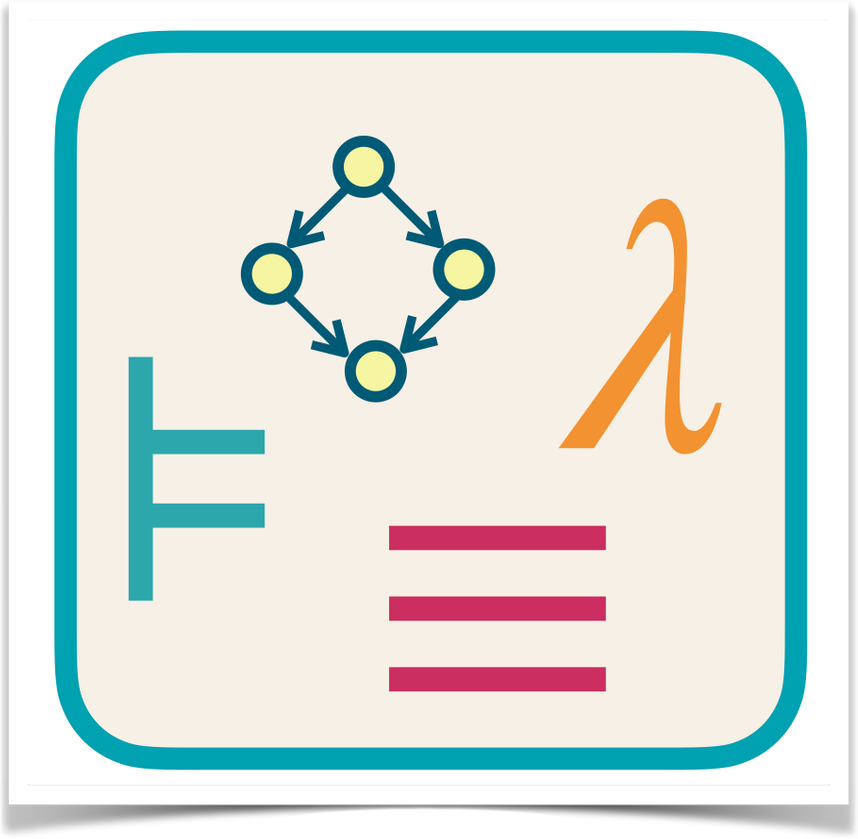


LPL 2026/27 (063AA, 9CFU)

Programming Languages

Fabio Gadducci

and Laboratory

Vincenzo Ciancia

08a - Complete Partial Orders

Consistency of expressions

Consistency?

$\langle a, \sigma \rangle \longrightarrow n$

$\mathcal{A}[\![a]\!]\sigma = n$

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$$\langle a, \sigma \rangle \longrightarrow n \quad \stackrel{?}{\Leftrightarrow} \quad \mathcal{A}[\![a]\!]\sigma = n$$

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structural induction!

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structural induction!

$$\forall x \in \mathsf{Ide}.\ P(x) \qquad\qquad \forall n \in \mathbb{Z}.\ P(n)$$

$$\frac{\forall a_0, a_1.\ P(a_0) \wedge P(a_1) \Rightarrow P(a_0\ \mathsf{op}\ a_1)}{\forall a.\ P(a)}$$

Base cases

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$$\forall a_0, a_1.\ P(a_0) \wedge P(a_1) \Rightarrow P(a_0 \text{ op } a_1)$$

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and thus $n = n_0\ op\ n_1 = \mathcal{A}[\![a_0]\!]\sigma\ op\ \mathcal{A}[\![a_1]\!]\sigma$

Denotational semantics of commands?

Recursive definitions

for divergence

$$\mathcal{C}[\![\cdot]\!] : \mathsf{Com} \to \mathbb{M} \to \mathbb{M} \cup \{\bot\}$$

$$\begin{aligned}
\mathcal{C}[\![\mathbf{skip}]\!]\sigma &\triangleq \sigma \\
\mathcal{C}[\![x := a]\!]\sigma &\triangleq \sigma[\mathcal{A}[\![a]\!]\sigma / x] \\
\mathcal{C}[\![c_0; c_1]\!]\sigma &\triangleq \mathcal{C}[\![c_1]\!](\mathcal{C}[\![c_0]\!]\sigma)
\end{aligned}$$

almost…

$$\mathcal{C}[\![\mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1]\!]\sigma \triangleq \begin{cases} \mathcal{C}[\![c_0]\!]\sigma & \text{if } \mathcal{B}[\![b]\!]\sigma \\ \mathcal{C}[\![c_1]\!]\sigma & \text{otherwise} \end{cases}$$

$$\mathcal{C}[\![\mathbf{while}\ b\ \mathbf{do}\ c]\!]\sigma \triangleq \begin{cases} \sigma & \text{if } \neg\mathcal{B}[\![b]\!]\sigma \\ \mathcal{C}[\![\mathbf{while}\ b\ \mathbf{do}\ c]\!](\mathcal{C}[\![c]\!]\sigma) & \text{otherwise} \end{cases}$$

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how do we know one solution exists? how do we know it is unique?

The general problem

$$f : D \to D$$

a fixed point of f is $d \in D$ such that $d = f(d)$

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three questions:

- under which hypotheses $F_f \neq \varnothing$?
- if $F_f \neq \varnothing$, can we select a preferred element $\mathit{fix}(f) \in F_f$?
- and can we compute $\mathit{fix}(f)$?

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Can we compute the "best" solution?

- and can we compute $\mathit{fix}(f)$?

Example

$$\{n \mid n = f(n)\}$$

$D = \mathbb{N}$ F_f $fix(f)$

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F_f

$fix(f)$

$f(n) \triangleq n + 1$

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$\varnothing$

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Example

$\{n \mid n = f(n)\}$

$D = \mathbb{N}$	F_f	$fix(f)$
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$$f(n) \triangleq \sum_{i \in \mathsf{div}(n)} i$$

where $\mathsf{div}(x) \triangleq \{1\} \cup \{d \mid 1 < d < x, x\%d = 0\}$

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Example

$D = \wp(\mathbb{N})$

F_f $^{\{S \mid S = f(S)\}}$

$fix(f)$

Example

$D = \wp(\mathbb{N})$

$f(S) \triangleq S \cap \{1\}$

F_f $^{\{S \mid S = f(S)\}}$

$fix(f)$

Example

$D = \wp(\mathbb{N})$

$f(S) \triangleq S \cap \{1\}$

$\overset{\{S \mid S = f(S)\}}{F_f}$

$\{ \varnothing , \{1\} \}$

$fix(f)$

Example

$D = \wp(\mathbb{N})$	$\overset{\{S \mid S = f(S)\}}{F_f}$	$fix(f)$
$f(S) \triangleq S \cap \{1\}$	$\{ \varnothing , \{1\} \}$	$\varnothing$

Example

$D = \wp(\mathbb{N})$	$F_f^{\{S \mid S = f(S)\}}$	$fix(f)$
$f(S) \triangleq S \cap \{1\}$	$\{ \varnothing , \{1\} \}$	$\varnothing$
$f(S) \triangleq \mathbb{N} \setminus S$		

Example

$D = \wp(\mathbb{N})$	$F_f^{\{S \mid S = f(S)\}}$	$fix(f)$
$f(S) \triangleq S \cap \{1\}$	$\{ \varnothing , \{1\} \}$	$\varnothing$
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Example

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Example

$D = \wp(\mathbb{N})$	$F_f^{\{S \mid S = f(S)\}}$	$fix(f)$
$f(S) \triangleq S \cap \{1\}$	$\{ \varnothing , \{1\} \}$	$\varnothing$
$f(S) \triangleq \mathbb{N} \setminus S$	$\varnothing$	
$f(S) \triangleq S \cup \{1\}$	$\{T \mid 1 \in T\}$	

Example

$D = \wp(\mathbb{N})$	$\overset{\{S \mid S = f(S)\}}{F_f}$	$fix(f)$
$f(S) \triangleq S \cap \{1\}$	$\{\, \varnothing \,,\, \{1\} \,\}$	$\varnothing$
$f(S) \triangleq \mathbb{N} \setminus S$	$\varnothing$	
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Example

$D = \wp(\mathbb{N})$	$\overset{\{S \mid S = f(S)\}}{F_f}$	$fix(f)$
$f(S) \triangleq S \cap \{1\}$	$\{\, \varnothing \,,\, \{1\} \,\}$	$\varnothing$
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$f(S) \triangleq S \cup \{1\}$	$\{T \mid 1 \in T\}$	$\{1\}$
$f(S) \triangleq \{n \mid \exists m \in S, n \leq m\}$		

Example

	$\{S \mid S = f(S)\}$ F_f	$fix(f)$
$D = \wp(\mathbb{N})$		
$f(S) \triangleq S \cap \{1\}$	$\{\varnothing, \{1\}\}$	$\varnothing$
$f(S) \triangleq \mathbb{N} \setminus S$	$\varnothing$	
$f(S) \triangleq S \cup \{1\}$	$\{T \mid 1 \in T\}$	$\{1\}$
$f(S) \triangleq \{n \mid \exists m \in S, n \leq m\}$	$\{[0,k] \mid k \in \mathbb{N}\} \cup \{\varnothing, \mathbb{N}\}$	

Example

	$\{S \mid S = f(S)\}$ F_f	$fix(f)$
$D = \wp(\mathbb{N})$		
$f(S) \triangleq S \cap \{1\}$	$\{\varnothing, \{1\}\}$	$\varnothing$
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$f(S) \triangleq S \cup \{1\}$	$\{T \mid 1 \in T\}$	$\{1\}$
$f(S) \triangleq \{n \mid \exists m \in S, n \leq m\}$	$\{[0,k] \mid k \in \mathbb{N}\} \cup \{\varnothing, \mathbb{N}\}$	$\varnothing$

Ingredients

a partial order (to compare elements)

order preserving functions

iterative approximations

a base case

a limit solution

limit preserving functions

Partial orders

Partially ordered set (Poset or just PO)

a set

a binary relation

$$(P, \sqsubseteq) \qquad \sqsubseteq \subseteq P \times P$$

Partially ordered set (Poset or just PO)

a set

a binary relation

$$(P, \sqsubseteq) \qquad \sqsubseteq \subseteq P \times P$$

reflexive

Partially ordered set (Poset or just PO)

a set

a binary relation

$$(P, \sqsubseteq) \qquad \sqsubseteq \subseteq P \times P$$

reflexive $\forall p \in P.$ $p \sqsubseteq p$

Partially ordered set (Poset or just PO)

a set

a binary relation

$$(P, \sqsubseteq) \qquad \sqsubseteq \subseteq P \times P$$

reflexive $\forall p \in P.$ $p \sqsubseteq p$

antisymmetric

Partially ordered set (Poset or just PO)

a set

a binary relation

$$(P, \sqsubseteq) \qquad \sqsubseteq \subseteq P \times P$$

reflexive	$\forall p \in P.$	$p \sqsubseteq p$
antisymmetric	$\forall p, q \in P.$	$p \sqsubseteq q \ \wedge \ q \sqsubseteq p \ \Rightarrow \ p = q$

Partially ordered set (Poset or just PO)

a set

a binary relation

$$(P, \sqsubseteq) \qquad \sqsubseteq \subseteq P \times P$$

reflexive	$\forall p \in P.$	$p \sqsubseteq p$
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transitive		

Partially ordered set (Poset or just PO)

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transitive	$\forall p, q, r \in P.$	$p \sqsubseteq q \ \wedge \ q \sqsubseteq r \ \Rightarrow \ p \sqsubseteq r$

Partially ordered set (Poset or just PO)

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$$(P, \sqsubseteq) \qquad \sqsubseteq \subseteq P \times P$$

reflexive	$\forall p \in P.$	$p \sqsubseteq p$
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$p \sqsubseteq q$ means that p and q are **comparable** and that p is less than (or equal to) q

Partially ordered set (Poset or just PO)

a set

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$p \sqsubseteq q$ means that p and q are **comparable** and that p is less than (or equal to) q

$p \sqsubset q$ means $p \sqsubseteq q \ \wedge \ p \neq q$

Partially ordered set (Poset or just PO)

a set

a binary relation

$(P, \sqsubseteq)$ $\quad \sqsubseteq \subseteq P \times P$

reflexive	$\forall p \in P.$	$p \sqsubseteq p$
antisymmetric	$\forall p, q \in P.$	$p \sqsubseteq q \ \wedge \ q \sqsubseteq p \ \Rightarrow \ p = q$
transitive	$\forall p, q, r \in P.$	$p \sqsubseteq q \ \wedge \ q \sqsubseteq r \ \Rightarrow \ p \sqsubseteq r$



$p \sqsubseteq q$ means that p and q are **comparable** and that p is less than (or equal to) q

$p \sqsubset q$ means $p \sqsubseteq q \ \wedge \ p \neq q$

Total orders

$(P, \sqsubseteq)$ PO

total

Total orders

$(P, \sqsubseteq)$ PO

total $\forall p, q \in P. \quad p \sqsubseteq q \ \vee \ q \sqsubseteq p$

Total orders

$(P, \sqsubseteq)$ PO

total $\forall p, q \in P. \quad p \sqsubseteq q \ \vee \ q \sqsubseteq p$

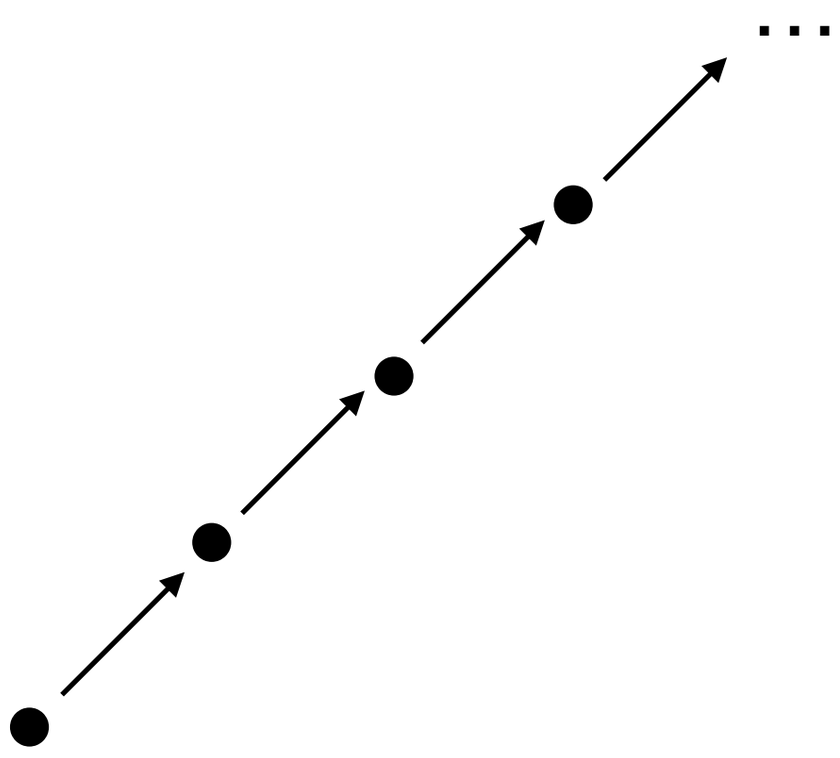
a PO where any two elements are **comparable**

Total orders

$(P, \sqsubseteq)$ PO

total $\forall p, q \in P. \quad p \sqsubseteq q \ \vee \ q \sqsubseteq p$

a PO where any two elements are **comparable**

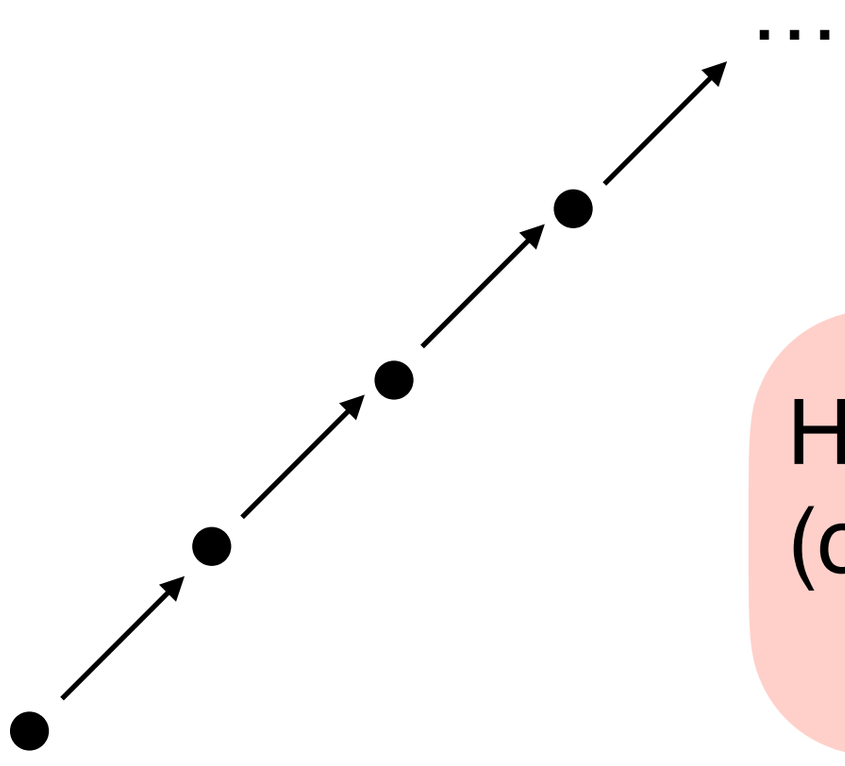


Total orders

$(P, \sqsubseteq)$ PO

total $\forall p, q \in P. \quad p \sqsubseteq q \ \vee \ q \sqsubseteq p$

a PO where any two elements are **comparable**



Hasse diagram notation
(omit: reflexive arcs,
transitive arcs)

Discrete orders

$(P, \sqsubseteq)$ PO

discrete

Discrete orders

$(P, \sqsubseteq)$ PO

discrete $\forall p, q \in P. \quad p \sqsubseteq q \Leftrightarrow p = q$

Discrete orders

$(P, \sqsubseteq)$ PO

discrete $\forall p, q \in P. \quad p \sqsubseteq q \Leftrightarrow p = q$

each element is **comparable** only to itself

Discrete orders

$(P, \sqsubseteq)$ PO

discrete $\forall p, q \in P. \quad p \sqsubseteq q \Leftrightarrow p = q$

each element is **comparable** only to itself



Flat orders

$(P, \sqsubseteq)$ PO

flat

Flat orders

$(P, \sqsubseteq)$ PO

flat $\forall p, q \in P. \quad p \sqsubseteq q \;\Leftrightarrow\; p = q \;\vee\; p = \bot$

Flat orders

$(P, \sqsubseteq)$ PO

flat $\quad \forall p, q \in P. \quad p \sqsubseteq q \;\Leftrightarrow\; p = q \;\vee\; p = \bot$

each element is **comparable** only to itself

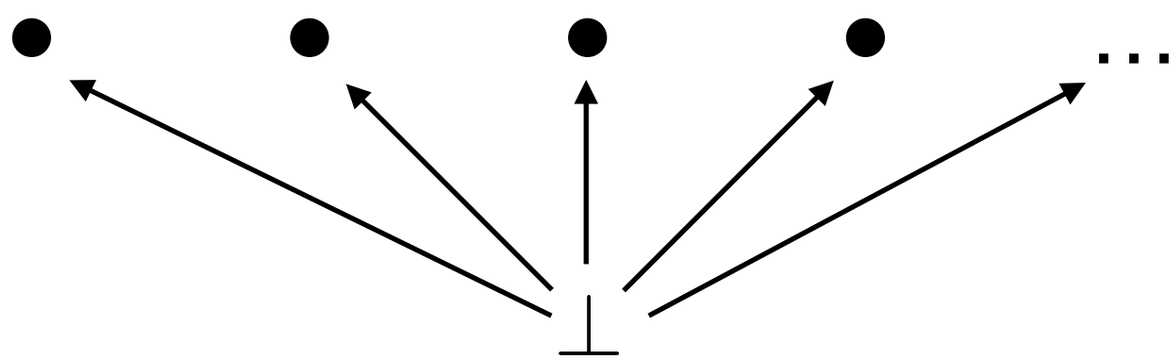
and with a distinguished (smaller) element $\bot$

Flat orders

$(P, \sqsubseteq)$ PO

flat $\quad \forall p, q \in P. \quad p \sqsubseteq q \;\Leftrightarrow\; p = q \;\vee\; p = \bot$

each element is **comparable** only to itself and with a distinguished (smaller) element $\bot$



Exercise

$(\mathbb{N}, \leq)$ PO? Total? Discrete? Flat?



Exercise

$(\mathbb{N}, \leq)$ PO? Total? Discrete? Flat?

