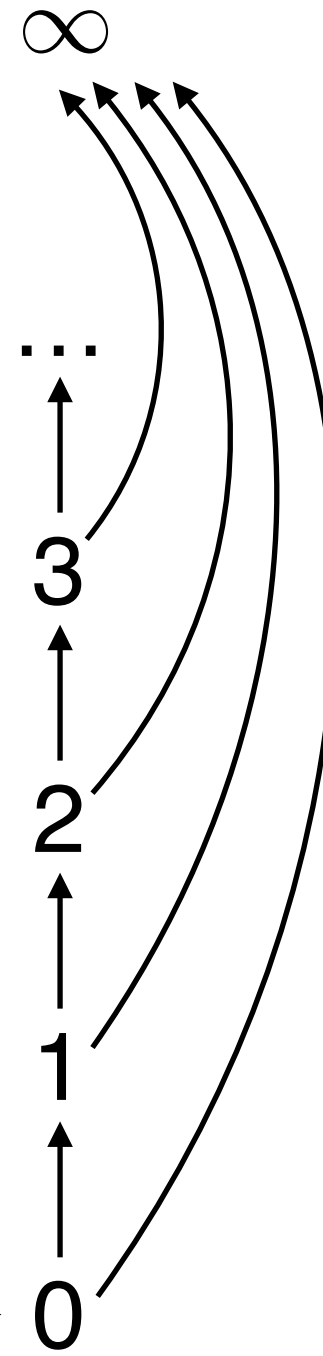
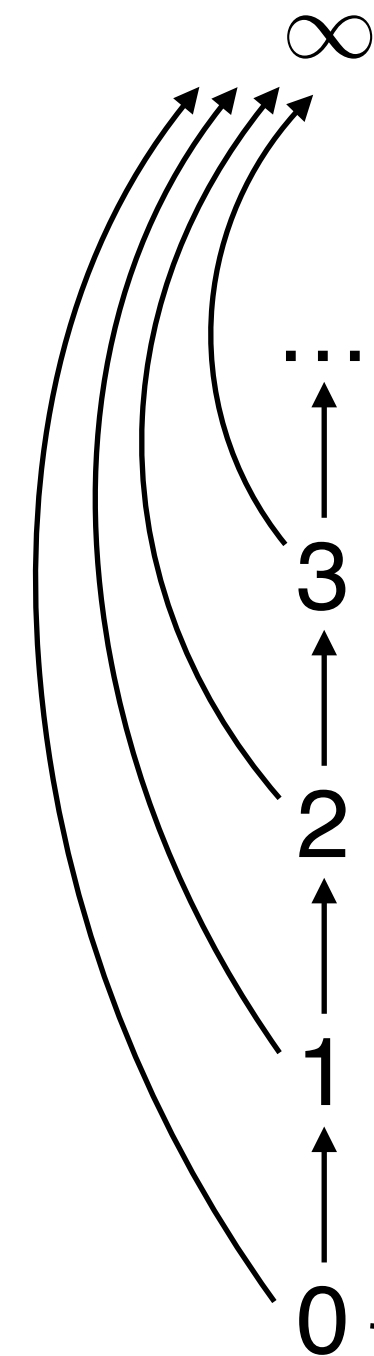


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$(\mathbb{N} \cup \{\infty\}, \leq)$



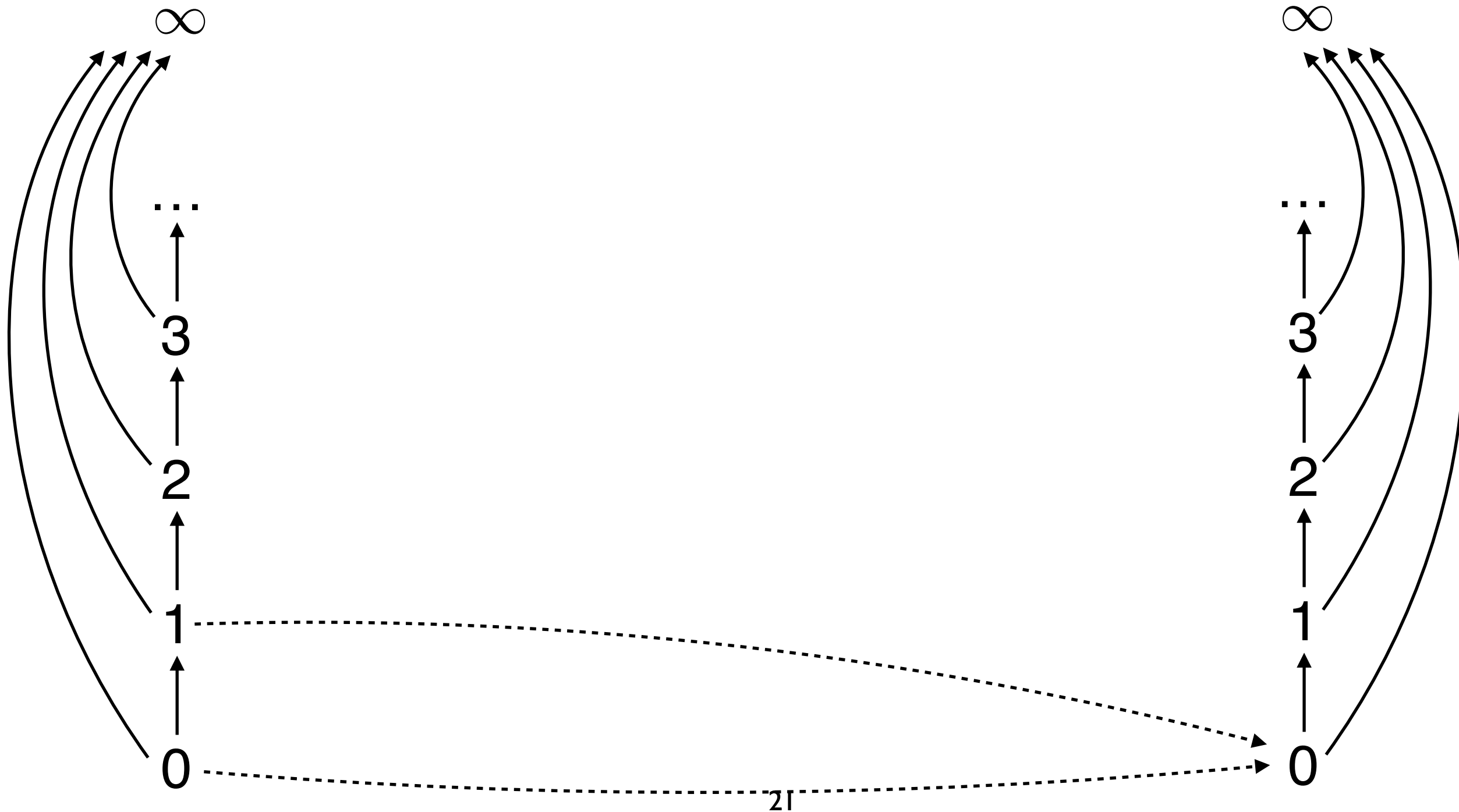
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$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$



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$(\mathbb{N} \cup \{\infty\}, \leq)$