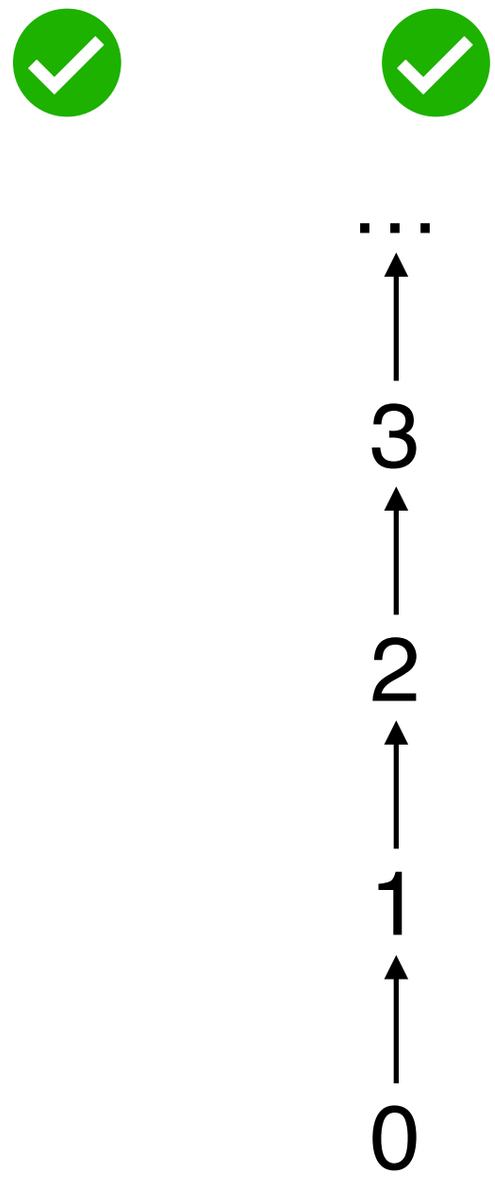




Exercise

$(\mathbb{N}, \leq)$

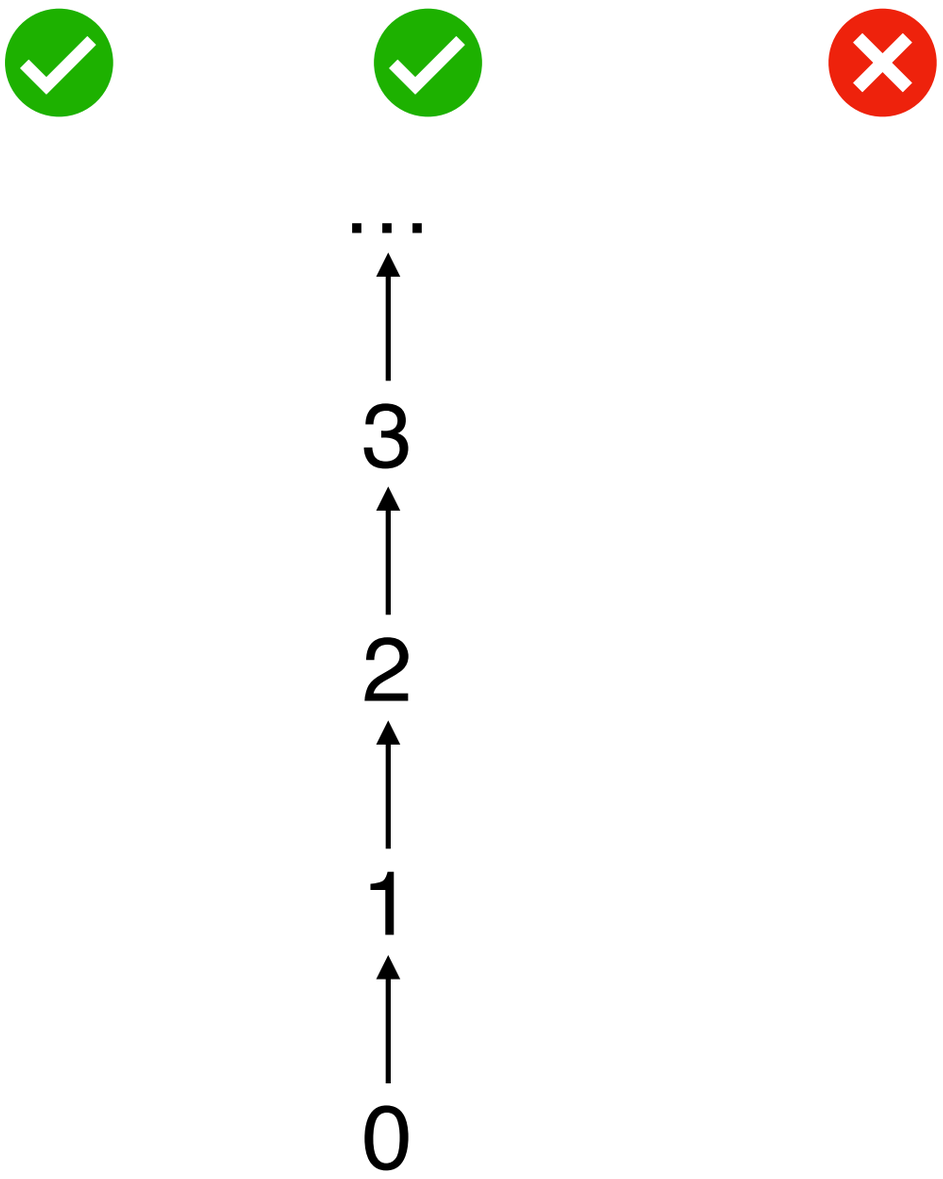
PO? Total? Discrete? Flat?



Exercise

$(\mathbb{N}, \leq)$

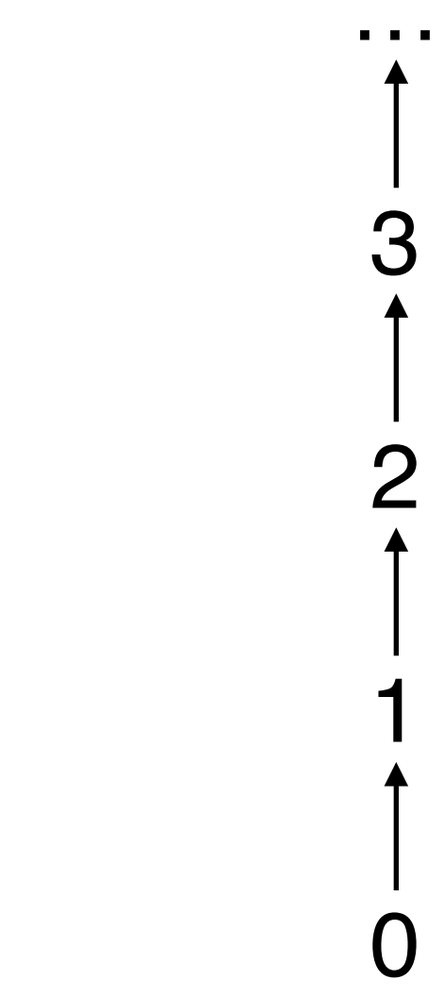
PO? Total? Discrete? Flat?



Exercise

$(\mathbb{N}, \leq)$

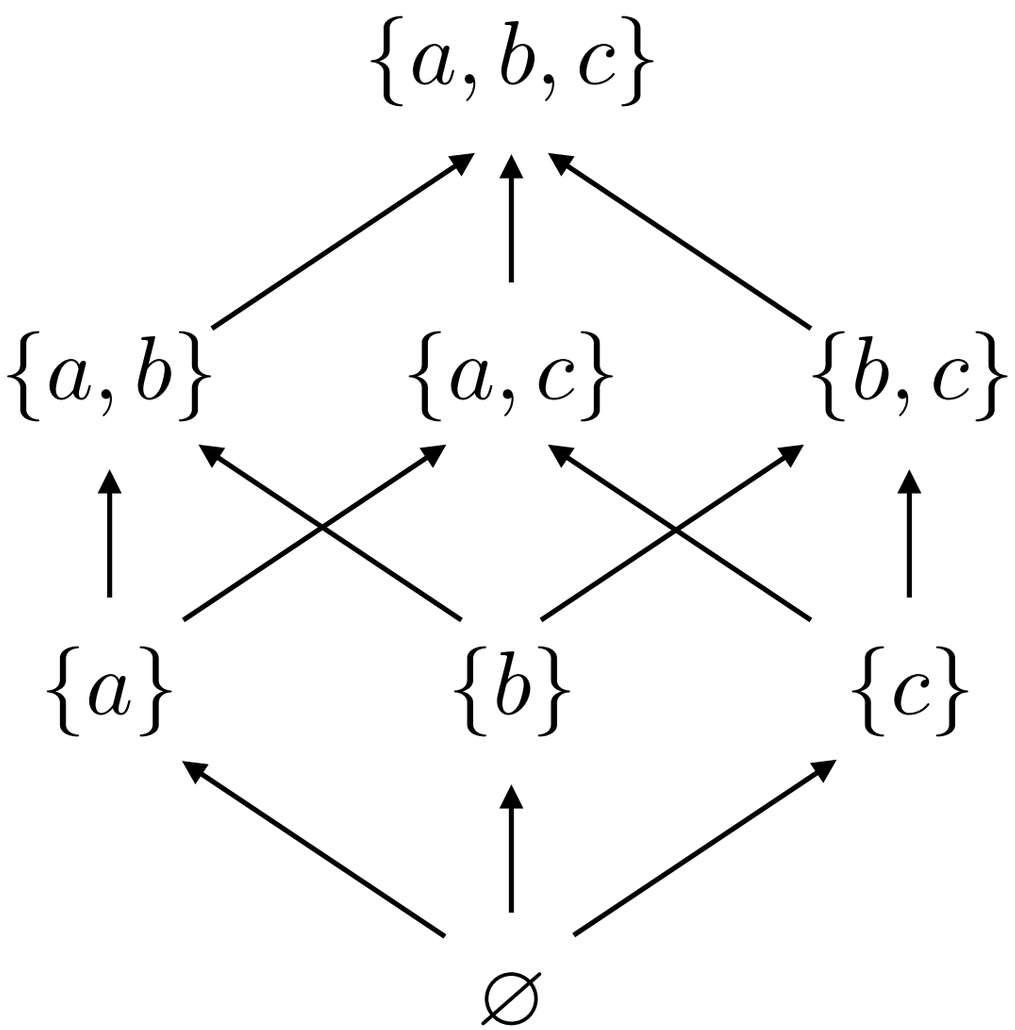
PO? ✅ Total? ✅ Discrete? ❌ Flat? ❌



Exercise

$(\wp(S), \subseteq)$ PO? Total? Discrete? Flat?

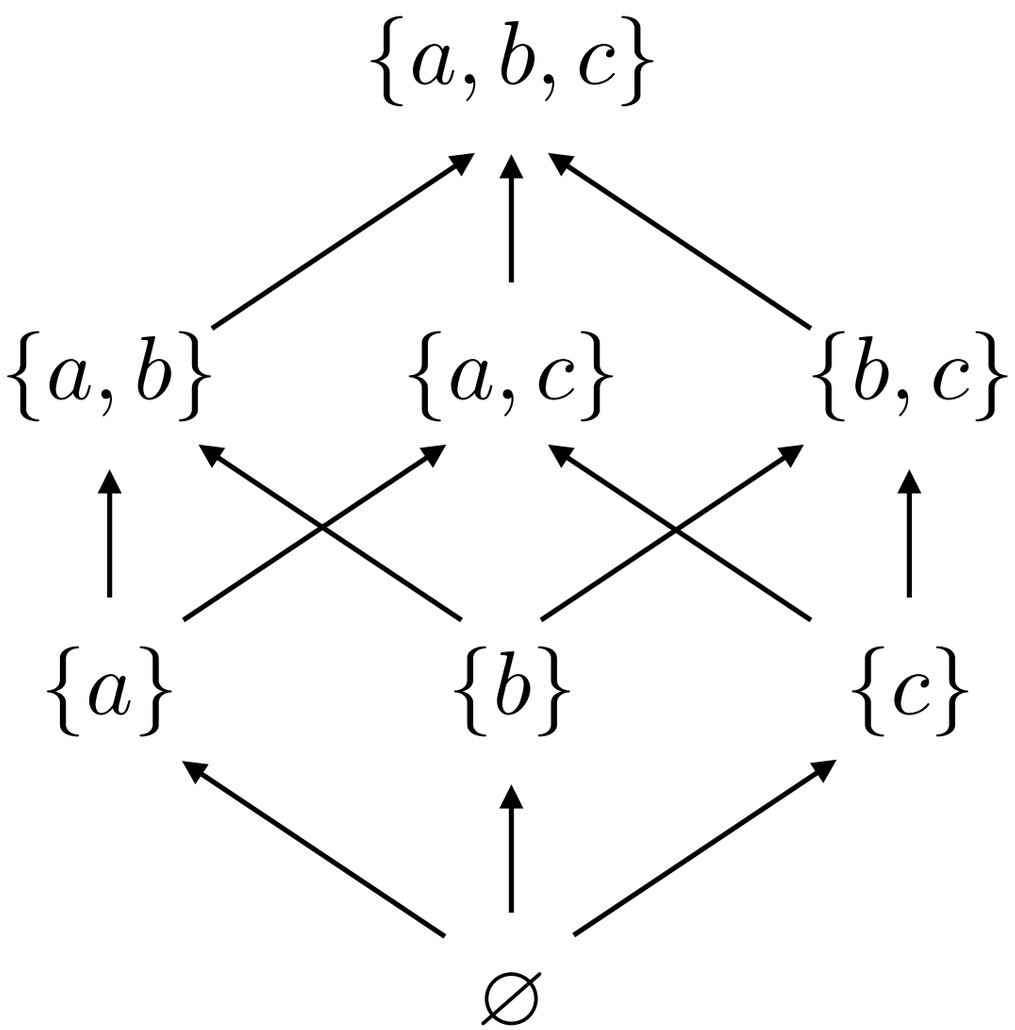
example: $S = \{a, b, c\}$



Exercise

$(\wp(S), \subseteq)$ PO? Total? Discrete? Flat?

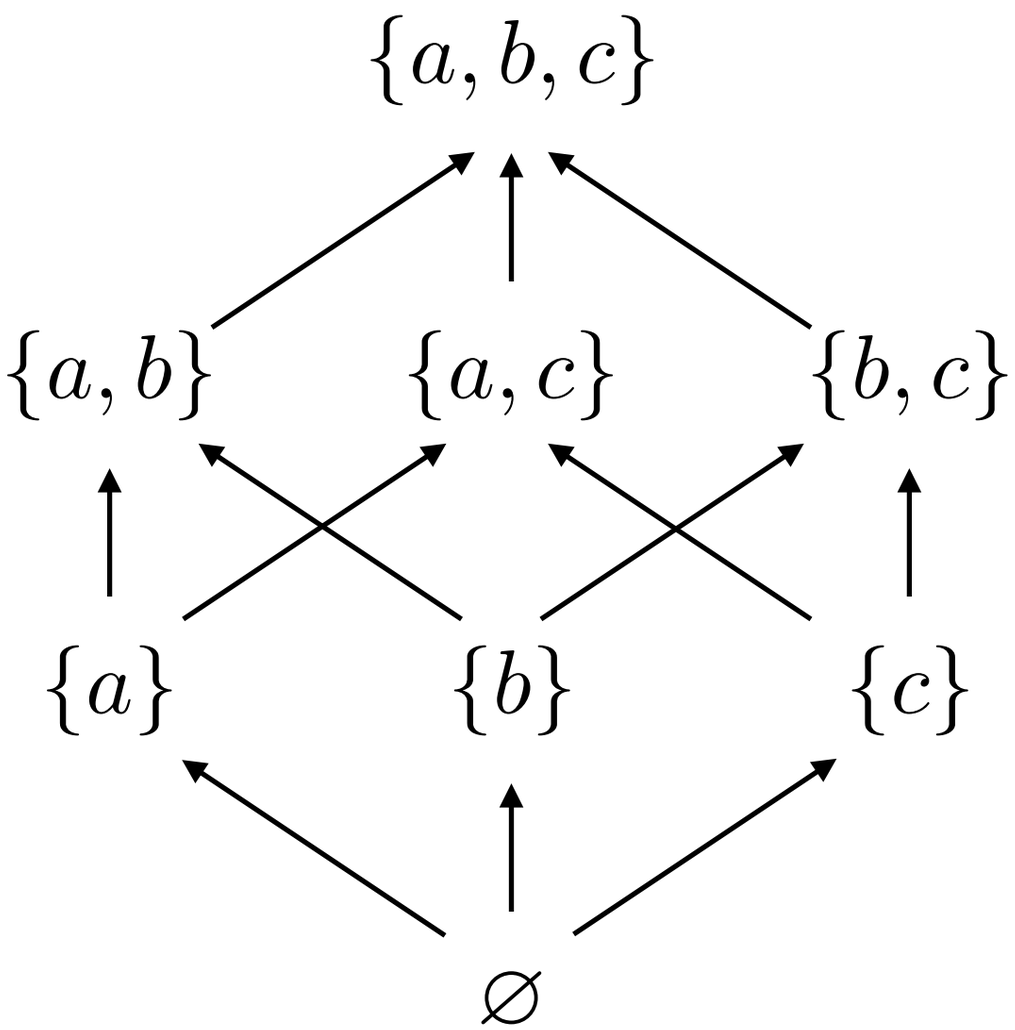
example: $S = \{a, b, c\}$



Exercise

$(\wp(S), \subseteq)$ PO? Total? Discrete? Flat?

example: $S = \{a, b, c\}$



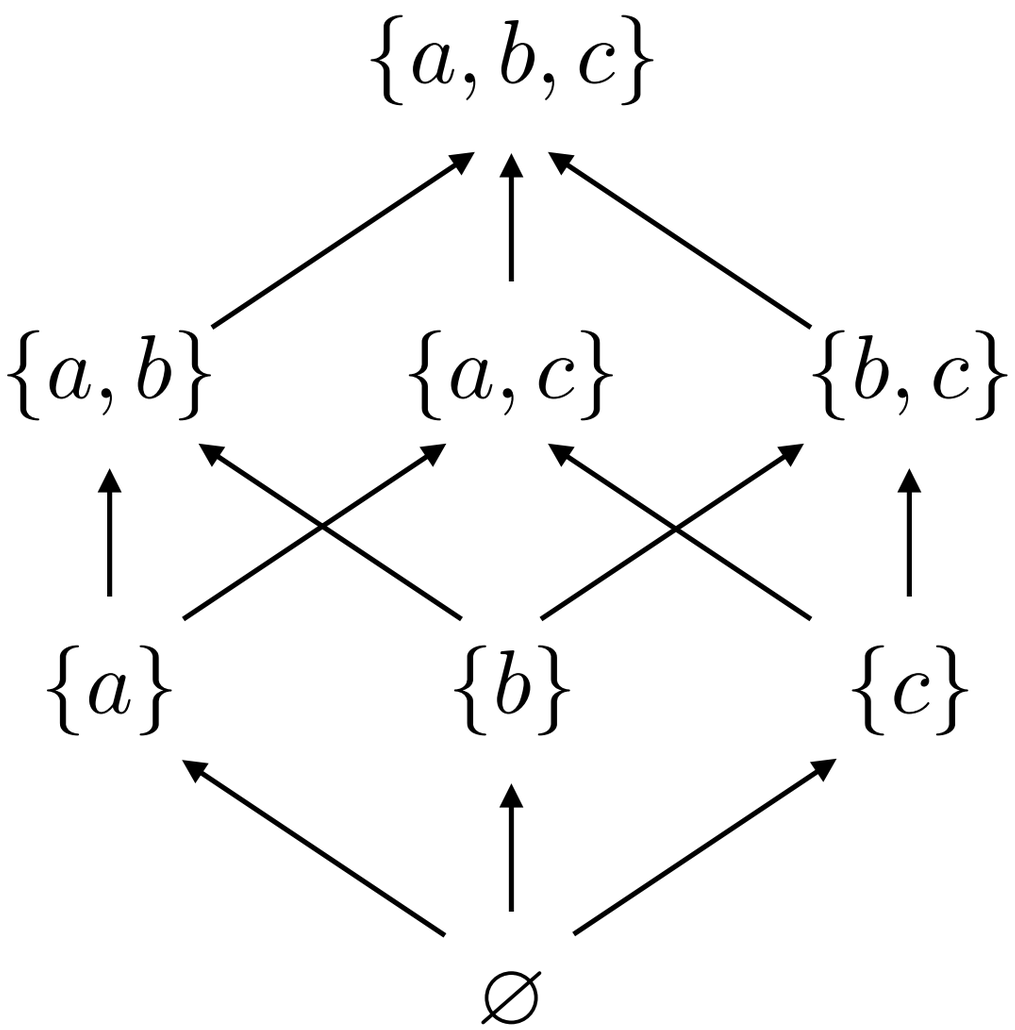
$\{a, b\} \not\subseteq \{b, c\}$

$\{a, b\} \not\supseteq \{b, c\}$

Exercise

$(\wp(S), \subseteq)$ PO? Total? Discrete? Flat?

example: $S = \{a, b, c\}$



$\{a, b\} \not\subseteq \not\supseteq \{b, c\}$

$\{a\} \not\subseteq \not\supseteq \{b\}$

Exercise

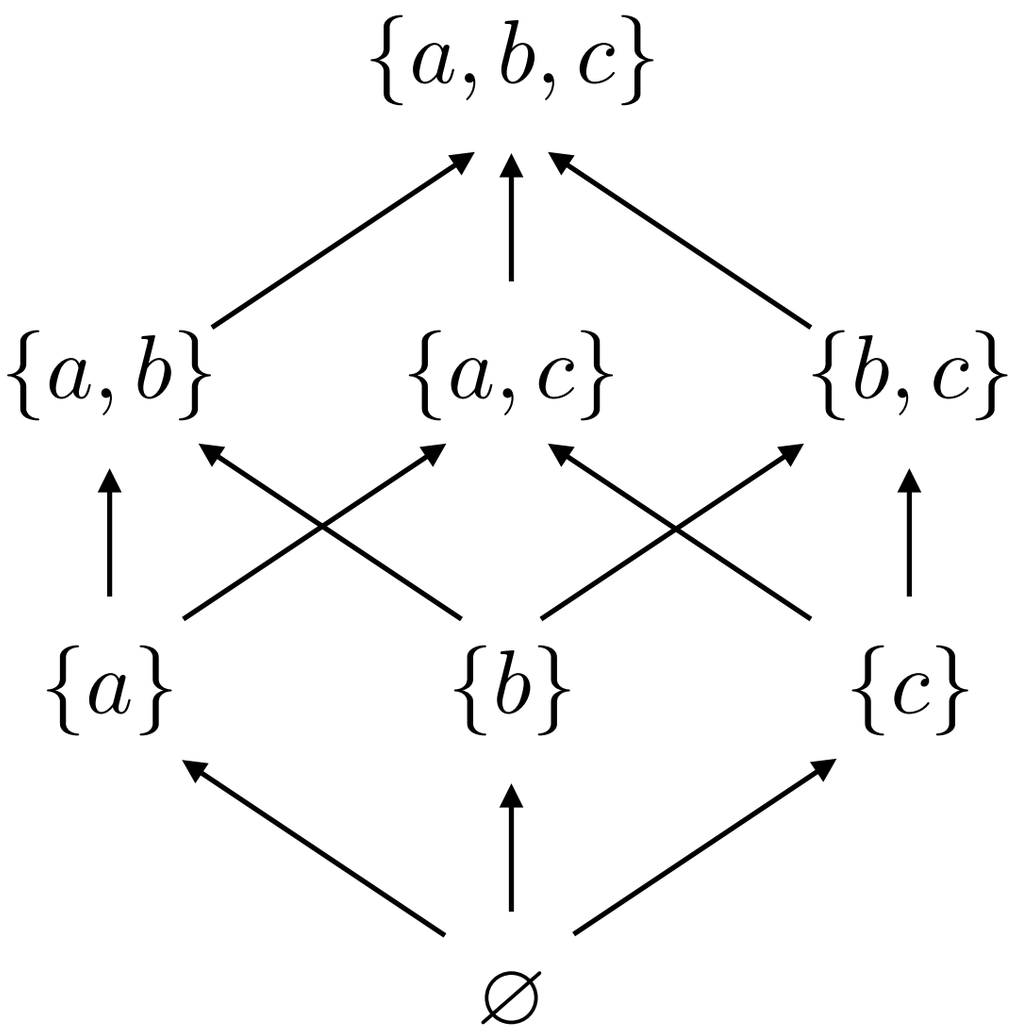
$(\wp(S), \subseteq)$

PO? Total? Discrete? Flat?



$|S| < 2$

example: $S = \{a, b, c\}$



$\{a, b\} \not\subseteq \not\supseteq \{b, c\}$

$\{a\} \not\subseteq \not\supseteq \{b\}$

Exercise

$(\wp(S), \subseteq)$

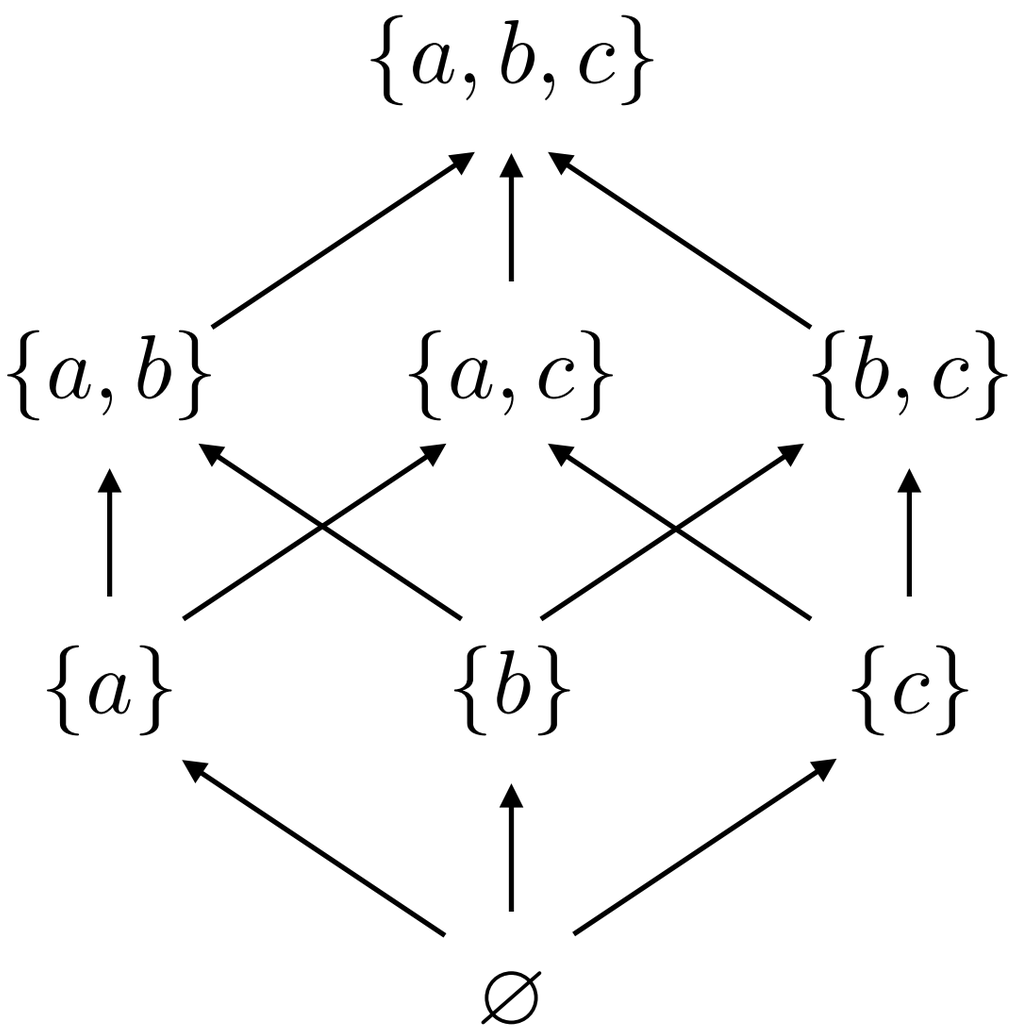
PO? ✅

Total? $|S| < 2$

Discrete?

Flat? $\wp(\varnothing) = \{\varnothing\}$

example: $S = \{a, b, c\}$



$\{a, b\} \not\subseteq \not\supseteq \{b, c\}$

$\{a\} \not\subseteq \not\supseteq \{b\}$

Exercise

$(\wp(S), \subseteq)$

PO? Total? Discrete? Flat?

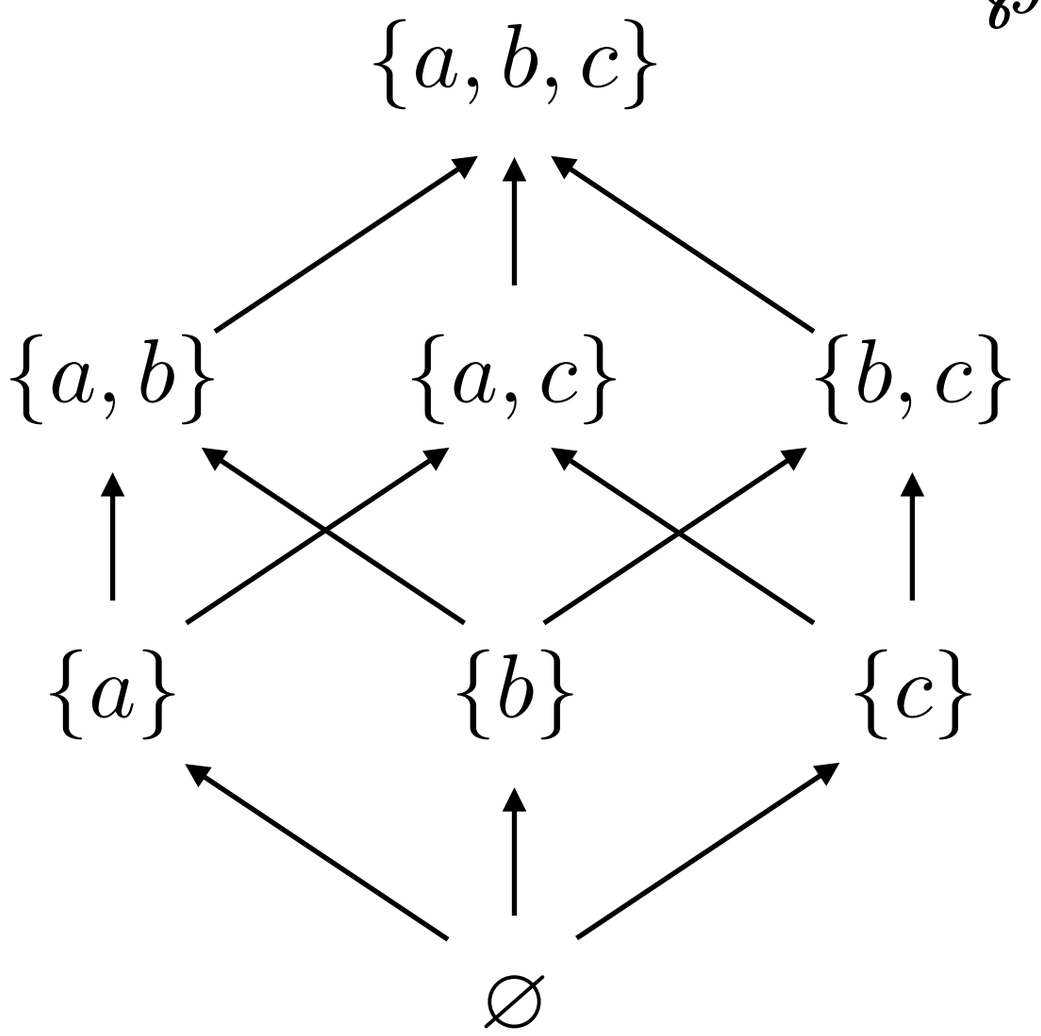


$|S| < 2$

example: $S = \{a, b, c\}$

$\wp(\varnothing) = \{\varnothing\}$

$\wp(\{a\}) = \{\varnothing, \{a\}\}$



$\{a, b\} \not\subseteq \not\supseteq \{b, c\}$

$\{a\} \not\subseteq \not\supseteq \{b\}$

Exercise

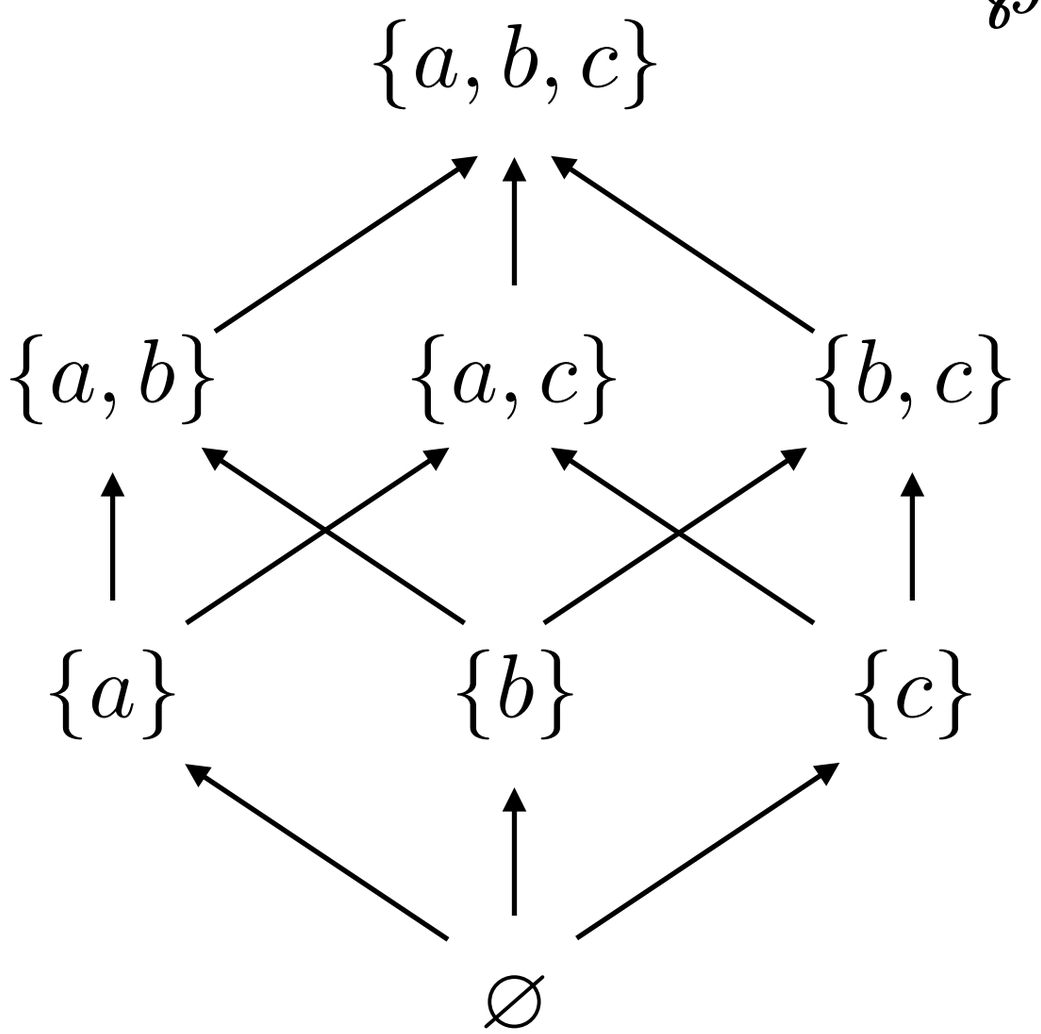
$(\wp(S), \subseteq)$

PO?	Total?	Discrete?	Flat?
✅	$\lvert S\rvert < 2$	$S = \varnothing$	

example: $S = \{a, b, c\}$

$\wp(\varnothing) = \{\varnothing\}$

$\wp(\{a\}) = \{\varnothing, \{a\}\}$



$\{a, b\} \not\subseteq \not\supseteq \{b, c\}$

$\{a\} \not\subseteq \not\supseteq \{b\}$

Exercise

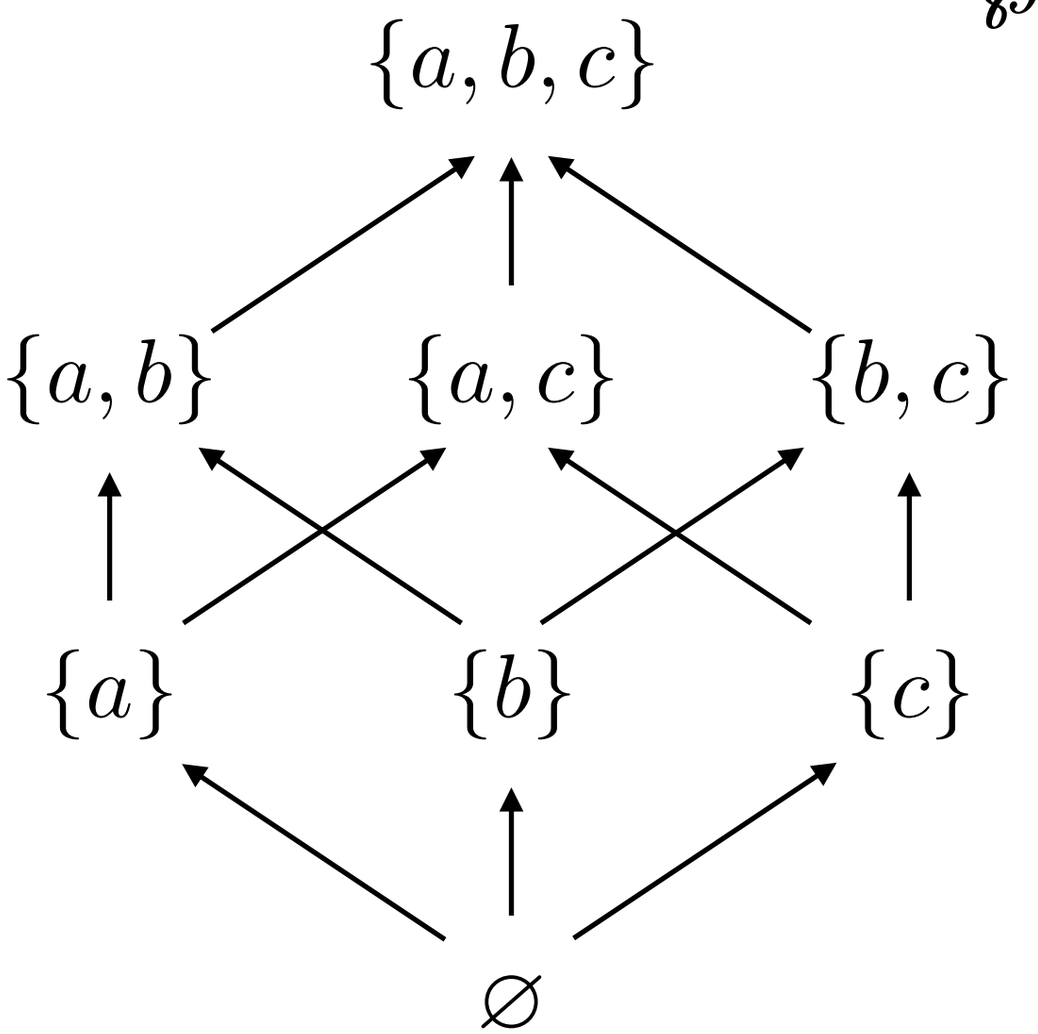
$(\wp(S), \subseteq)$

PO?	Total?	Discrete?	Flat?
✓	$\lvert S\rvert < 2$	$S = \varnothing$	$\lvert S\rvert < 2$

example: $S = \{a, b, c\}$

$\wp(\varnothing) = \{\varnothing\}$

$\wp(\{a\}) = \{\varnothing, \{a\}\}$



$\{a, b\} \not\subseteq \not\supseteq \{b, c\}$

$\{a\} \not\subseteq \not\supseteq \{b\}$

Exercise

$(\mathbb{N}, =)$ PO? Total? Discrete? Flat?

Exercise

$(\mathbb{N}, =)$ PO? Total? Discrete? Flat?



Exercise

$(\mathbb{N}, =)$ PO? Total? Discrete? Flat?





Exercise

$(\mathbb{N}, =)$

PO? 

Total? 

Discrete? 

Flat?

Exercise

$(\mathbb{N}, =)$

PO? 

Total? 

Discrete? 

Flat? 

Exercise

$(\mathbb{N}, =)$

PO? Total? Discrete? Flat?



0 1 2 3 ...

Exercise

$(\mathbb{N} \cup \{\bot\}, \{(\bot, n) \mid n \in \mathbb{N}\}^*)$

PO? Total? Discrete? Flat?

Exercise

$(\mathbb{N} \cup \{\bot\}, \{(\bot, n) \mid n \in \mathbb{N}\}^*)$

PO? Total? Discrete? Flat?



Exercise

$(\mathbb{N} \cup \{\bot\}, \{(\bot, n) \mid n \in \mathbb{N}\}^*)$

PO? Total? Discrete? Flat?




Exercise

$(\mathbb{N} \cup \{\bot\}, \{(\bot, n) \mid n \in \mathbb{N}\}^*)$

PO? Total? Discrete? Flat?





Exercise

$(\mathbb{N} \cup \{\bot\}, \{(\bot, n) \mid n \in \mathbb{N}\}^*)$

PO? Total? Discrete? Flat?

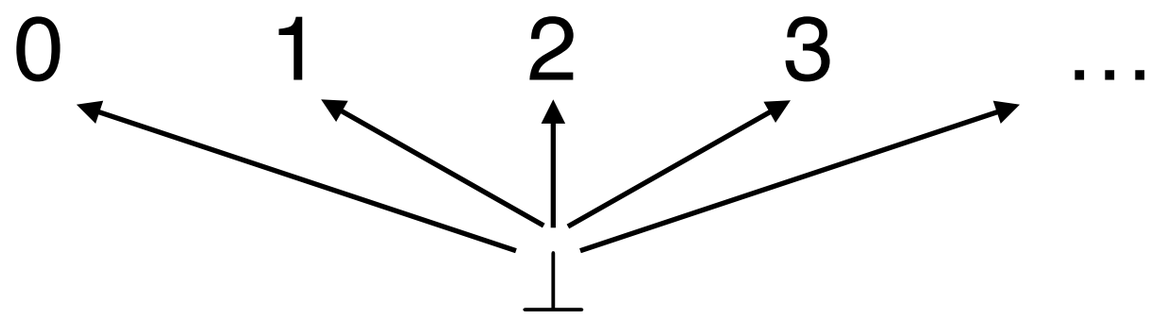





Exercise

$(\mathbb{N} \cup \{\bot\}, \{(\bot, n) \mid n \in \mathbb{N}\}^*)$

PO? ✓ Total? ✗ Discrete? ✗ Flat? ✓



Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$ PO? Total? Discrete? Flat?

Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$ PO? Total? Discrete? Flat?



Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$ PO? Total? Discrete? Flat?





Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

PO? 

Total? 

Discrete? 

Flat?

Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

PO? 

Total? 