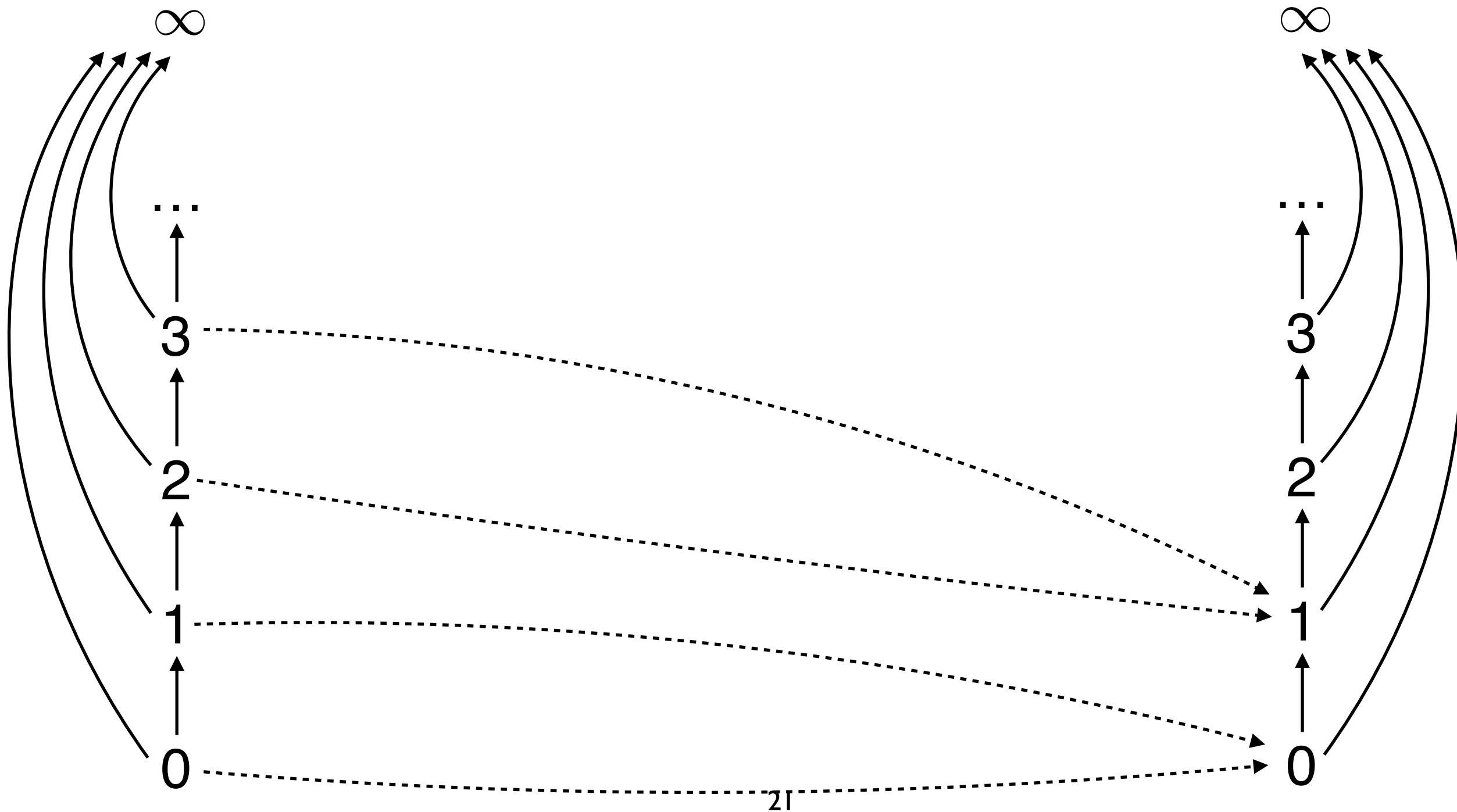


Example

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$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$