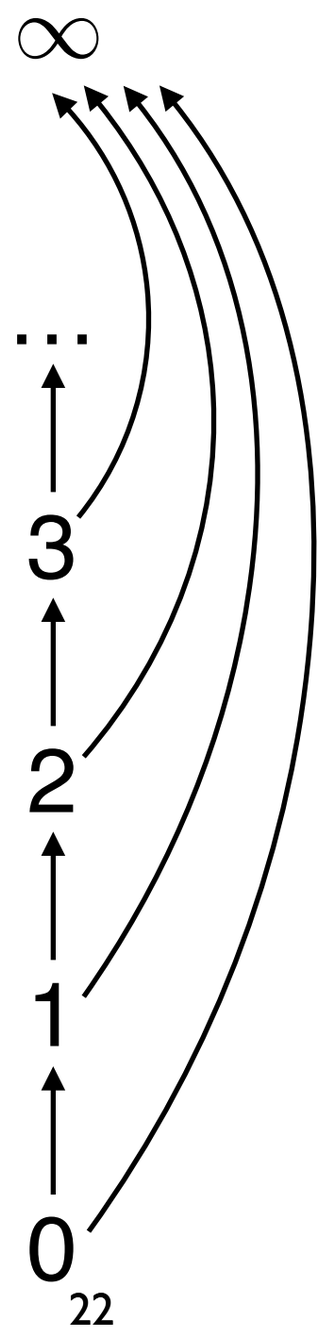
Discrete? 

Flat? 

Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

PO? ✓ Total? ✓ Discrete? ✗ Flat? ✗



Exercise

	PO?	Total?	Discrete?	Flat?
$(\mathbb{N}, <)$				
$(\mathbb{Z}, \leq)$				
$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$				
$(T_\Sigma, \prec)$				
$(\mathbb{N}, \neq)$				

Exercise

	PO?	Total?	Discrete?	Flat?
$(\mathbb{N}, <)$	✗			
$(\mathbb{Z}, \leq)$				
$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$				
$(T_\Sigma, \prec)$				
$(\mathbb{N}, \neq)$				

Exercise

	PO?	Total?	Discrete?	Flat?
$(\mathbb{N}, <)$	✗	✗	✗	✗
$(\mathbb{Z}, \leq)$				
$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$				
$(T_\Sigma, \prec)$				
$(\mathbb{N}, \neq)$				

Exercise

	PO?	Total?	Discrete?	Flat?
$(\mathbb{N}, <)$	✗	✗	✗	✗
$(\mathbb{Z}, \leq)$	✓			

$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$

$(T_\Sigma, \prec)$

$(\mathbb{N}, \neq)$

Exercise

	PO?	Total?	Discrete?	Flat?
$(\mathbb{N}, <)$	✗	✗	✗	✗
$(\mathbb{Z}, \leq)$	✓	✓		

$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$

$(T_\Sigma, \prec)$

$(\mathbb{N}, \neq)$

Exercise

	PO?	Total?	Discrete?	Flat?
$(\mathbb{N}, <)$	✗	✗	✗	✗
$(\mathbb{Z}, \leq)$	✓	✓	✗	

$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$

$(T_\Sigma, \prec)$

$(\mathbb{N}, \neq)$

Exercise

	PO?	Total?	Discrete?	Flat?
$(\mathbb{N}, <)$	✗	✗	✗	✗
$(\mathbb{Z}, \leq)$	✓	✓	✗	✗

$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$

$(T_\Sigma, \prec)$

$(\mathbb{N}, \neq)$

Exercise

	PO?	Total?	Discrete?	Flat?
$(\mathbb{N}, <)$	✗	✗	✗	✗
$(\mathbb{Z}, \leq)$	✓	✓	✗	✗
$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$	✓			
$(T_\Sigma, \prec)$				
$(\mathbb{N}, \neq)$				

Exercise

	PO?	Total?	Discrete?	Flat?
$(\mathbb{N}, <)$	✗	✗	✗	✗
$(\mathbb{Z}, \leq)$	✓	✓	✗	✗
$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$	✓	✓		
$(T_\Sigma, \prec)$				
$(\mathbb{N}, \neq)$				

Exercise

	PO?	Total?	Discrete?	Flat?
$(\mathbb{N}, <)$	✗	✗	✗	✗
$(\mathbb{Z}, \leq)$	✓	✓	✗	✗
$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$	✓	✓	✗	
$(T_\Sigma, \prec)$				
$(\mathbb{N}, \neq)$				

Exercise

	PO?	Total?	Discrete?	Flat?
$(\mathbb{N}, <)$	✗	✗	✗	✗
$(\mathbb{Z}, \leq)$	✓	✓	✗	✗
$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$	✓	✓	✗	✗
$(T_\Sigma, \prec)$				
$(\mathbb{N}, \neq)$				

Exercise

	PO?	Total?	Discrete?	Flat?
$(\mathbb{N}, <)$	✗	✗	✗	✗
$(\mathbb{Z}, \leq)$	✓	✓	✗	✗
$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$	✓	✓	✗	✗
$(T_\Sigma, \prec)$	✗			
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Exercise

	PO?	Total?	Discrete?	Flat?
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$(\mathbb{N}, \neq)$	✗			

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Subset of a PO

$(P, \sqsubseteq_P)$ PO $\quad Q \subseteq P$

Subset of a PO

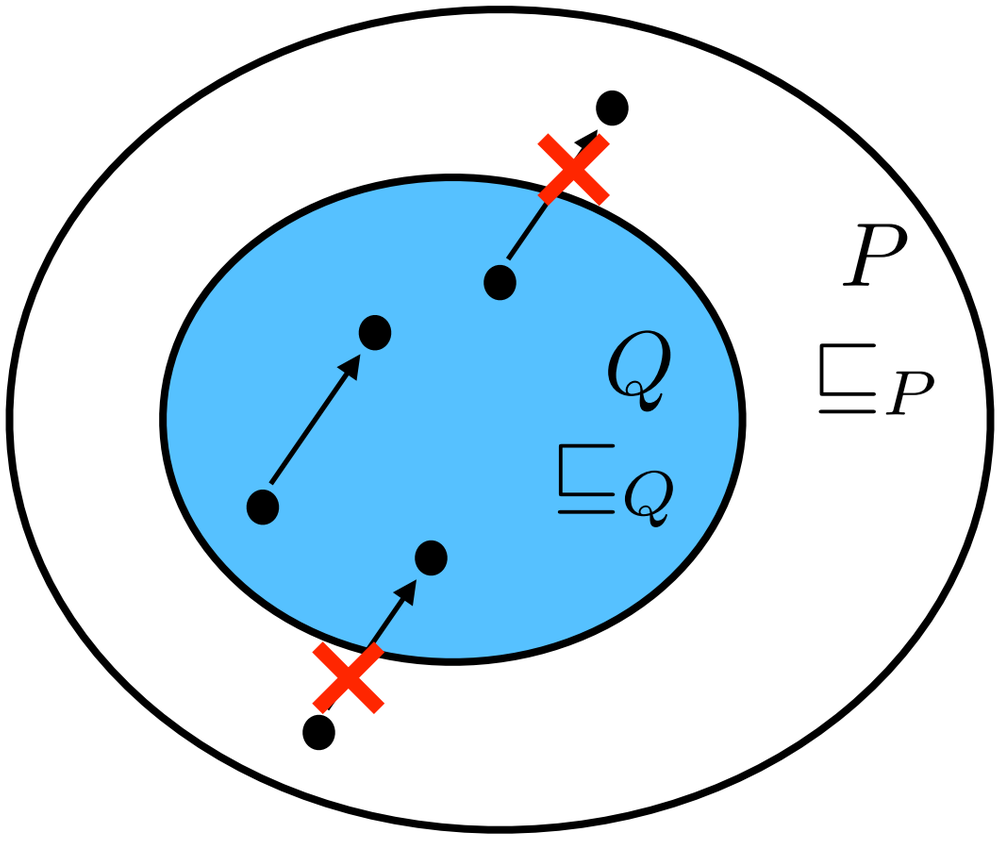
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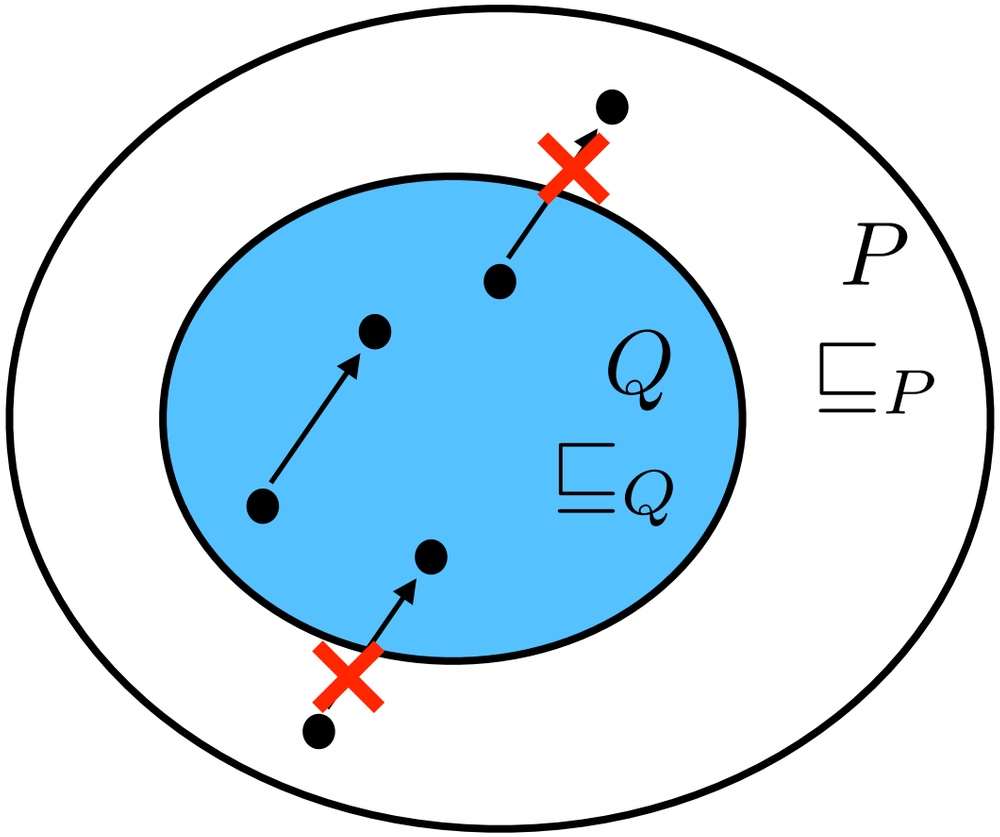


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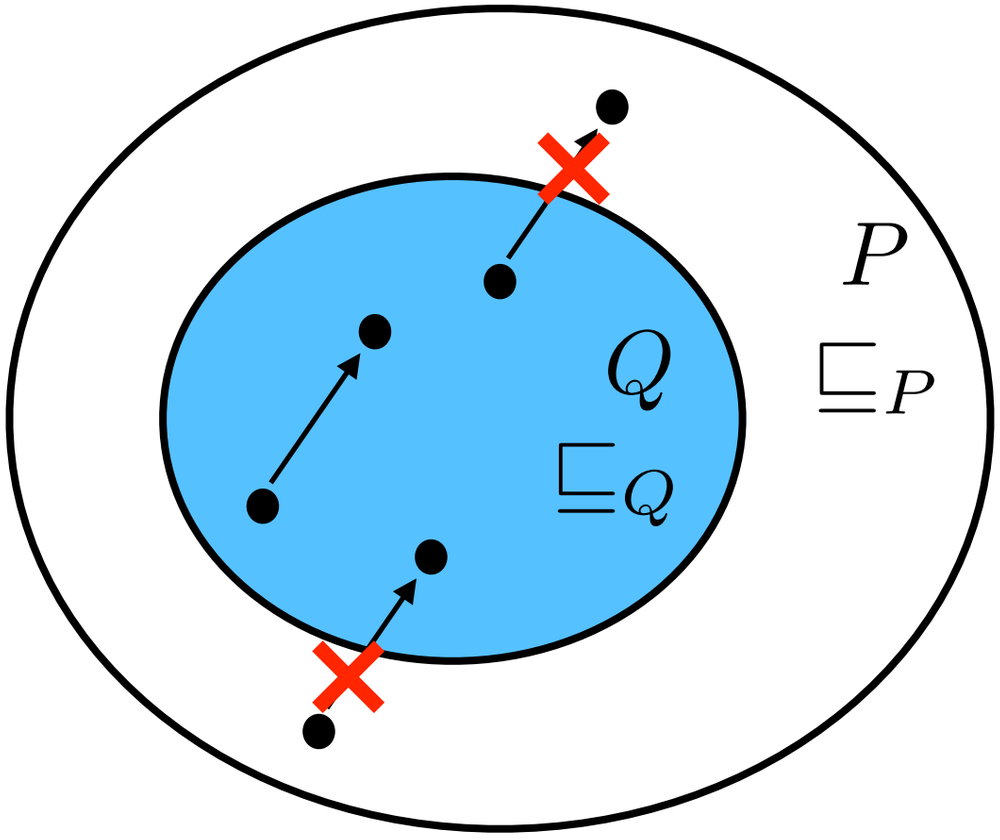
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TH. $(Q, \sqsubseteq_Q)$ is a PO

TH. if $(P, \sqsubseteq_P)$ is total, then $(Q, \sqsubseteq_Q)$ is total



PO $\sqsubseteq$	w.f. $\prec$
reflexive	

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Element properties (least, minimal, …)

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TH. (uniqueness of least element)

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by antisymmetry

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$(P, \sqsubseteq)$ PO

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bottom?

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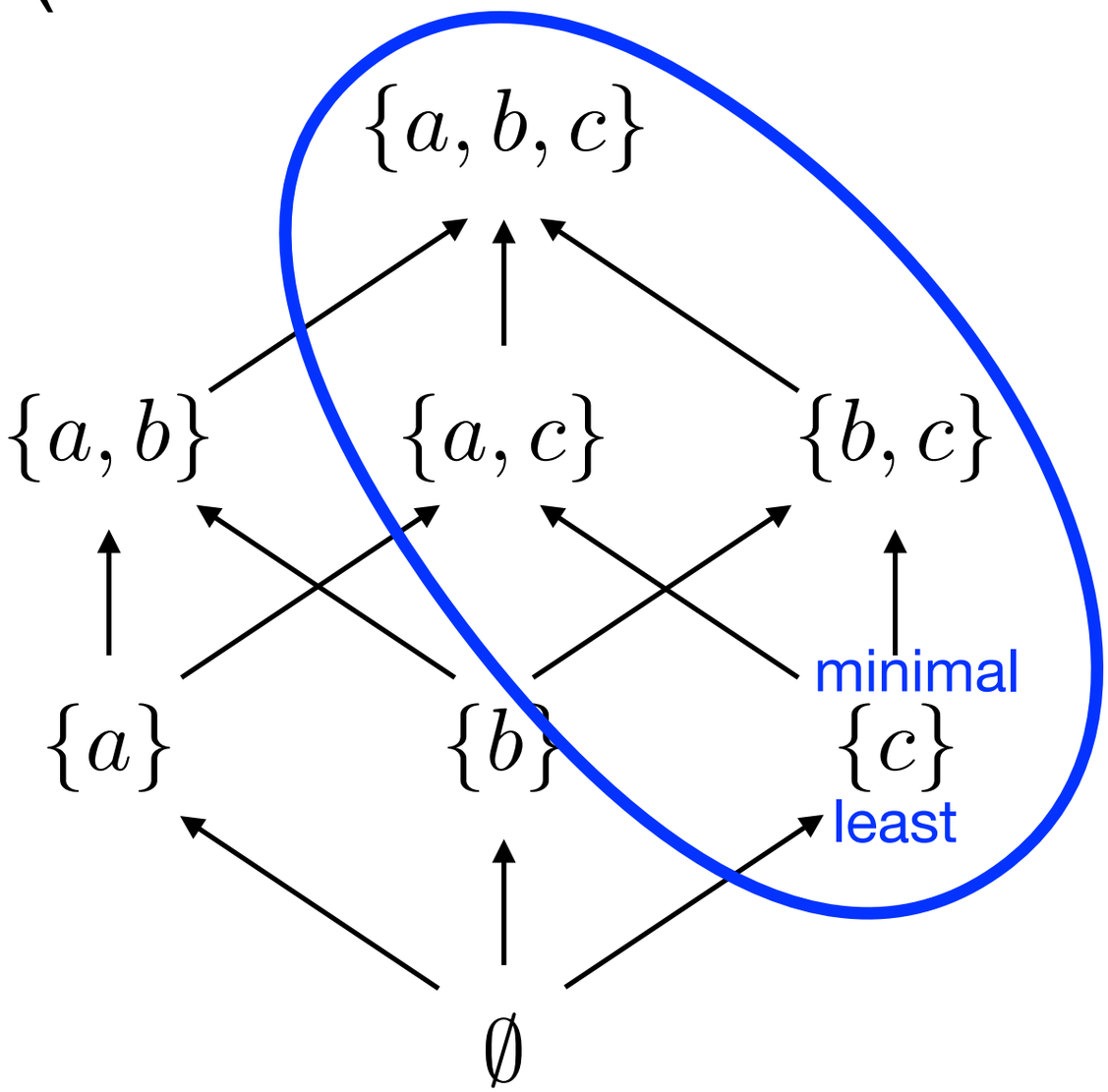
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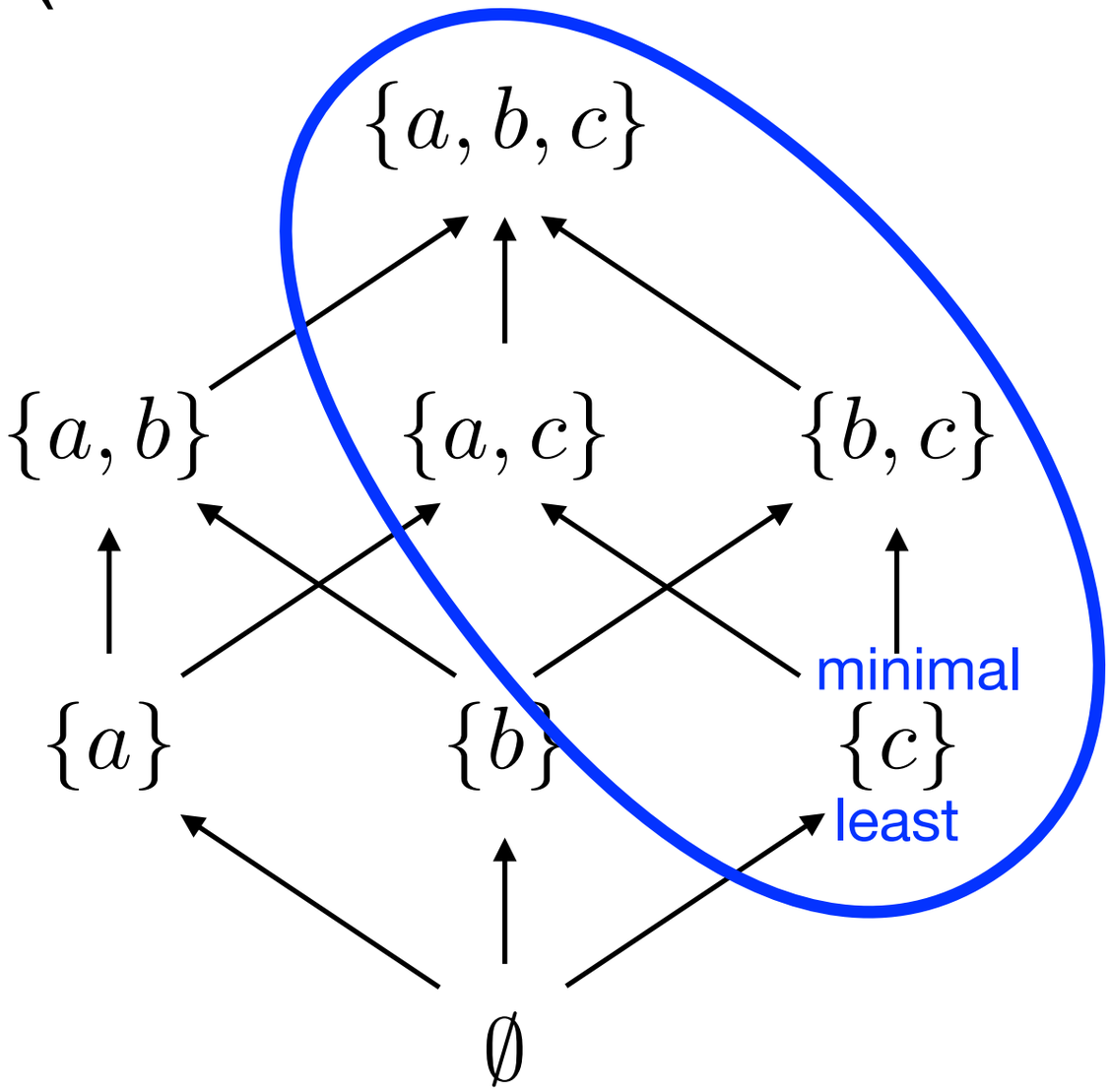