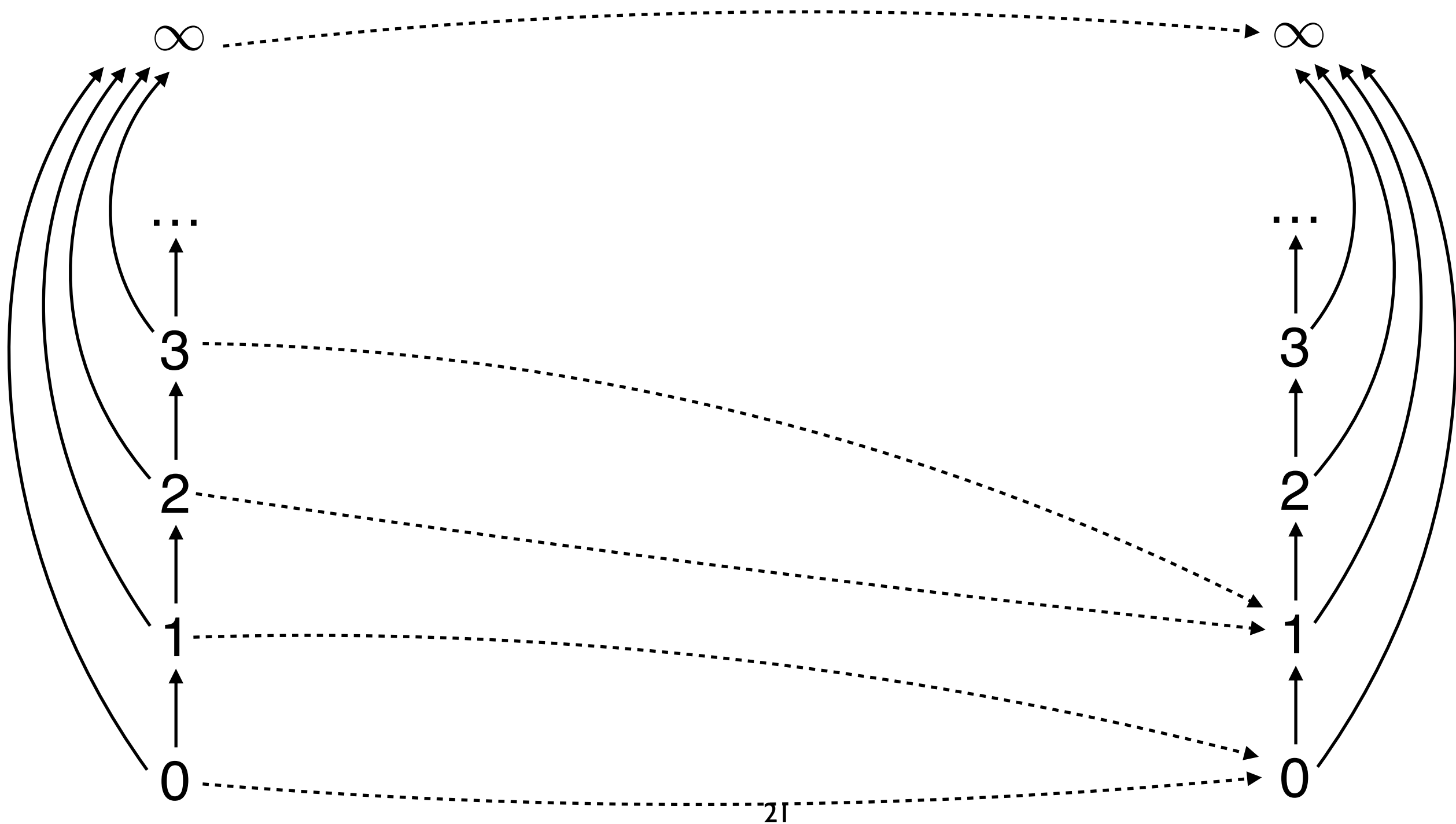
Example

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$$f(n) = n/2$$

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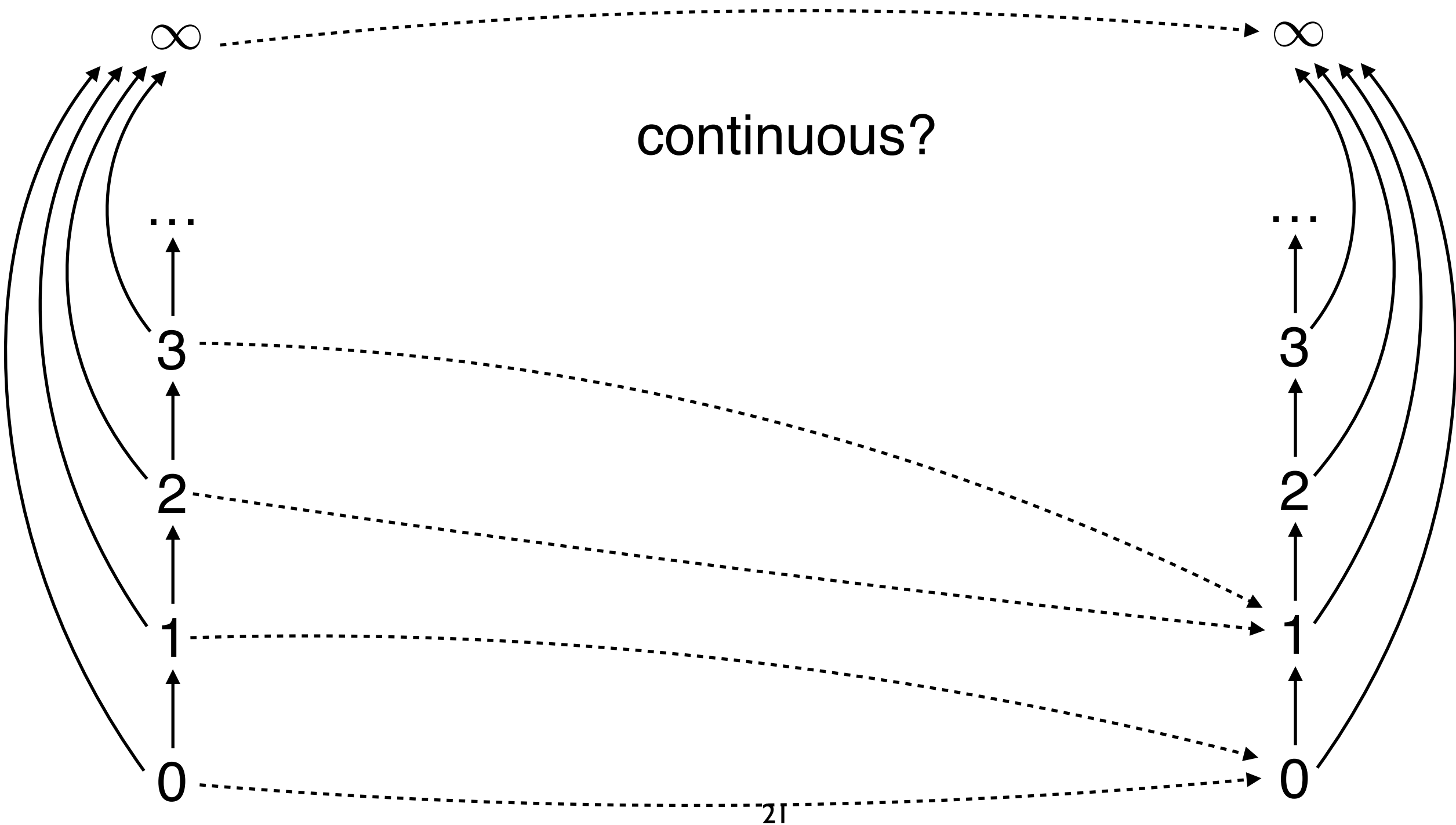
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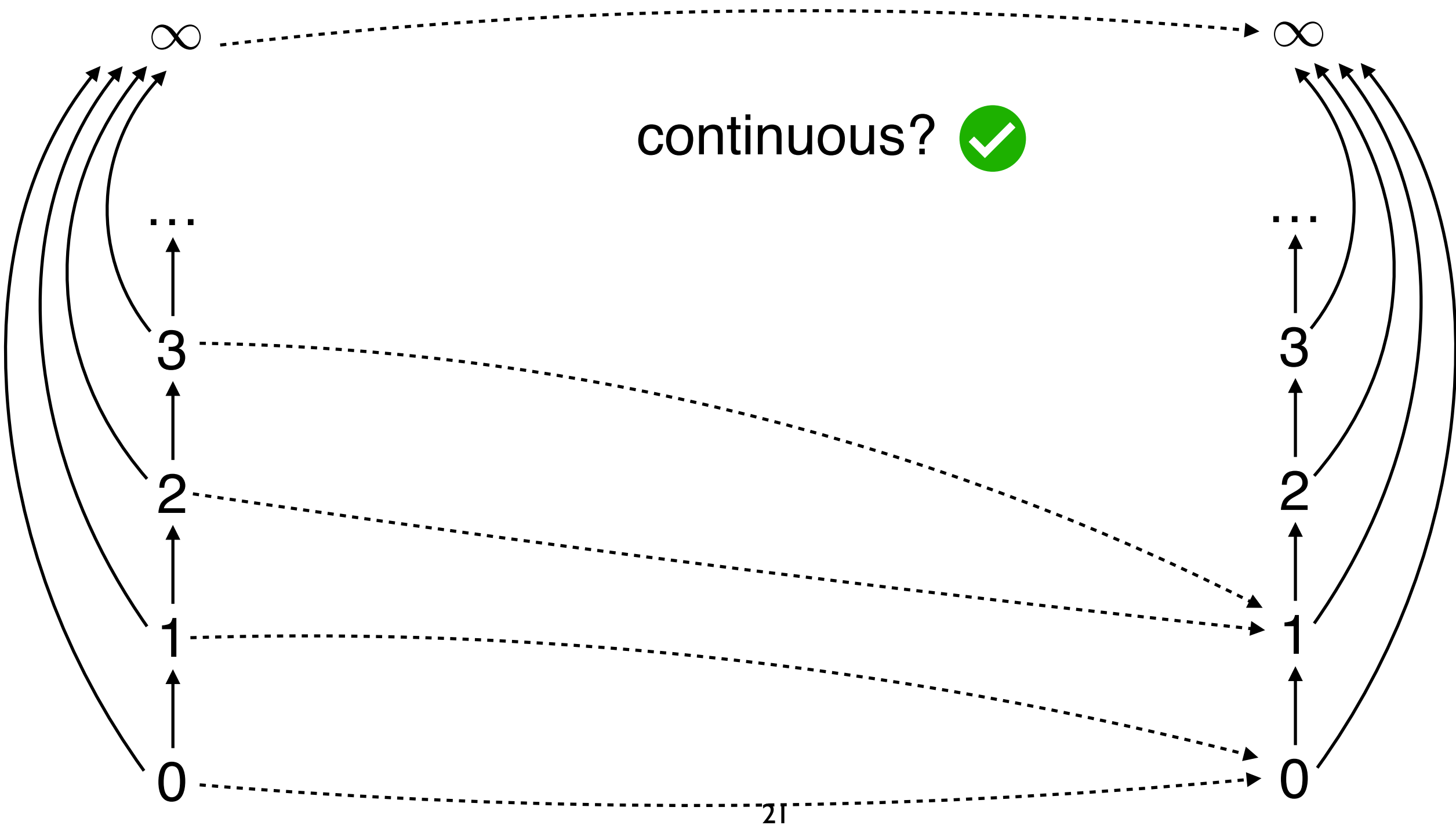
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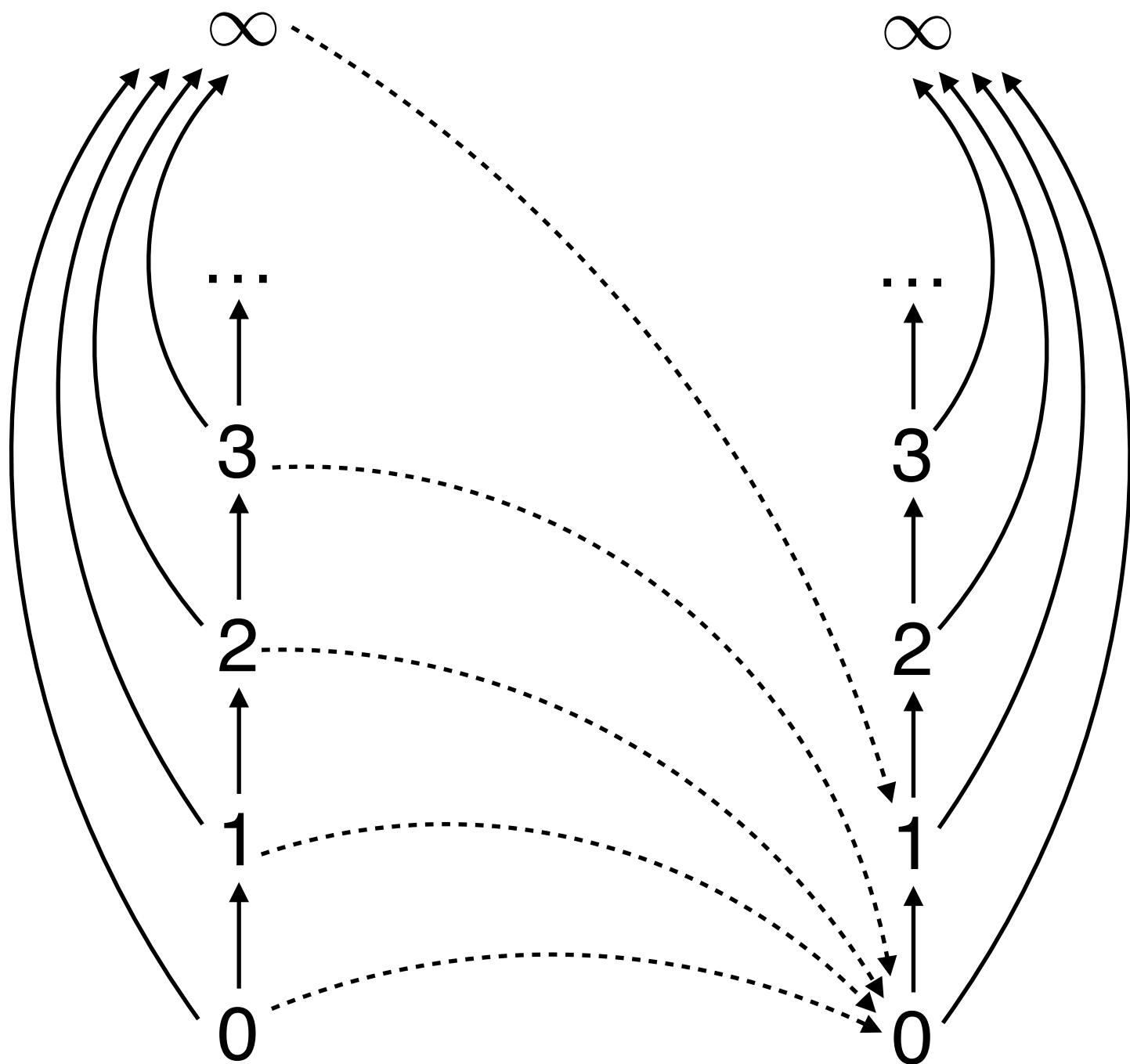


Example

$(\mathbb{N} \cup \{\infty\}, \leq)$

monotone function, not continuous

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{N} \\ 1 & \text{if } x = \infty \end{cases}$$



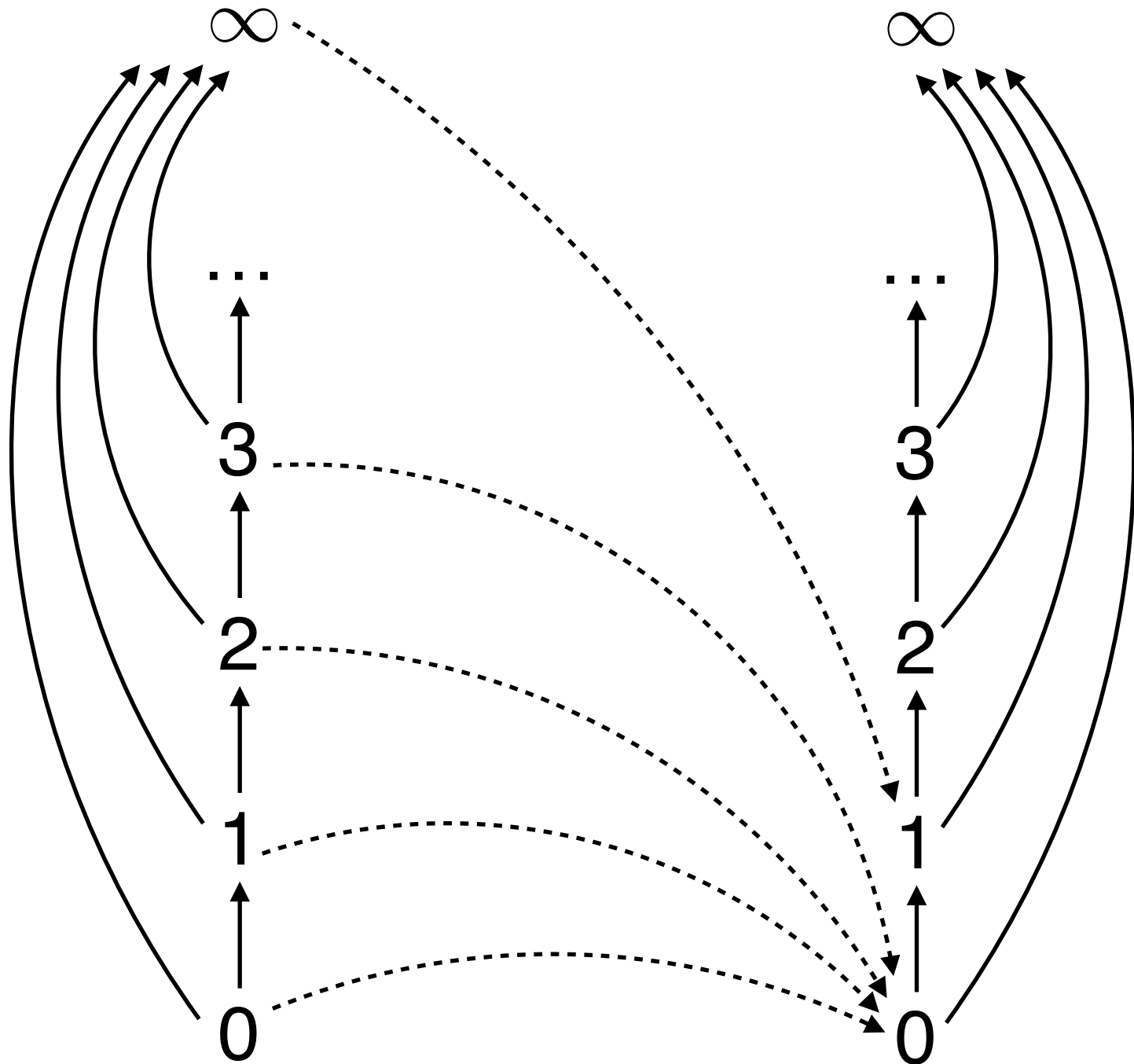
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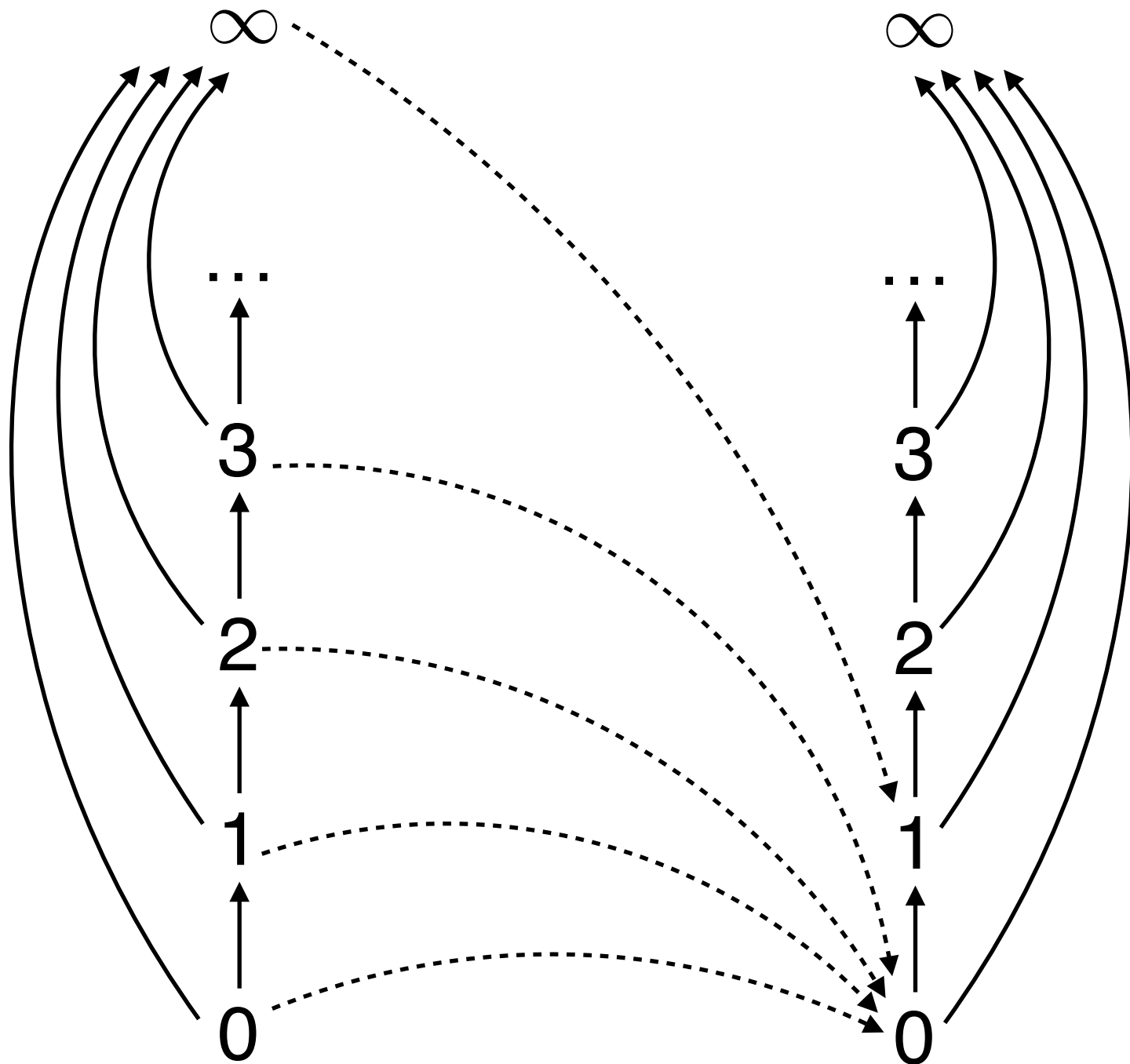