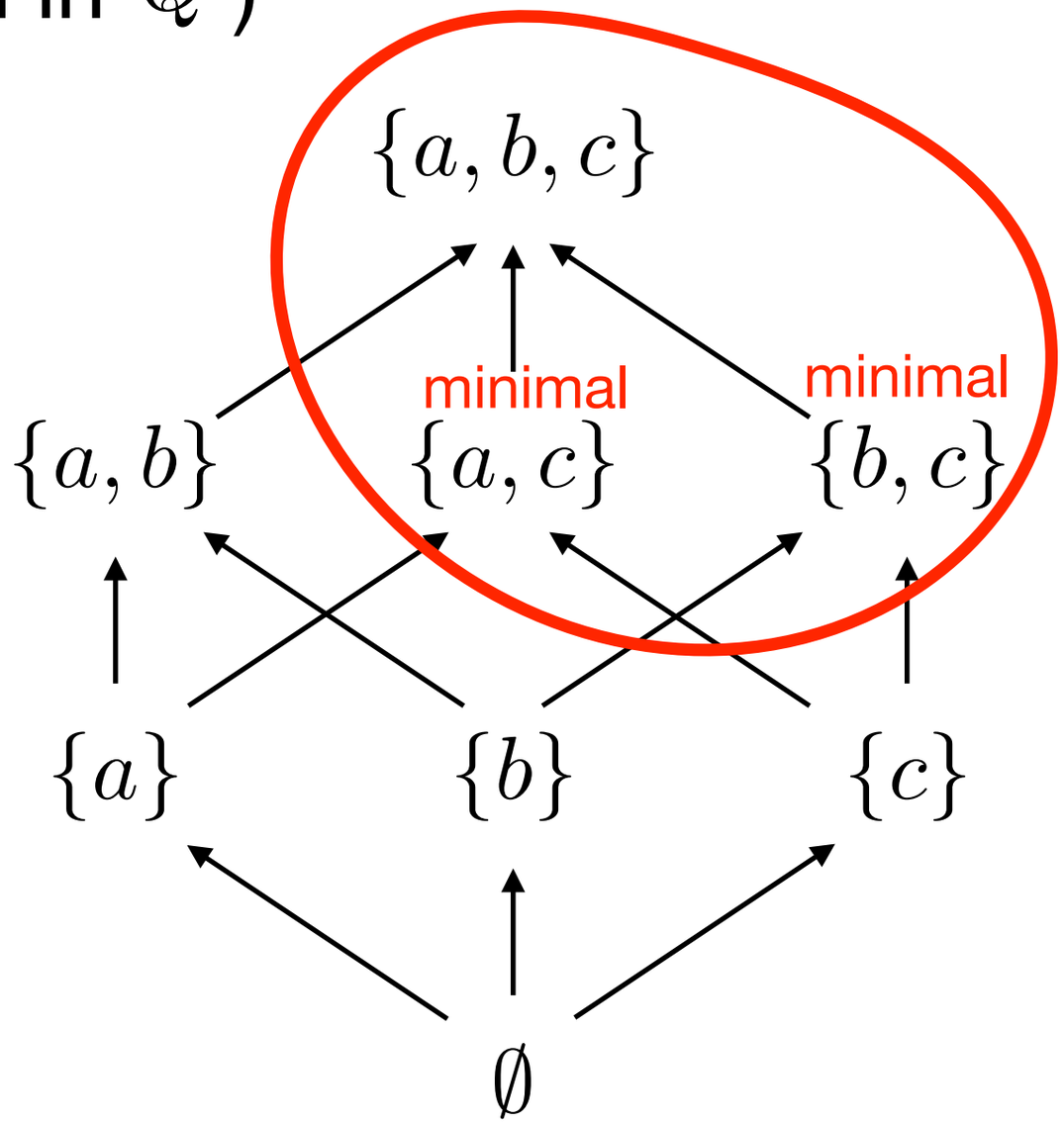
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TH. $(P, \sqsubseteq)$ PO implies $(P, \sqsupseteq)$ PO

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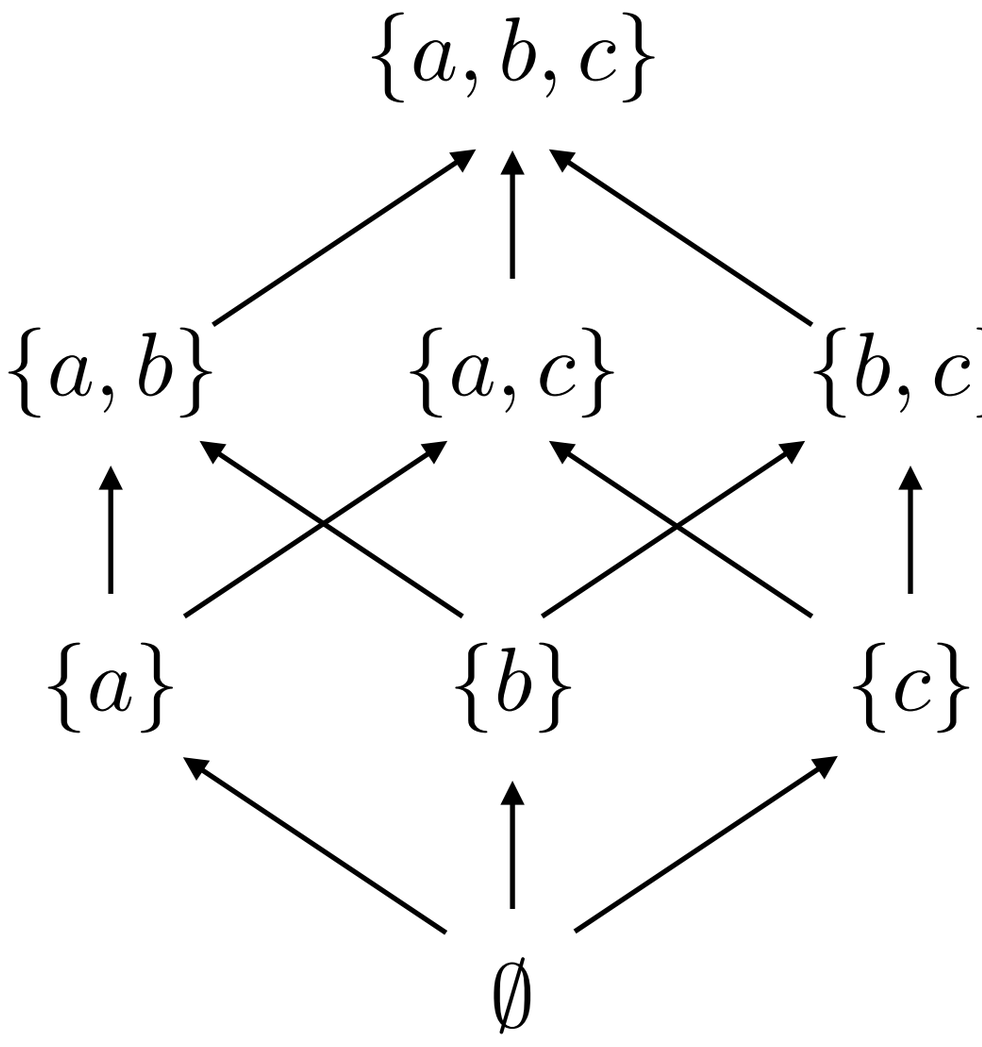
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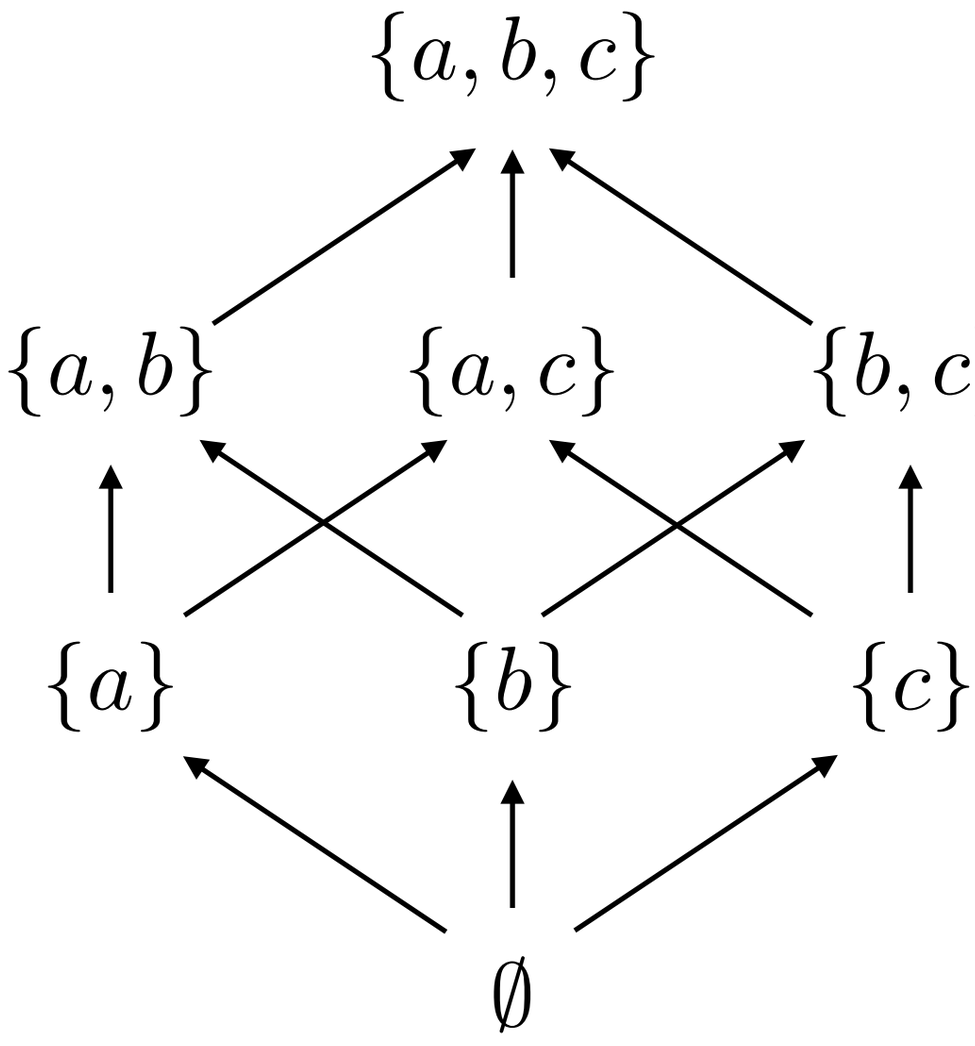
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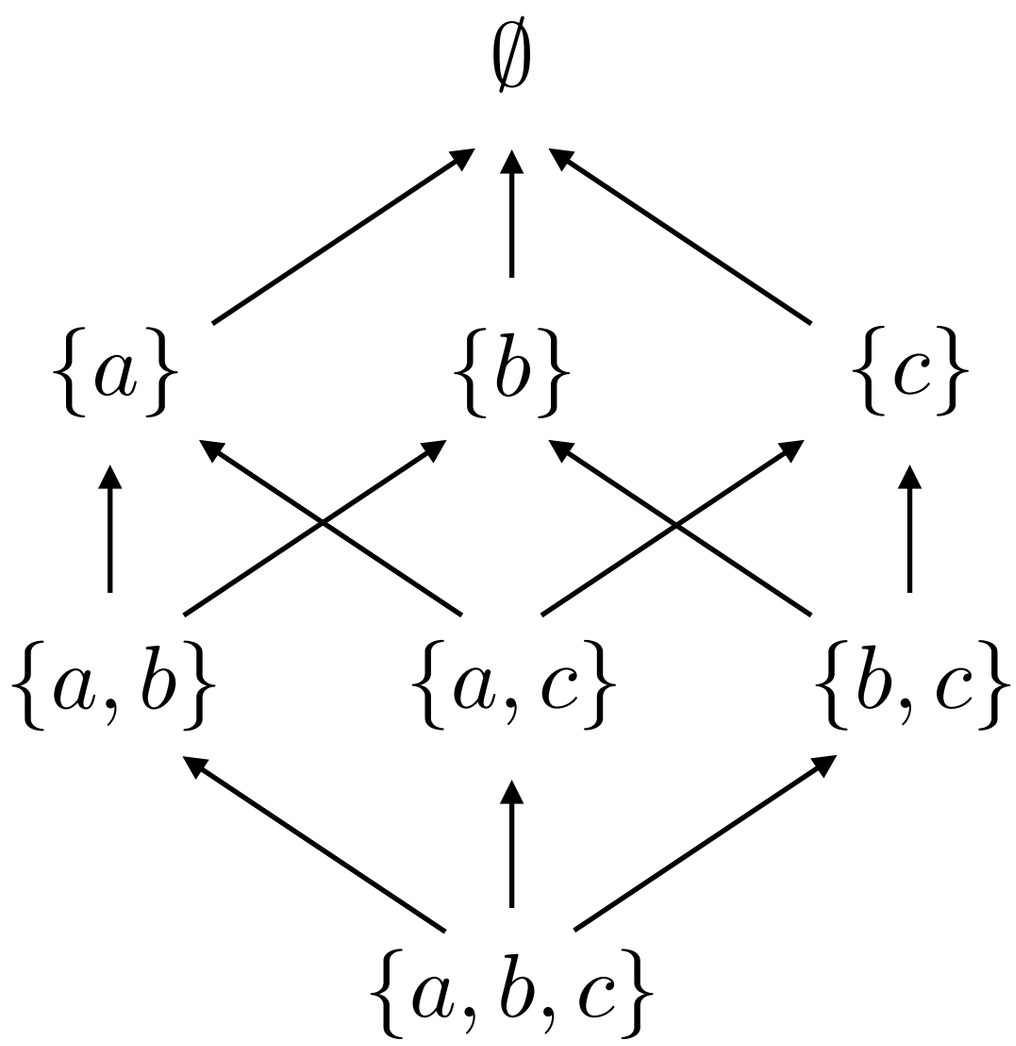
maximal element: minimal element of Q w.r.t. $(P, \sqsupseteq)$

Reversed powerset

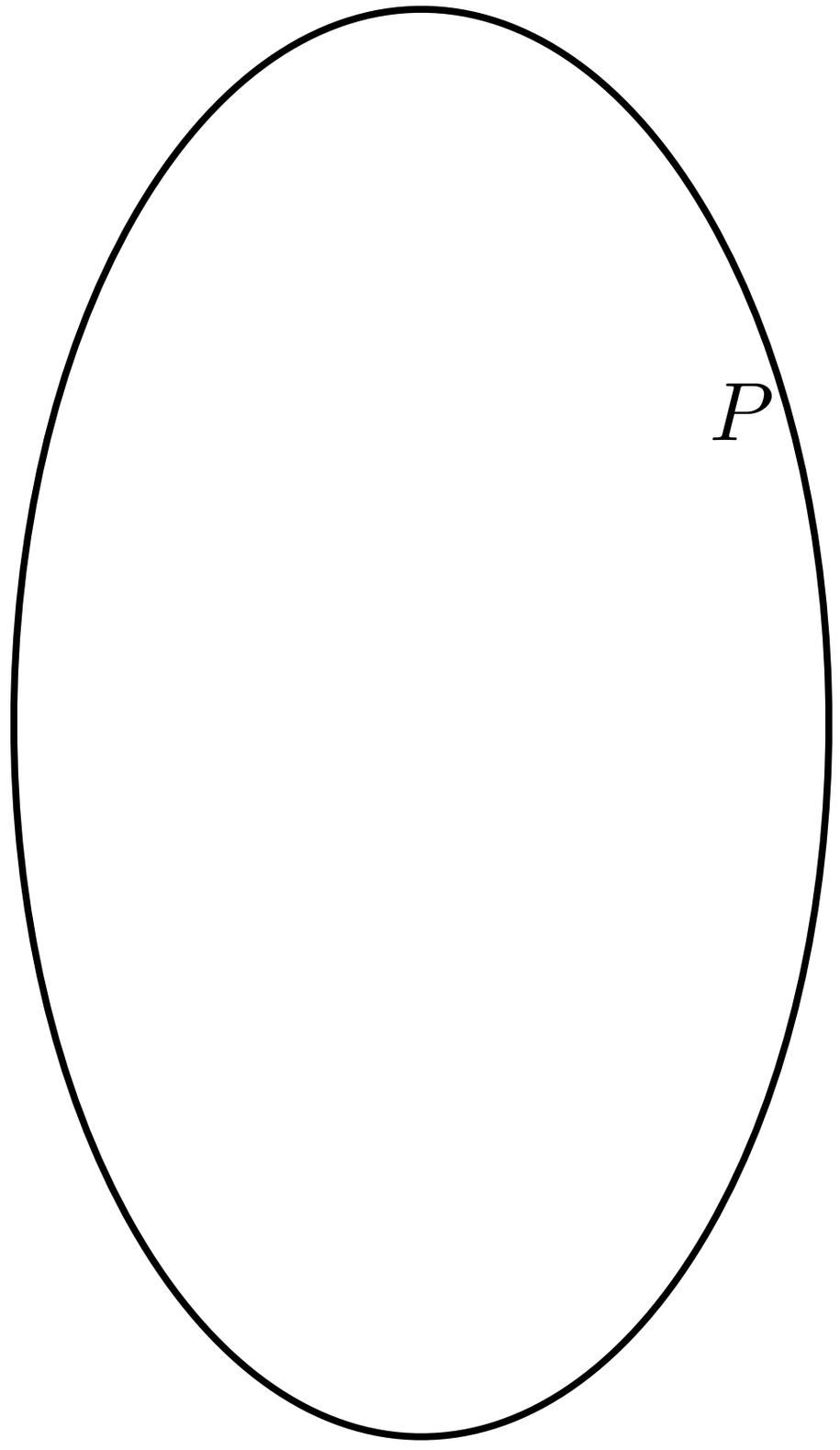


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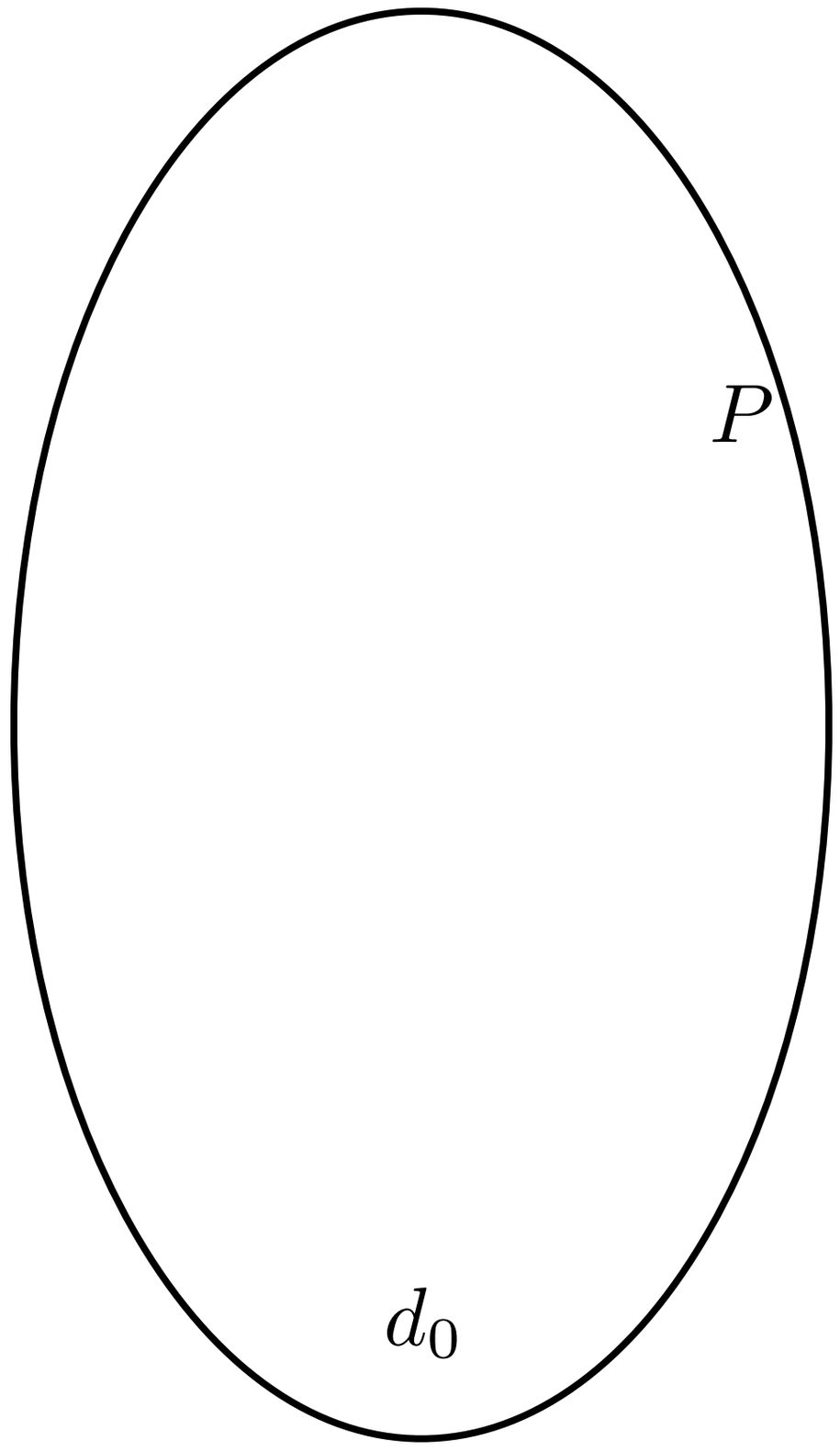