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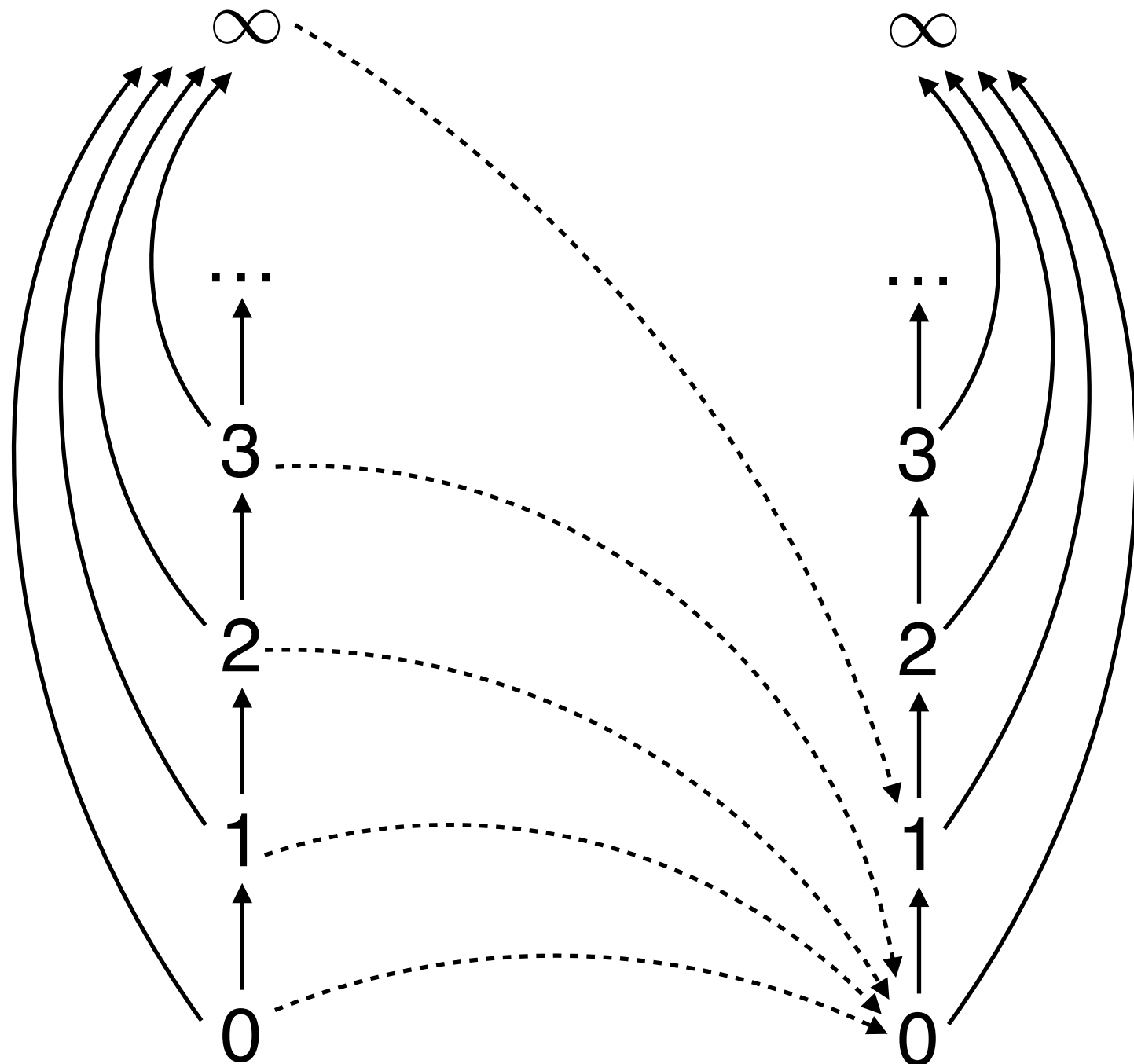
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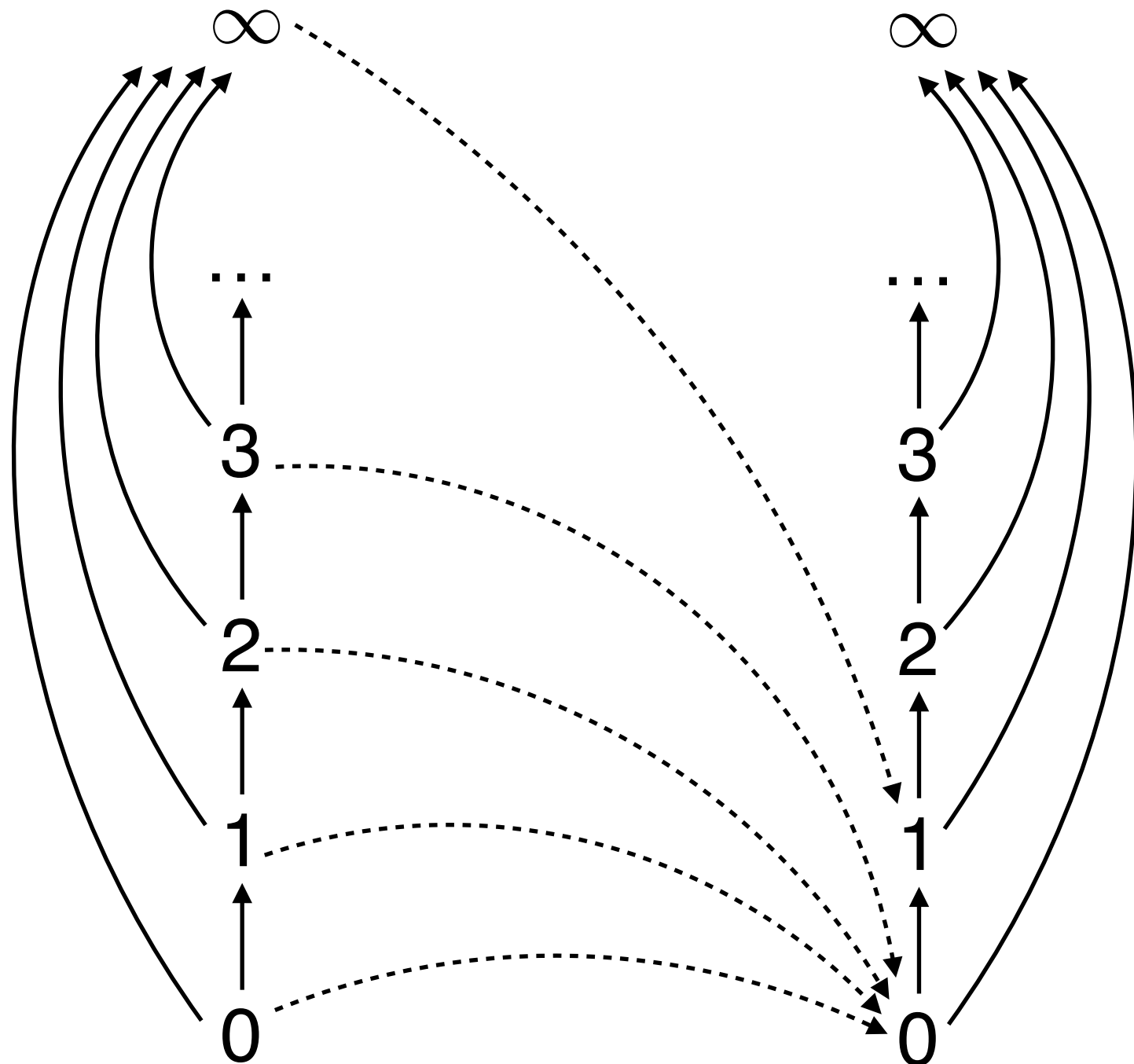
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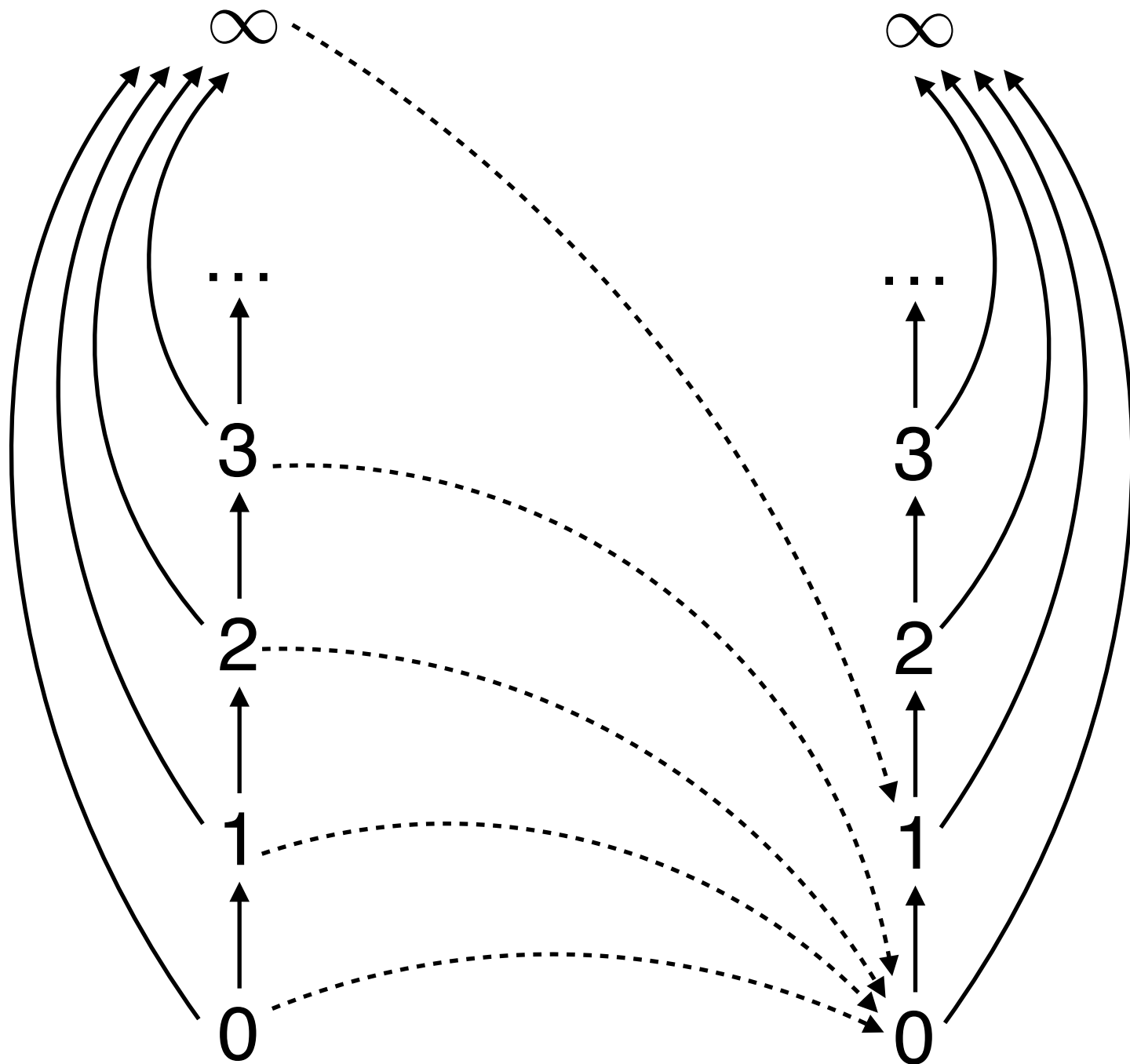
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$$f(d_i) = 0$$

$$\bigsqcup_{i \in \mathbb{N}} f(d_i) = \bigsqcup_{i \in \mathbb{N}} 0 = 0$$

Exercise

$$(\wp(\mathbb{N}), \subseteq) \quad f(S) = \{ m \in \mathbb{N} \mid \exists n \in S, m \leq n \} \quad (\wp(\mathbb{N}), \subseteq)$$

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Lemma

$(D, \sqsubseteq_D)$ CPO
no infinite chains

$(E, \sqsubseteq_E)$ PO

$f : D \rightarrow E$
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$$\{f(d_i)\}_{i \in \mathbb{N}} \text{ is finite} \Rightarrow \bigsqcup_{i \in \mathbb{N}} f(d_i) = f(d_k) = f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right)$$

Composition

TH. Any composition of continuous functions is continuous

$(D, \sqsubseteq_D)$ CPO $f : D \rightarrow E$ continuous
 $(E, \sqsubseteq_E)$ CPO $g : E \rightarrow F$ continuous
 $(F, \sqsubseteq_F)$ CPO

Composition

TH. Any composition of continuous functions is continuous

$(D, \sqsubseteq_D)$ CPO	$f : D \rightarrow E$ continuous	$\Rightarrow$	$h = g \circ f : D \rightarrow F$ continuous
$(E, \sqsubseteq_E)$ CPO	$g : E \rightarrow F$ continuous		
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$$\begin{array}{lcl} (D, \sqsubseteq_D) \text{ CPO} & f : D \rightarrow E \text{ continuous} & \\ (E, \sqsubseteq_E) \text{ CPO} & g : E \rightarrow F \text{ continuous} & \Rightarrow h = g \circ f : D \rightarrow F \\ (F, \sqsubseteq_F) \text{ CPO} & & \text{continuous} \end{array}$$

proof.

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 &= \bigsqcup_{i \in \mathbb{N}} h(d_i)
 \end{aligned}$$

Kleene's fixpoint theorem

Repeated application

$$f : D \rightarrow D$$

$$f^0(d) \triangleq d$$

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Lemma

$(D, \sqsubseteq)$ $\text{PO}_\perp$ $f : D \rightarrow D$ monotone

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assume $P(n) \triangleq f^n(\perp) \sqsubseteq f^{n+1}(\perp)$

we want to prove $P(n+1) \triangleq f^{n+1}(\perp) \sqsubseteq f^{n+2}(\perp)$

$$f^n(\perp) \sqsubseteq f^{n+1}(\perp)$$

$\Downarrow$

$$f^{n+1}(\perp) = f(f^n(\perp)) \sqsubseteq f(f^{n+1}(\perp)) = f^{n+2}(\perp)$$

Towards Kleene's th.

when $(D, \sqsubseteq)$ is a $\text{CPO}_\perp$

then $\{f^n(\perp)\}_{n \in \mathbb{N}}$ is a chain

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not necessarily
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not necessarily
a chain!