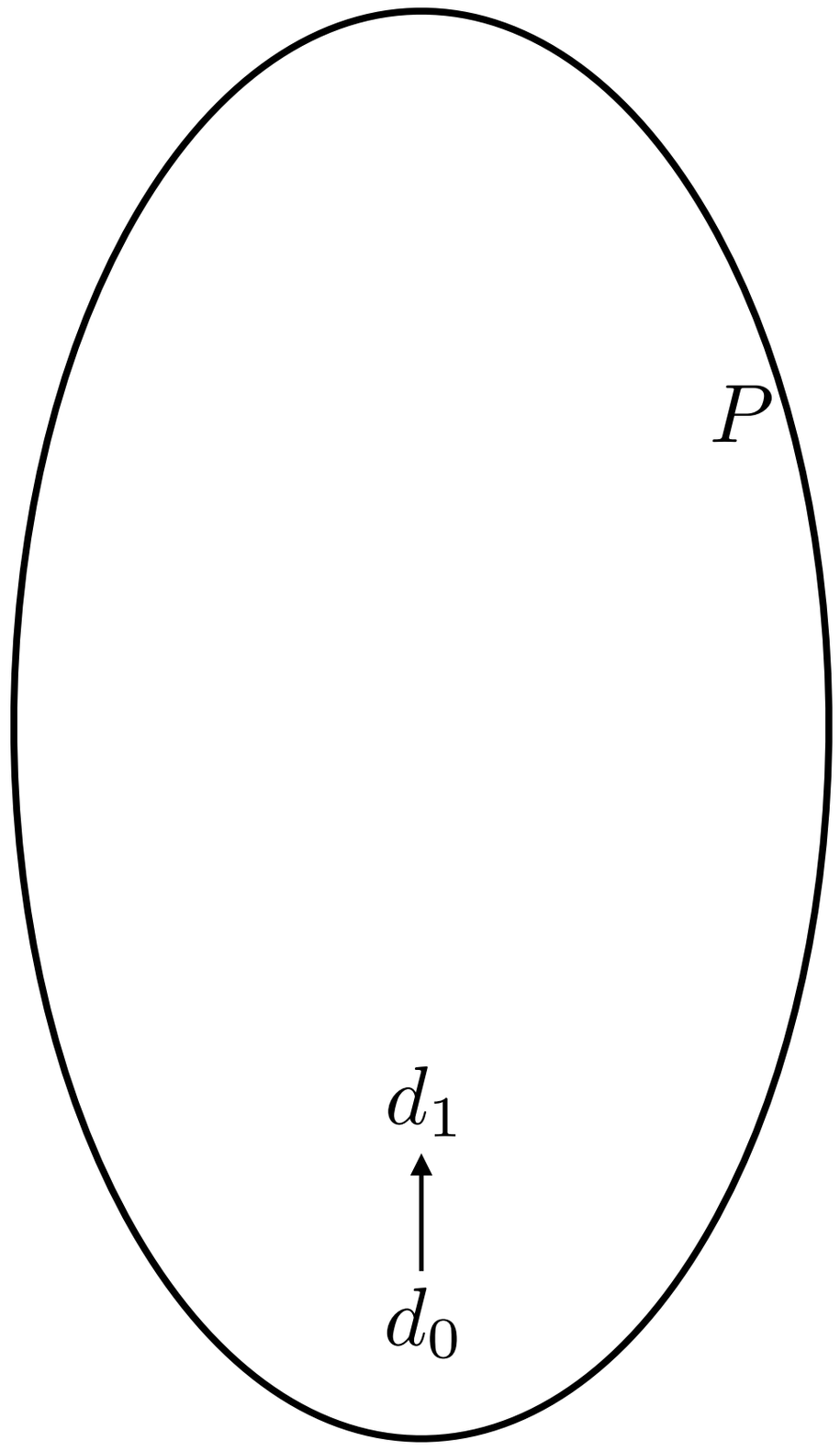
Limit: idea



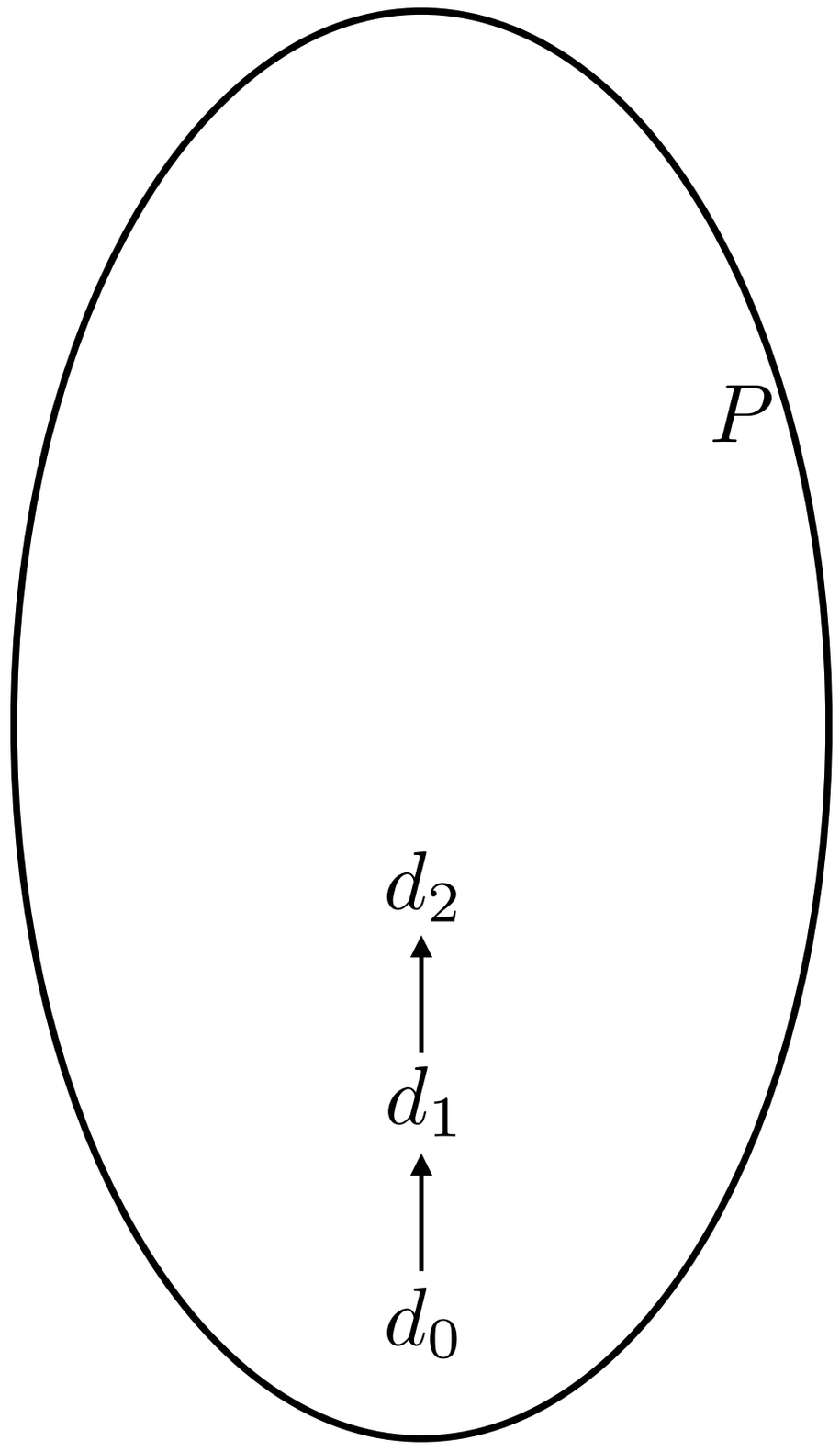
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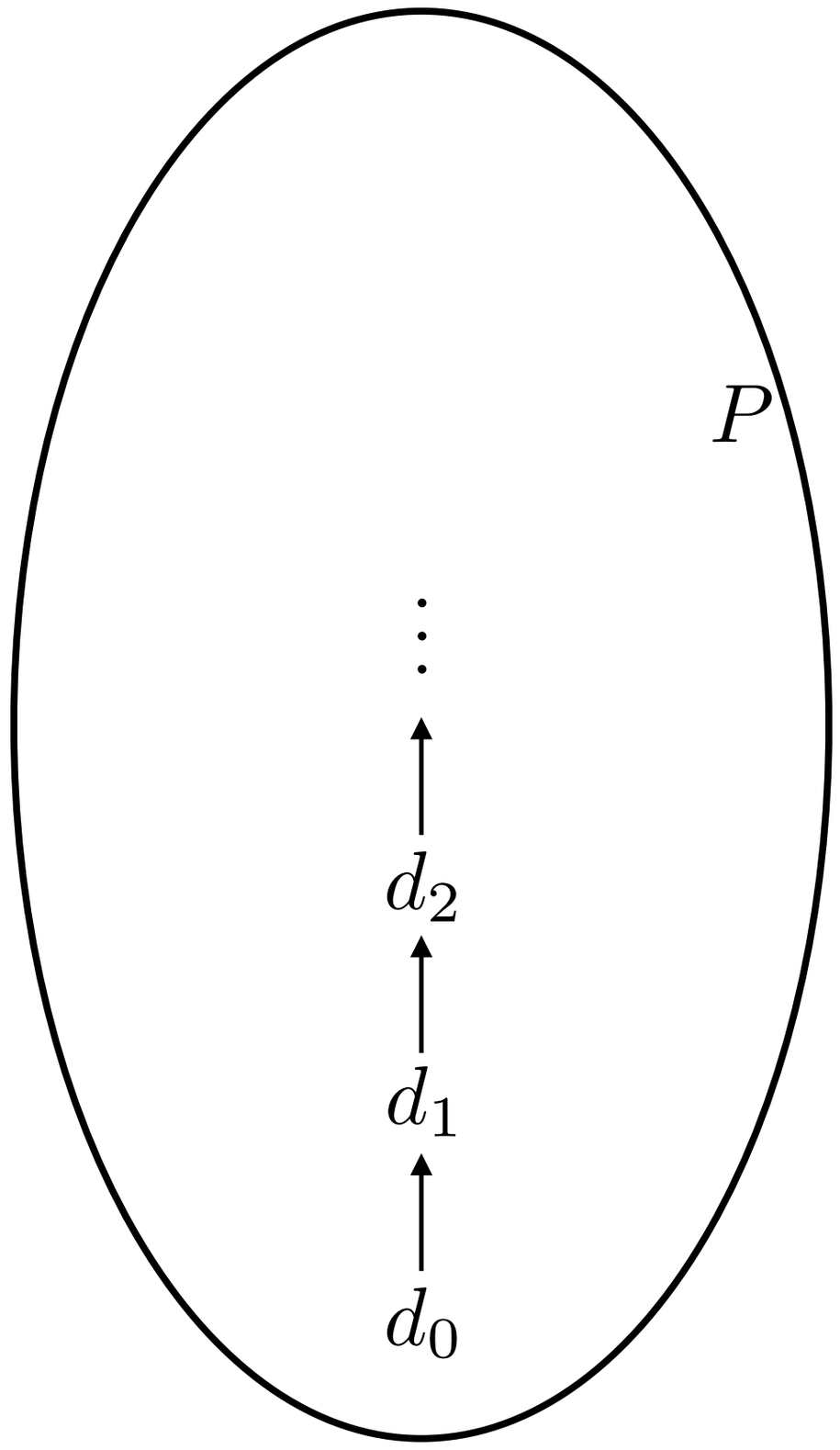
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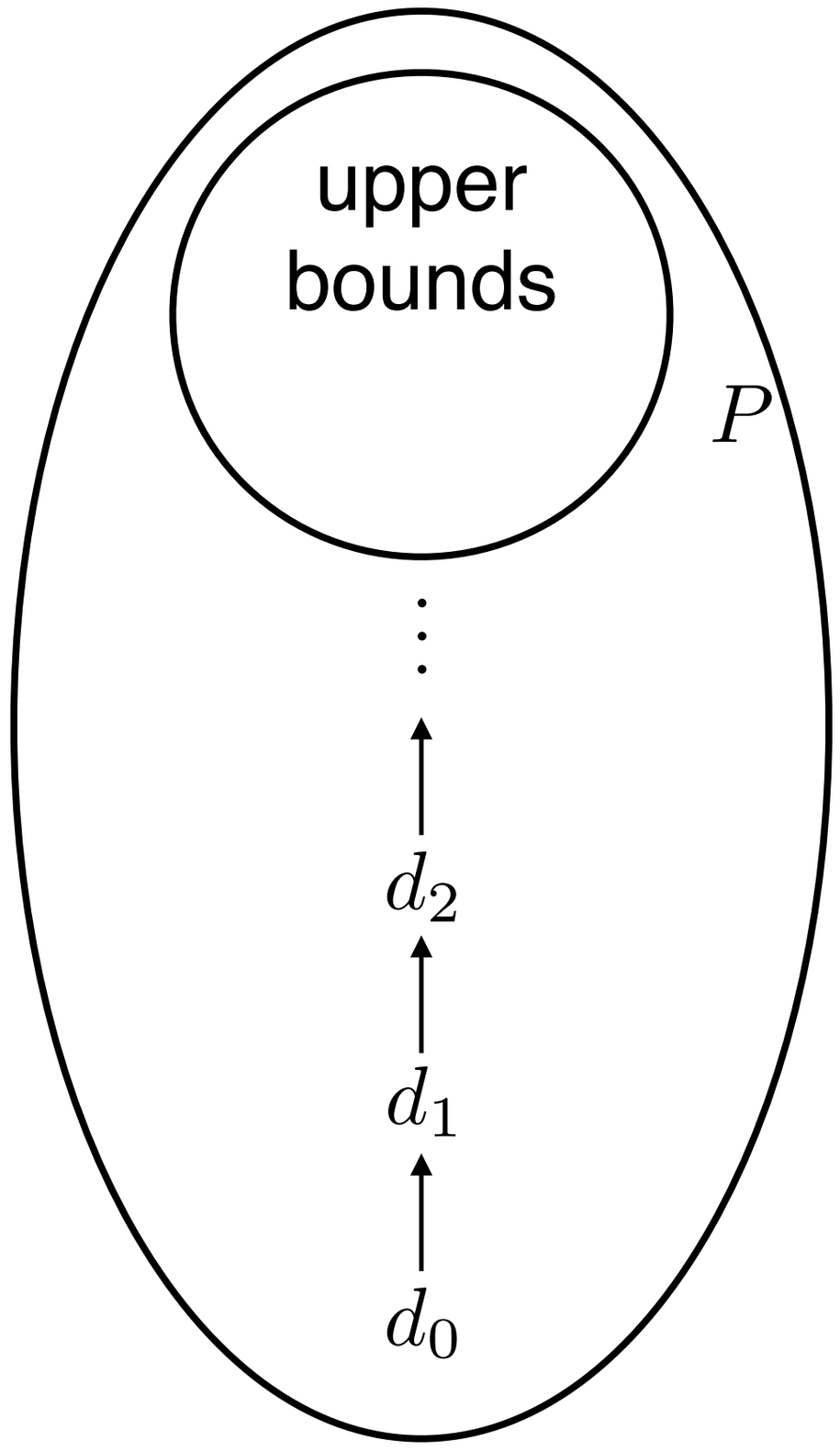
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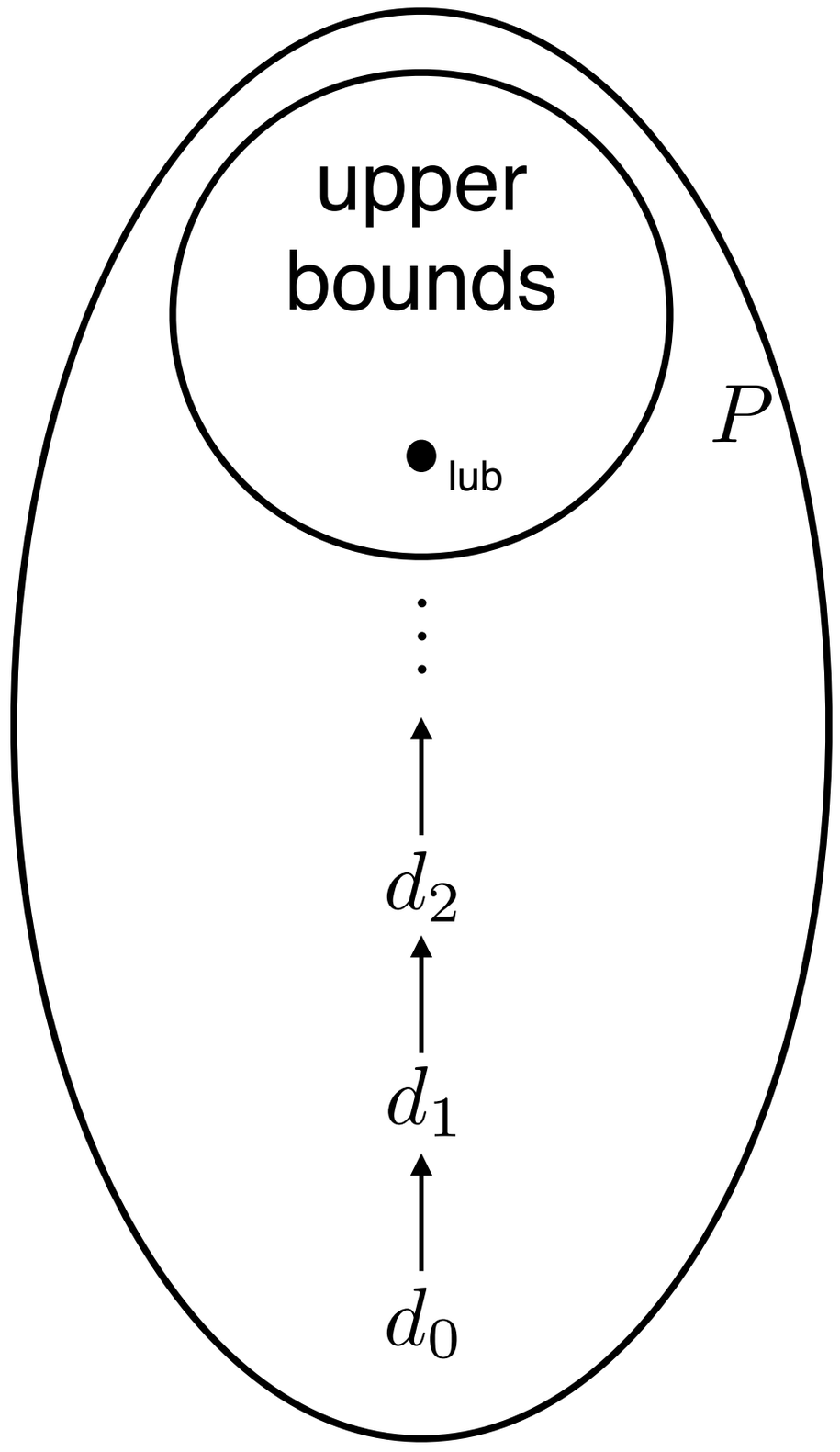
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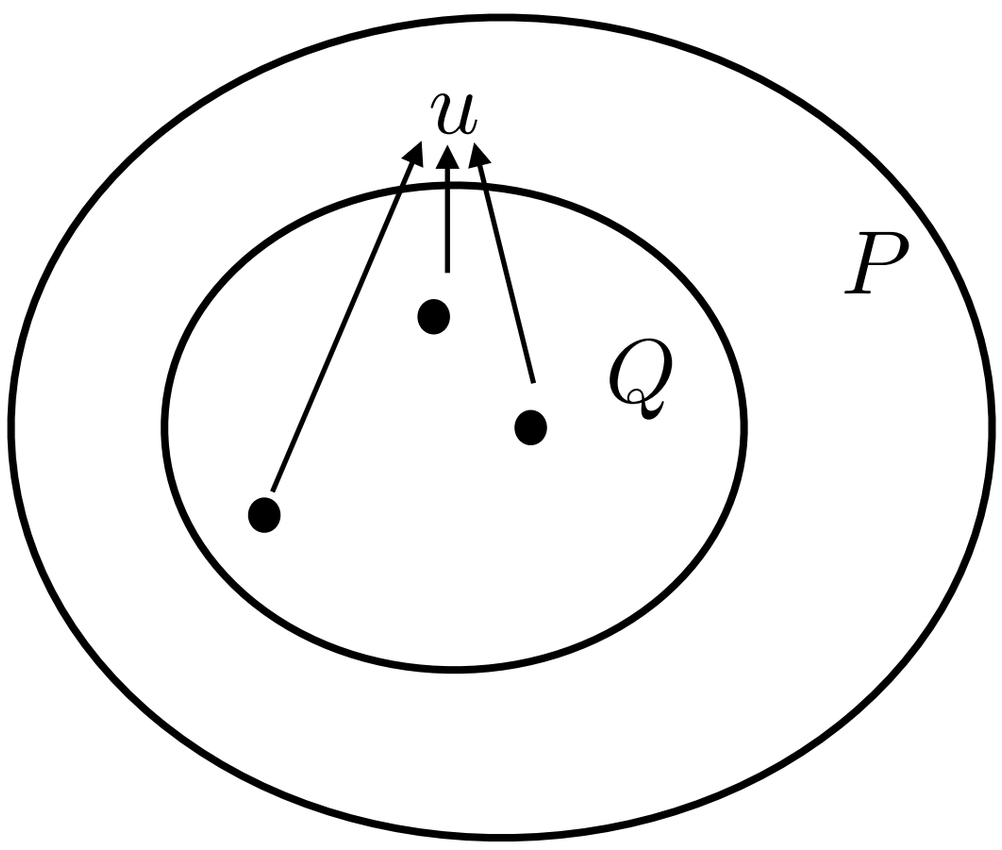
(all the elements of Q are smaller than u)