Kleene's fix point theorem states that
if f is continuous, then the limit of the above chain
is the least fixpoint of f

Pre-fixpoints

$(D, \sqsubseteq)$ PO $f : D \rightarrow D$ monotone

fixpoint $p \in D$ $f(p) = p$

Pre-fixpoints

$(D, \sqsubseteq)$ PO $f : D \rightarrow D$ monotone

fixpoint $p \in D$ $f(p) = p$

pre-fixpoint $p \in D$ $f(p) \sqsubseteq p$

Pre-fixpoints

$(D, \sqsubseteq)$ PO $f : D \rightarrow D$ monotone

fixpoint $p \in D$ $f(p) = p$

pre-fixpoint $p \in D$ $f(p) \sqsubseteq p$

Clearly any fixpoint is also a pre-fixpoint

Kleene's theorem

$(D, \sqsubseteq) \text{ CPO}_\perp$ $f : D \rightarrow D$ continuous

let $\text{fix}(f) \triangleq \bigsqcup_{n \in \mathbb{N}} f^n(\perp)$

1. $\text{fix}(f)$ is a fix point of f

2. $\text{fix}(f)$ is the least pre-fixpoint of f

Kleene's theorem

$(D, \sqsubseteq) \text{ CPO}_\perp \quad f : D \rightarrow D \text{ continuous}$

let $\text{fix}(f) \triangleq \bigsqcup_{n \in \mathbb{N}} f^n(\perp)$

1. $\text{fix}(f)$ is a fix point of f

$$f(\text{fix}(f)) = \text{fix}(f)$$

2. $\text{fix}(f)$ is the least pre-fixpoint of f

Kleene's theorem

$(D, \sqsubseteq)$ CPO $_{\perp}$ $f : D \rightarrow D$ continuous

let $fix(f) \triangleq \bigsqcup_{n \in \mathbb{N}} f^n(\perp)$

1. $fix(f)$ is a fix point of f

$$f(fix(f)) = fix(f)$$

2. $fix(f)$ is the least pre-fixpoint of f

$$\forall d \in D. f(d) \sqsubseteq d \Rightarrow fix(f) \sqsubseteq d$$

Kleene's theorem

$(D, \sqsubseteq)$ CPO $_{\perp}$ $f : D \rightarrow D$ continuous

let $fix(f) \triangleq \bigsqcup_{n \in \mathbb{N}} f^n(\perp)$

1. $fix(f)$ is a fix point of f

$$f(fix(f)) = fix(f)$$

2. $fix(f)$ is the least pre-fixpoint of f

$$\forall d \in D. f(d) \sqsubseteq d \Rightarrow fix(f) \sqsubseteq d$$

if d is a pre-fixpoint then $fix(f)$ is smaller than d

Kleene's theorem: 1

$$1. \ f(\text{fix}(f)) = \text{fix}(f)$$

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$$1. \ f(\text{fix}(f)) = \text{fix}(f)$$

proof.

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$$= \bigsqcup_{n \in \mathbb{N}} f^n(\perp) \quad \text{by prefix independence of limits}$$

Kleene's theorem: 1

$$1. \quad f(\text{fix}(f)) = \text{fix}(f)$$

proof.

$$\begin{aligned} f(\text{fix}(f)) &= f\left(\bigsqcup_{n \in \mathbb{N}} f^n(\perp)\right) && \text{by def of } \text{fix} \\ &= \bigsqcup_{n \in \mathbb{N}} f(f^n(\perp)) && \text{by continuity} \\ &= \bigsqcup_{n \in \mathbb{N}} f^{n+1}(\perp) && \text{by def of } f^n \\ &= \bigsqcup_{n \in \mathbb{N}} f^n(\perp) && \text{by prefix independence of limits} \\ &= \text{fix}(f) && \text{by def of } \text{fix} \end{aligned}$$

Kleene's theorem: 2

$$2. \quad \forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

Kleene's theorem: 2

proof.

$$2. \quad \forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

Kleene's theorem: 2

$$2. \quad \forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

proof.

we prove that any pre-fixpoint is an upper bound of the chain

$$\{f^n(\perp)\}_{n \in \mathbb{N}}$$

Kleene's theorem: 2

$$2. \quad \forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

proof.

we prove that any pre-fixpoint is an upper bound of the chain

$$\{f^n(\perp)\}_{n \in \mathbb{N}}$$

by definition $\text{fix}(f)$ is the lub of the same chain
and thus smaller than any other upper bound

Kleene's theorem: 2

$$2. \quad \forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

Kleene's theorem: 2

$$2. \quad \forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

take any $d \in D$ such that $f(d) \sqsubseteq d$

Kleene's theorem: 2

$$2. \quad \forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

take any $d \in D$ such that $f(d) \sqsubseteq d$

we prove $\forall n \in \mathbb{N}. f^n(\perp) \sqsubseteq d$

Kleene's theorem: 2

$$2. \quad \forall d \in D. \quad f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

take any $d \in D$ such that $f(d) \sqsubseteq d$

we prove $\forall n \in \mathbb{N}. \quad f^n(\perp) \sqsubseteq d$ (d is an upper bound)

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$$P(n) \triangleq f^n(\perp) \sqsubseteq d$$

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$P(n) \triangleq f^n(\perp) \sqsubseteq d$ by mathematical induction

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$$\forall n \in \mathbb{N}. P(n) \Rightarrow P(n+1)$$

Kleene's theorem: 2

$$2. \quad \forall d \in D. \quad f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

take any $d \in D$ such that $f(d) \sqsubseteq d$

we prove $\forall n \in \mathbb{N}. \quad f^n(\perp) \sqsubseteq d$ (d is an upper bound)

$P(n) \triangleq f^n(\perp) \sqsubseteq d$ by mathematical induction

$$P(0) \triangleq f^0(\perp) \sqsubseteq d \qquad f^0(\perp) = \perp \sqsubseteq d$$

$$\forall n \in \mathbb{N}. \quad P(n) \Rightarrow P(n+1) \qquad \text{take a generic } n$$

Kleene's theorem: 2

$$2. \quad \forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

take any $d \in D$ such that $f(d) \sqsubseteq d$

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$$\forall n \in \mathbb{N}. P(n) \Rightarrow P(n+1)$$

take a generic n

assume $P(n) \triangleq f^n(\perp) \sqsubseteq d$

Kleene's theorem: 2

$$2. \quad \forall d \in D. \quad f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

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$\forall n \in \mathbb{N}. \quad P(n) \Rightarrow P(n+1)$ take a generic n
assume $P(n) \triangleq f^n(\perp) \sqsubseteq d$

we want to prove $P(n+1) \triangleq f^{n+1}(\perp) \sqsubseteq d$

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$$f^n(\perp) \sqsubseteq d$$

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assume $P(n) \triangleq f^n(\perp) \sqsubseteq d$

we want to prove $P(n+1) \triangleq f^{n+1}(\perp) \sqsubseteq d$

$$f^n(\perp) \sqsubseteq d \\ \text{(monot.)} \Downarrow$$

Kleene's theorem: 2

$$2. \quad \forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

take any $d \in D$ such that $f(d) \sqsubseteq d$

we prove $\forall n \in \mathbb{N}. f^n(\perp) \sqsubseteq d$ (d is an upper bound)

$P(n) \triangleq f^n(\perp) \sqsubseteq d$ by mathematical induction

$$P(0) \triangleq f^0(\perp) \sqsubseteq d \quad f^0(\perp) = \perp \sqsubseteq d$$

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assume $P(n) \triangleq f^n(\perp) \sqsubseteq d$

we want to prove $P(n+1) \triangleq f^{n+1}(\perp) \sqsubseteq d$

$$\begin{array}{c} f^n(\perp) \sqsubseteq d \\ \text{(monot.)} \downarrow \\ f(f^n(\perp)) \sqsubseteq f(d) \end{array}$$