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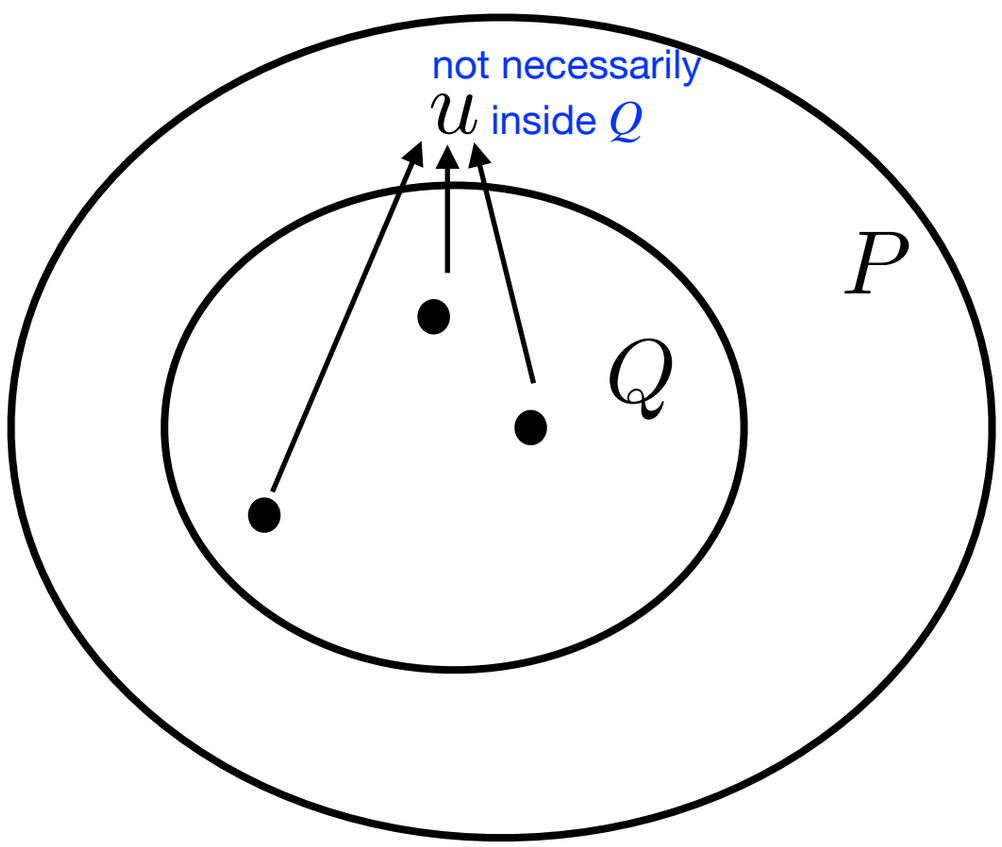


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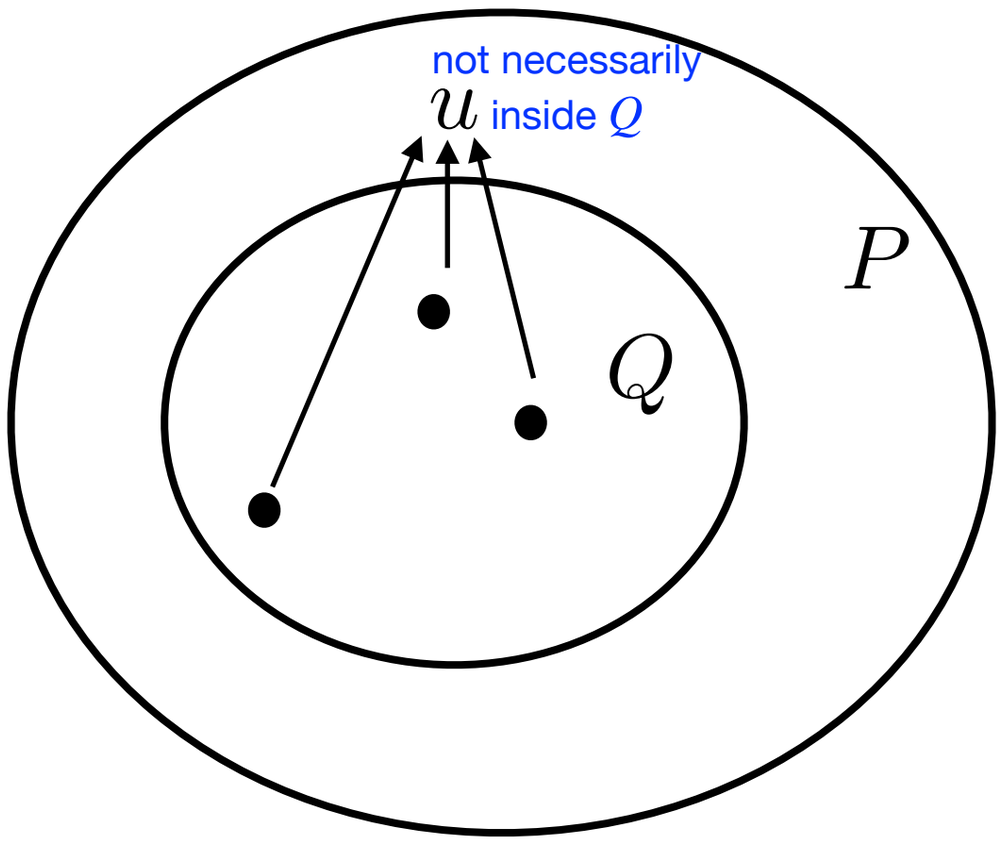
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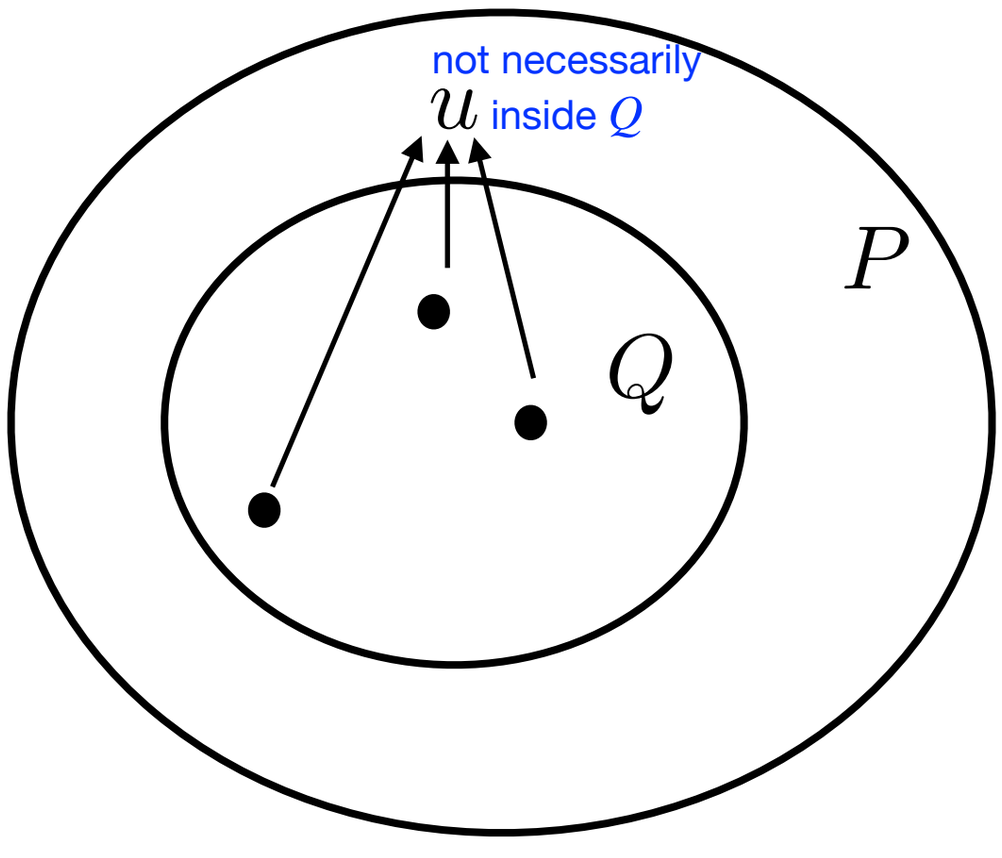


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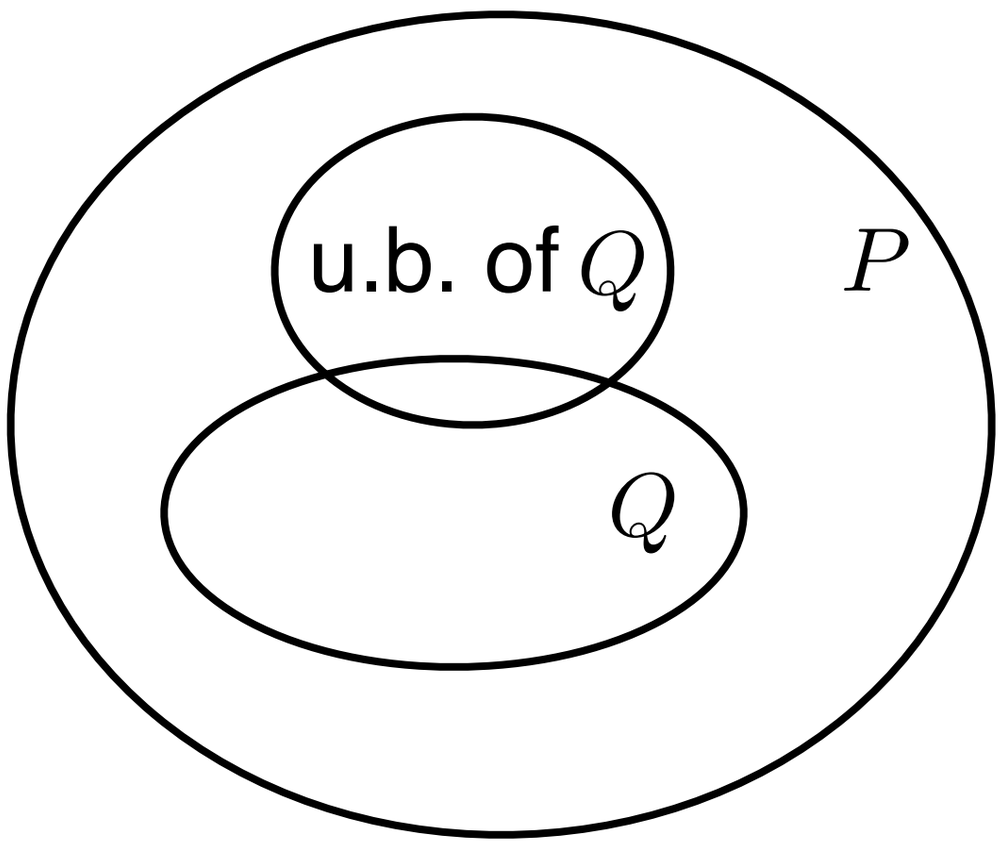
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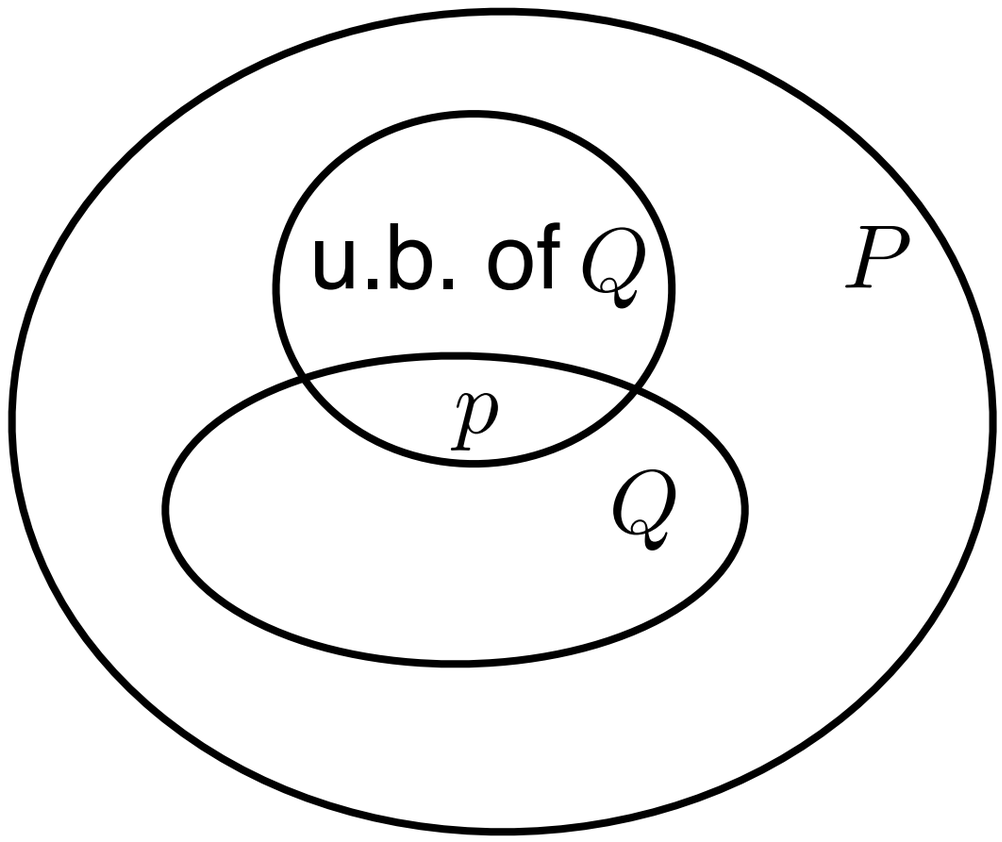
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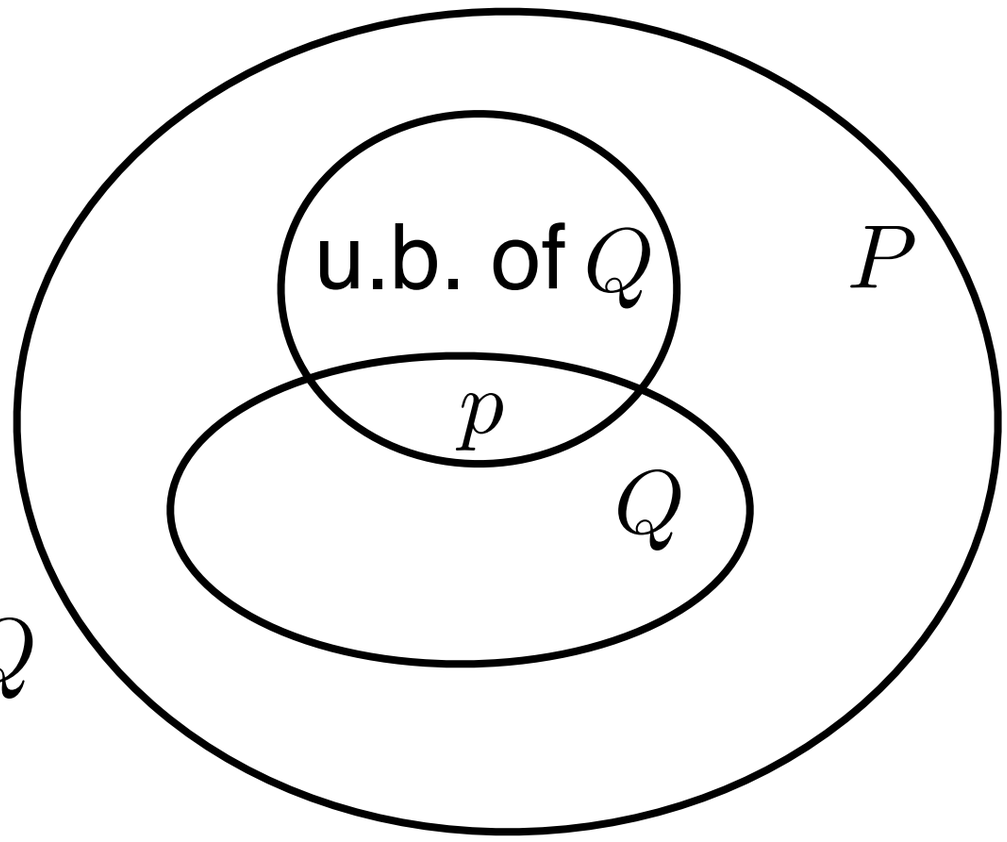
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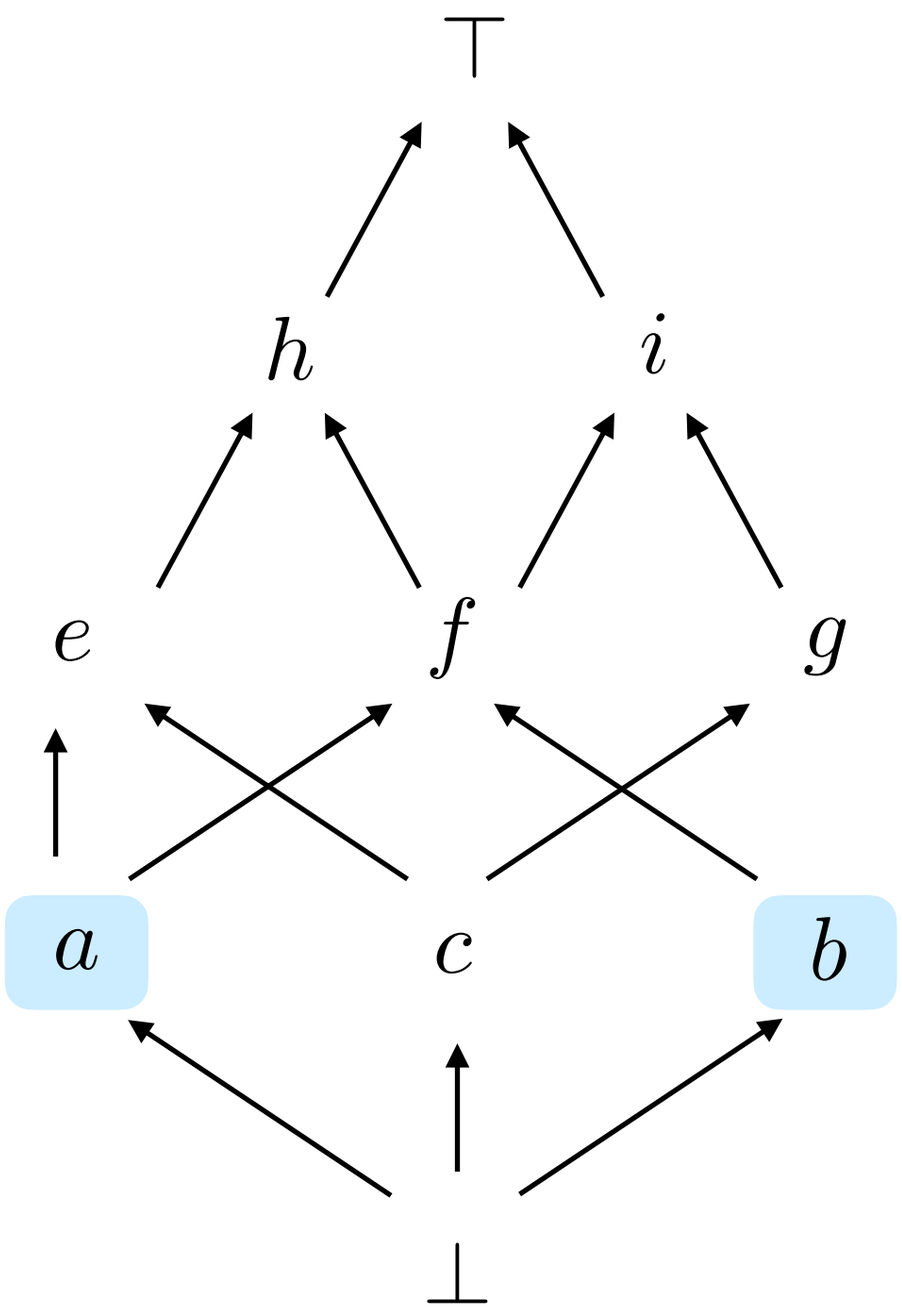
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p not necessarily an element of Q



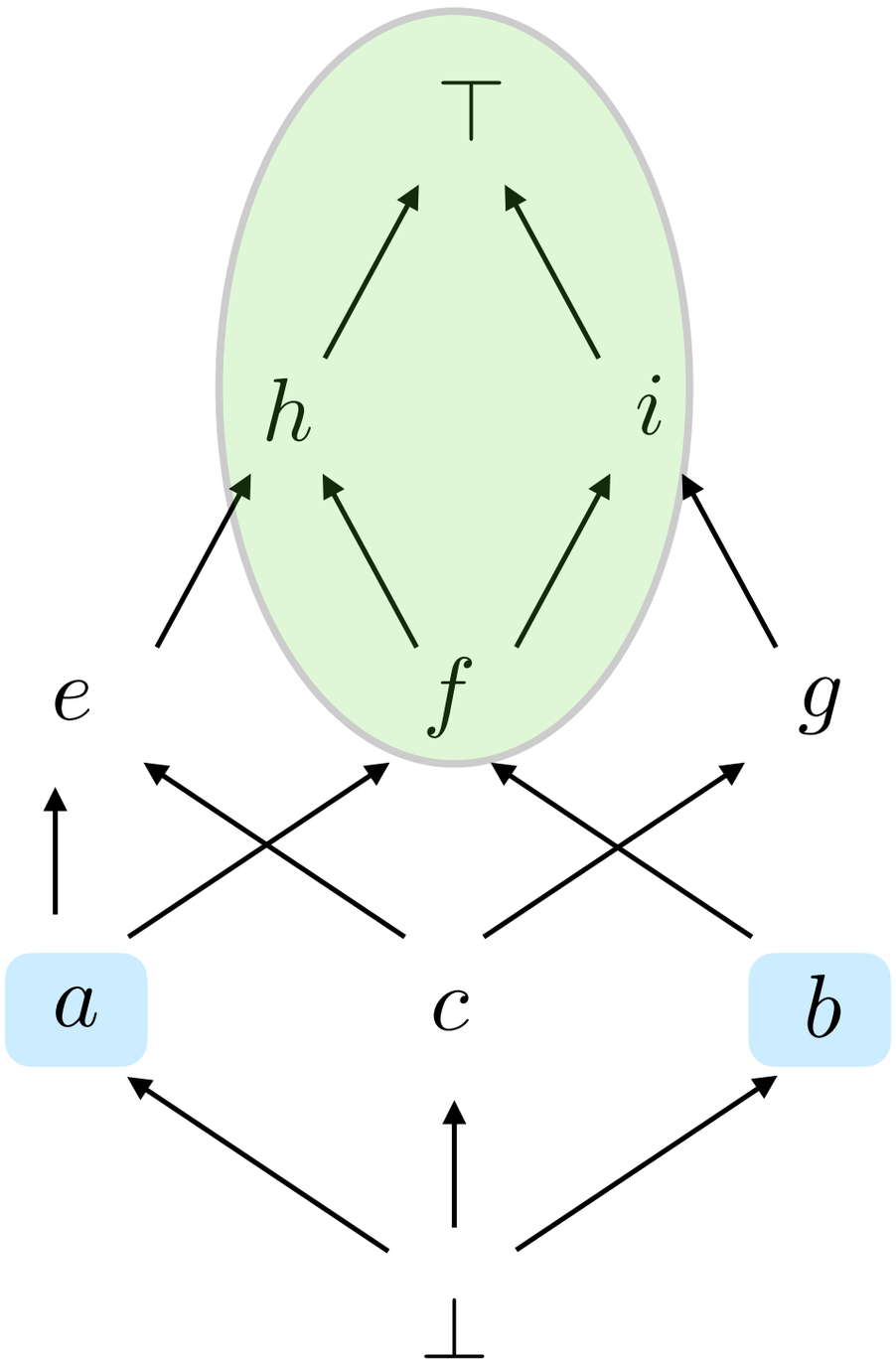
Exercise



Upper bounds of $\{a, b\}$?

lub?