Kleene's theorem: 2

$$2. \quad \forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

take any $d \in D$ such that $f(d) \sqsubseteq d$

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$P(n) \triangleq f^n(\perp) \sqsubseteq d$ by mathematical induction

$$P(0) \triangleq f^0(\perp) \sqsubseteq d \quad f^0(\perp) = \perp \sqsubseteq d$$

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$$\begin{array}{ccc} & f^n(\perp) \sqsubseteq d & \\ & \text{(monot.)} \downarrow & \\ f^{n+1}(\perp) & \stackrel{\text{(by def)}}{=} f(f^n(\perp)) & \sqsubseteq f(d) \end{array}$$

Kleene's theorem: 2

$$2. \quad \forall d \in D. \quad f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

take any $d \in D$ such that $f(d) \sqsubseteq d$

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$$P(0) \triangleq f^0(\perp) \sqsubseteq d \quad f^0(\perp) = \perp \sqsubseteq d$$

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assume $P(n) \triangleq f^n(\perp) \sqsubseteq d$

we want to prove $P(n+1) \triangleq f^{n+1}(\perp) \sqsubseteq d$

$$\begin{array}{ccccc} & & f^n(\perp) \sqsubseteq d & & \\ & & \downarrow \text{(monot.)} & & \\ f^{n+1}(\perp) & \stackrel{\text{(by def)}}{=} & f(f^n(\perp)) & \sqsubseteq & f(d) \sqsubseteq d \quad \text{(pre-fixpoint)} \end{array}$$

Example

$$n = 2 \cdot n$$

$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

$$\text{CPO}_{\perp}$$

Example

$$n = 2 \cdot n$$

$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

$$\text{CPO}_{\perp}$$

$$\begin{aligned} f(n) &= 2 \cdot n \\ f(\infty) &= \infty \end{aligned}$$

Example

$$n = 2 \cdot n$$

$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

$\text{CPO}_{\perp}$

$$\begin{aligned} f(n) &= 2 \cdot n \\ f(\infty) &= \infty \end{aligned}$$

monotone? ok

Example

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$$\begin{aligned} f(n) &= 2 \cdot n \\ f(\infty) &= \infty \end{aligned}$$

monotone? ok

continuous? ok

Example

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$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

$\text{CPO}_{\perp}$

$$\begin{aligned} f(n) &= 2 \cdot n \\ f(\infty) &= \infty \end{aligned}$$

monotone? ok

continuous? ok

$$f^0(0) = 0$$

Example

$$n = 2 \cdot n$$

$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

$\text{CPO}_{\perp}$

$$\begin{aligned} f(n) &= 2 \cdot n \\ f(\infty) &= \infty \end{aligned}$$

monotone? ok

continuous? ok

$$f^0(0) = 0$$

$$f^1(0) = f(0) = 2 \cdot 0 = 0$$

Example

$$n = 2 \cdot n$$

$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

$\text{CPO}_{\perp}$

$$\begin{aligned} f(n) &= 2 \cdot n \\ f(\infty) &= \infty \end{aligned}$$

monotone? ok

continuous? ok

$$f^0(0) = 0$$

$$f^1(0) = f(0) = 2 \cdot 0 = 0$$

fixpoint reached!

Example

$$n = n + 1$$

$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

$$\text{CPO}_{\perp}$$

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$$n = n + 1$$

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$$\begin{aligned} f(n) &= n + 1 \\ f(\infty) &= \infty \end{aligned}$$

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$$(\mathbb{N} \cup \{\infty\}, \leq)$$

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Example

$$n = n + 1$$

$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

$\text{CPO}_{\perp}$

$$\begin{aligned} f(n) &= n + 1 \\ f(\infty) &= \infty \end{aligned}$$

monotone? ok

continuous? ok

$$f^0(0) = 0$$

Example

$$n = n + 1$$

$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

$\text{CPO}_{\perp}$

$$\begin{aligned} f(n) &= n + 1 \\ f(\infty) &= \infty \end{aligned}$$

monotone? ok

continuous? ok

$$f^0(0) = 0$$

$$f^1(0) = f(0) = 0 + 1 = 1$$

Example

$$n = n + 1$$

$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

$\text{CPO}_{\perp}$

$$\begin{aligned} f(n) &= n + 1 \\ f(\infty) &= \infty \end{aligned}$$

monotone? ok

continuous? ok

$$f^0(0) = 0$$

$$f^1(0) = f(0) = 0 + 1 = 1$$

$$f^2(0) = f(f^1(0)) = f(1) = 1 + 1 = 2$$

Example

$$n = n + 1$$

$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

$\text{CPO}_{\perp}$

$$\begin{aligned} f(n) &= n + 1 \\ f(\infty) &= \infty \end{aligned}$$

monotone? ok

continuous? ok

$$f^0(0) = 0$$

$$f^1(0) = f(0) = 0 + 1 = 1$$

$$f^2(0) = f(f^1(0)) = f(1) = 1 + 1 = 2$$

$$f^3(0) = f(f^2(0)) = f(2) = 2 + 1 = 3$$

Example

$$n = n + 1$$

$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

$\text{CPO}_{\perp}$

$$\begin{aligned} f(n) &= n + 1 \\ f(\infty) &= \infty \end{aligned}$$

monotone? ok

continuous? ok

$$f^0(0) = 0$$

$$f^1(0) = f(0) = 0 + 1 = 1$$

$$f^2(0) = f(f^1(0)) = f(1) = 1 + 1 = 2$$

$$f^3(0) = f(f^2(0)) = f(2) = 2 + 1 = 3$$

$$\bigsqcup_{n \in \mathbb{N}} f^n(0) = \bigsqcup_{n \in \mathbb{N}} n = \infty$$

Example

$$n = n + 1$$

$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

$\text{CPO}_{\perp}$

$$\begin{aligned} f(n) &= n + 1 \\ f(\infty) &= \infty \end{aligned}$$

monotone? ok

continuous? ok

$$f^0(0) = 0$$

$$f^1(0) = f(0) = 0 + 1 = 1$$

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$$\bigsqcup_{n \in \mathbb{N}} f^n(0) = \bigsqcup_{n \in \mathbb{N}} n = \infty \quad \text{fixpoint}$$

Example

$$X = X \cap \{1\}$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset$$

$$\text{CPO}_{\perp}$$

Example

$$X = X \cap \{1\}$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset \quad \text{CPO}_{\perp}$$

$$f(X) = X \cap \{1\}$$

Example

$$X = X \cap \{1\}$$

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monotone? ok

Example

$$X = X \cap \{1\}$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset \quad \text{CPO}_{\perp}$$

$$f(X) = X \cap \{1\}$$

monotone? ok
continuous? ok

Example

$$X = X \cap \{1\}$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset \quad \text{CPO}_{\perp}$$

$$f(X) = X \cap \{1\}$$

monotone? ok
continuous? ok

$$f^0(\emptyset) = \emptyset$$

Example

$$X = X \cap \{1\}$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset \quad \text{CPO}_{\perp}$$

$$f(X) = X \cap \{1\}$$

monotone? ok
continuous? ok

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \emptyset \cap \{1\} = \emptyset$$

Example

$$X = X \cap \{1\}$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset \quad \text{CPO}_{\perp}$$

$$f(X) = X \cap \{1\}$$

monotone? ok
continuous? ok

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \emptyset \cap \{1\} = \emptyset$$

fixpoint reached!

Example

$$X = \mathbb{N} \setminus X$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset \quad \text{CPO}_{\perp}$$

Example

$$X = \mathbb{N} \setminus X$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset \quad \text{CPO}_{\perp}$$

$$f(X) = \mathbb{N} \setminus X$$

Example

$$X = \mathbb{N} \setminus X$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset \quad \text{CPO}_{\perp}$$

$$f(X) = \mathbb{N} \setminus X$$

monotone? NO

Example

$$X = \mathbb{N} \setminus X$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset \quad \text{CPO}_{\perp}$$

$$f(X) = \mathbb{N} \setminus X$$

monotone? NO

the larger X the smaller $f(X)$

Example

$$X = \mathbb{N} \setminus X$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset \quad \text{CPO}_{\perp}$$

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not a chain!

Example

$$X = X \cup \{1\}$$

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fixpoint reached!