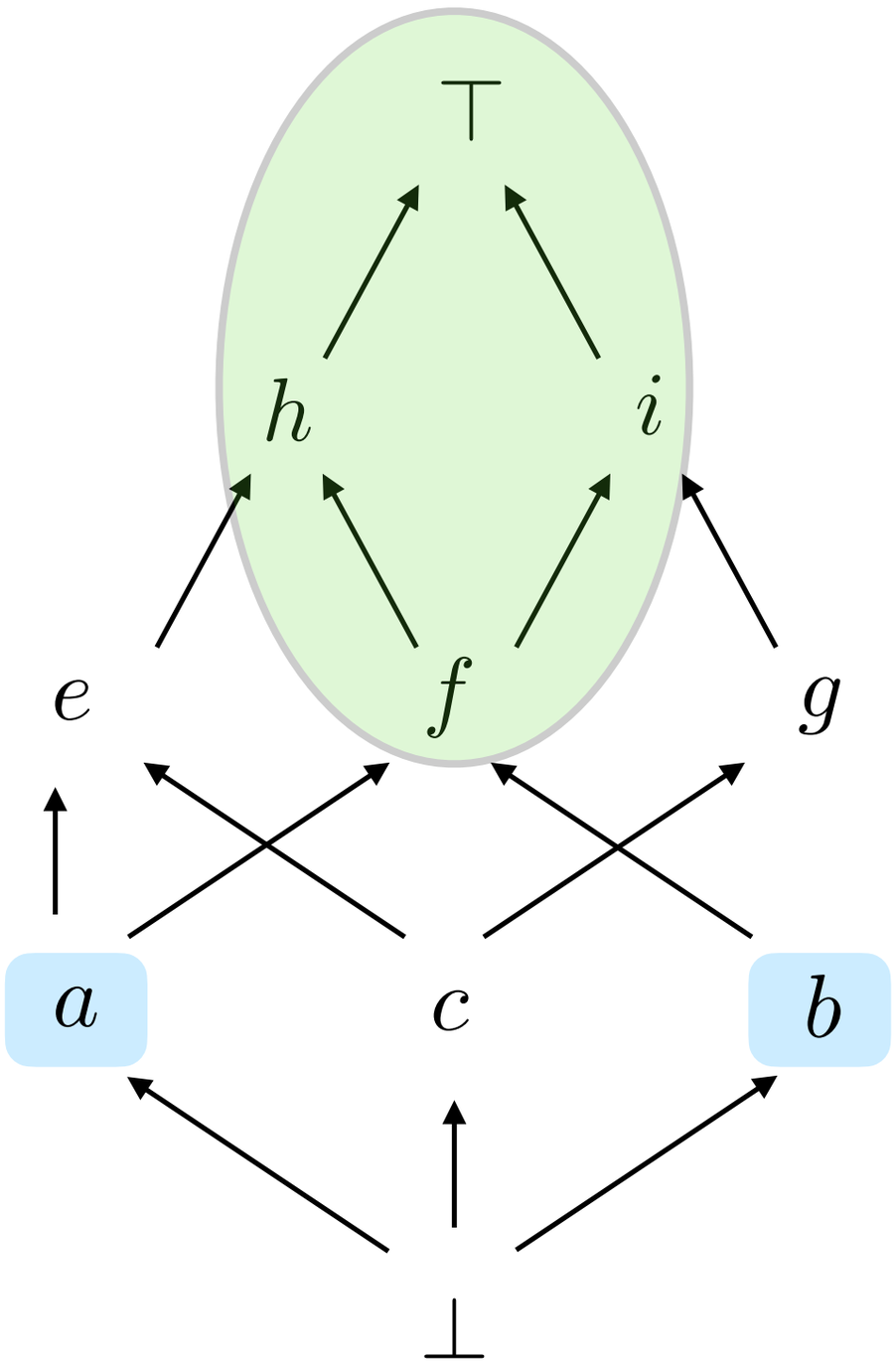
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Upper bounds of $\{a, b\}$? $\{f, h, i, \top\}$

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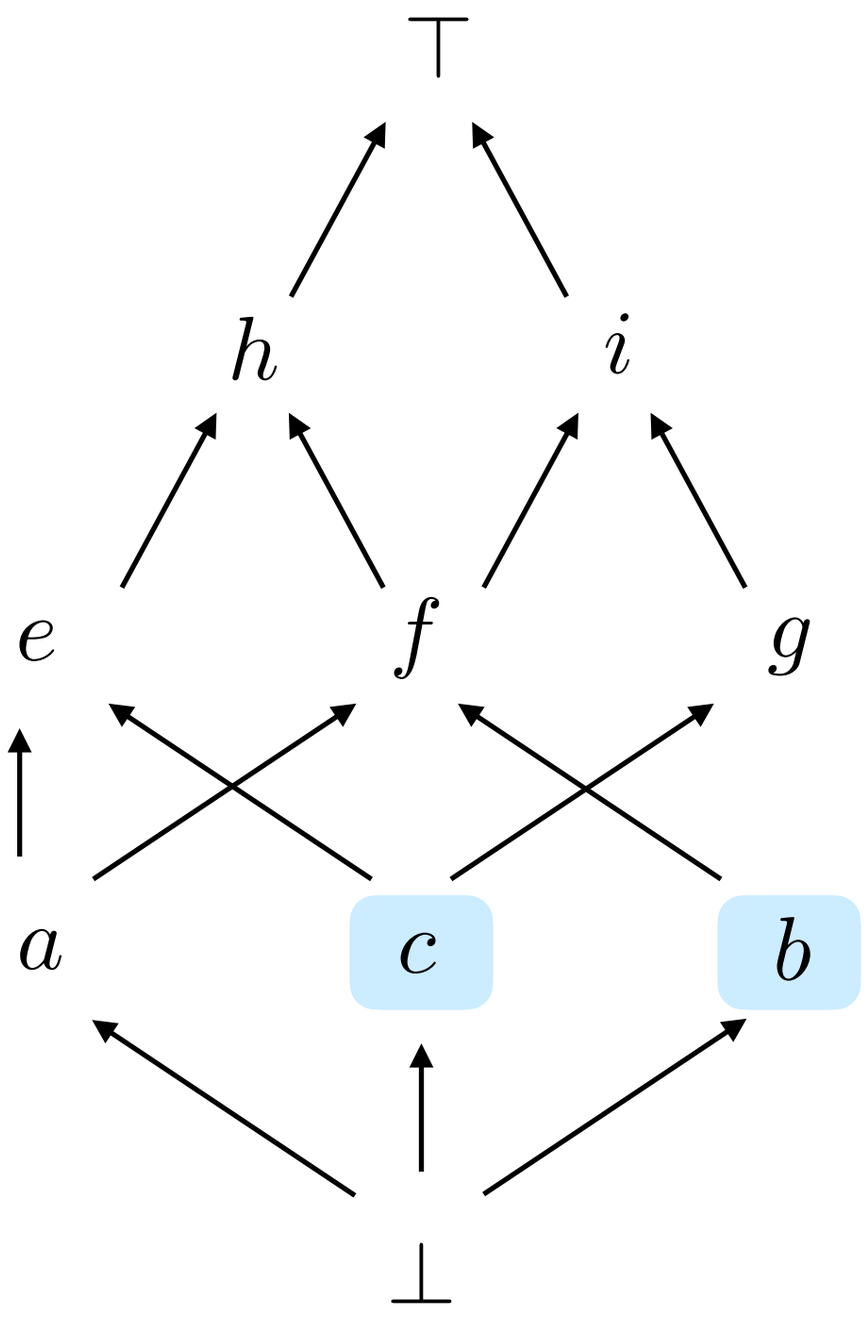
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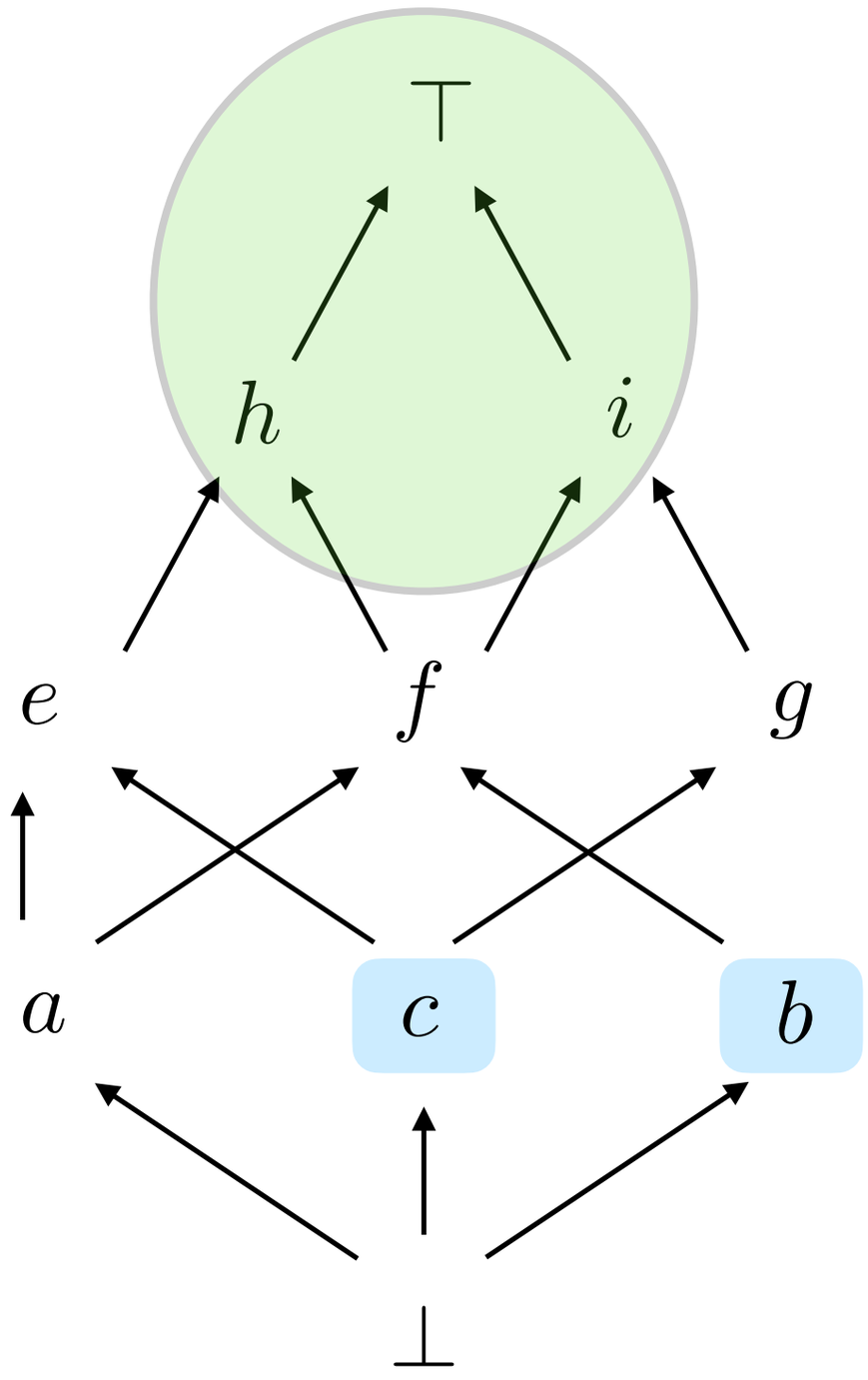
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Upper bounds of $\{b, c\}$?

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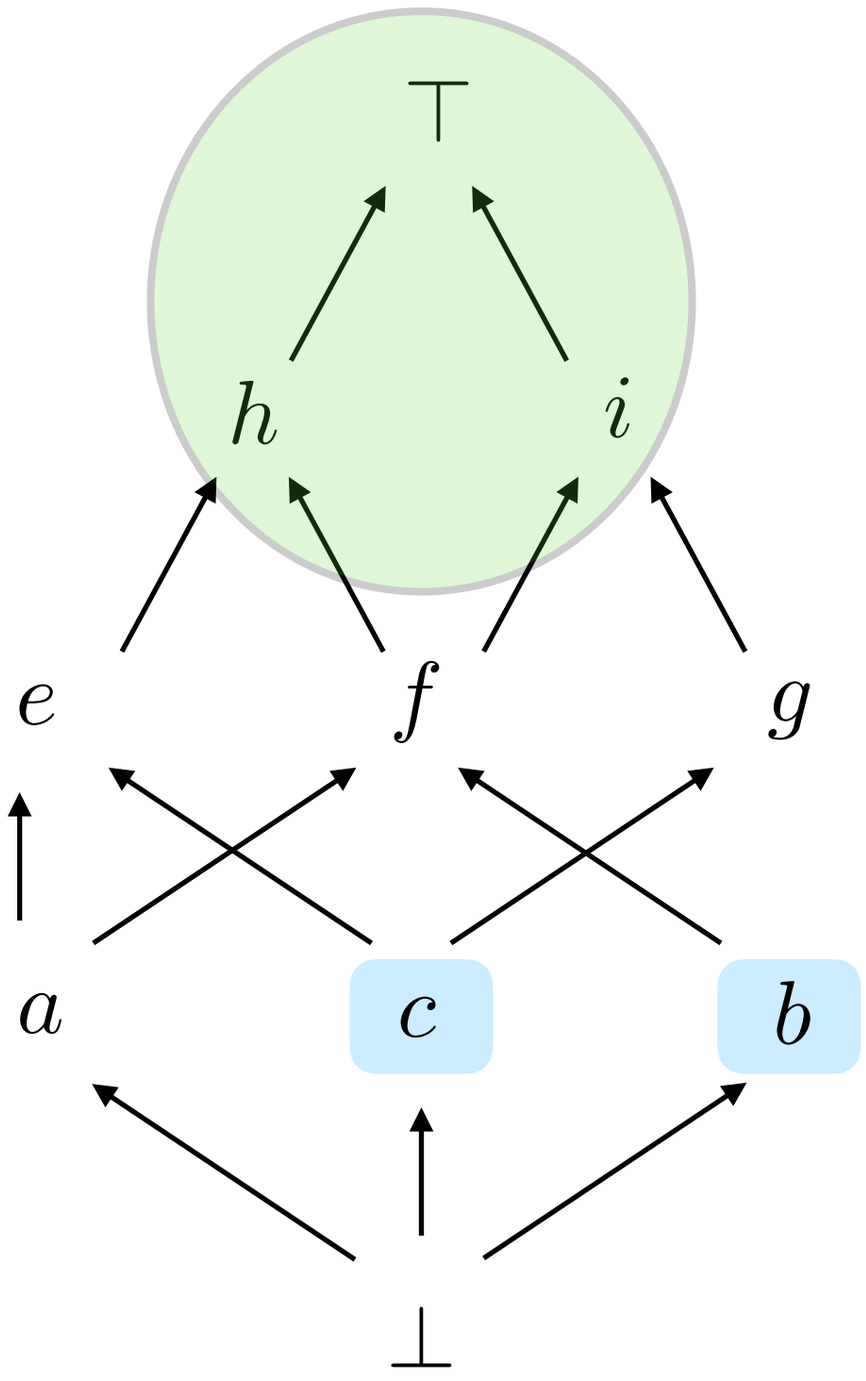
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lub? no lub!

Exercise

$(\mathbb{N}, \leq)$ $Q \subseteq \mathbb{N}$ lub?



Exercise

$(\mathbb{N}, \leq)$ $Q \subseteq \mathbb{N}$ lub? if Q finite



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$(\mathbb{N}, \leq)$ $Q \subseteq \mathbb{N}$ lub? if Q finite $lub\ Q = \max Q$



Exercise

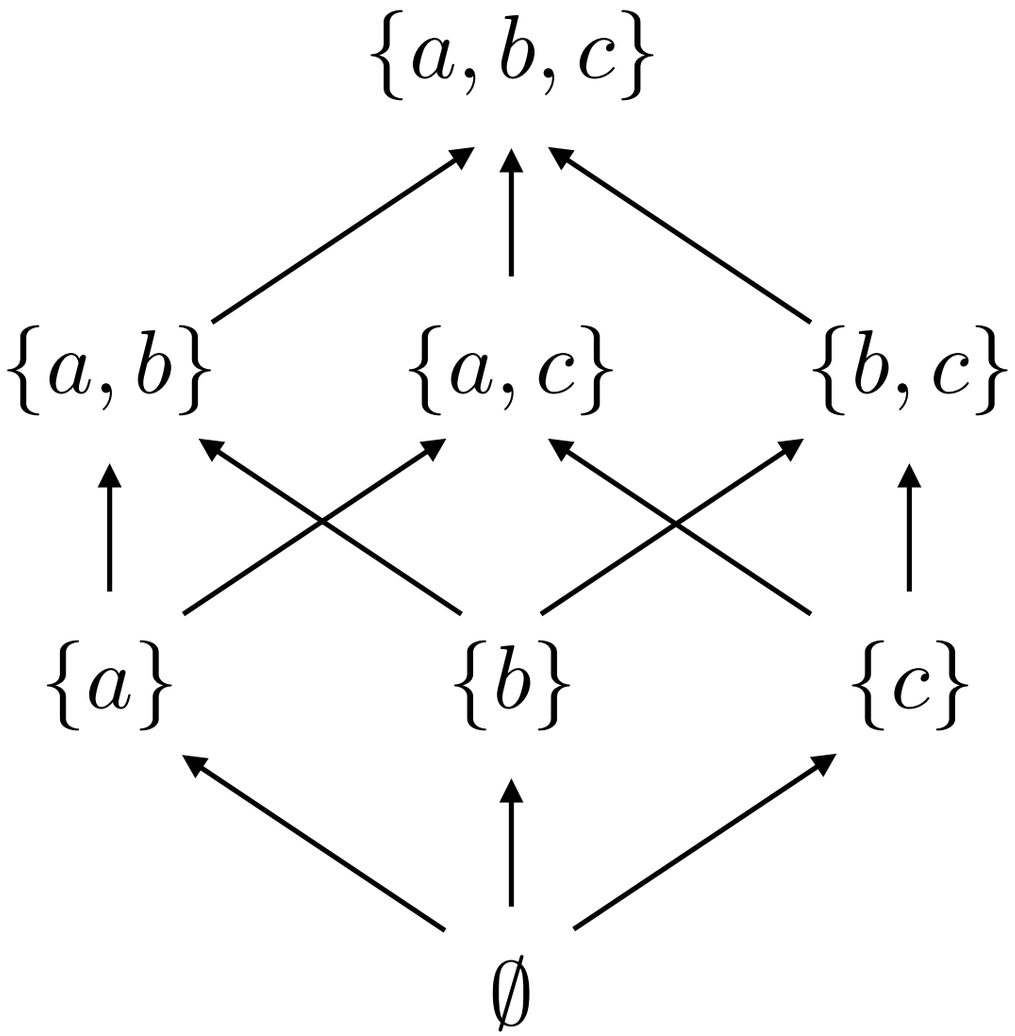
$(\mathbb{N}, \leq)$ $\quad Q \subseteq \mathbb{N}$ lub? if Q finite $lub\ Q = \max Q$

otherwise no lub



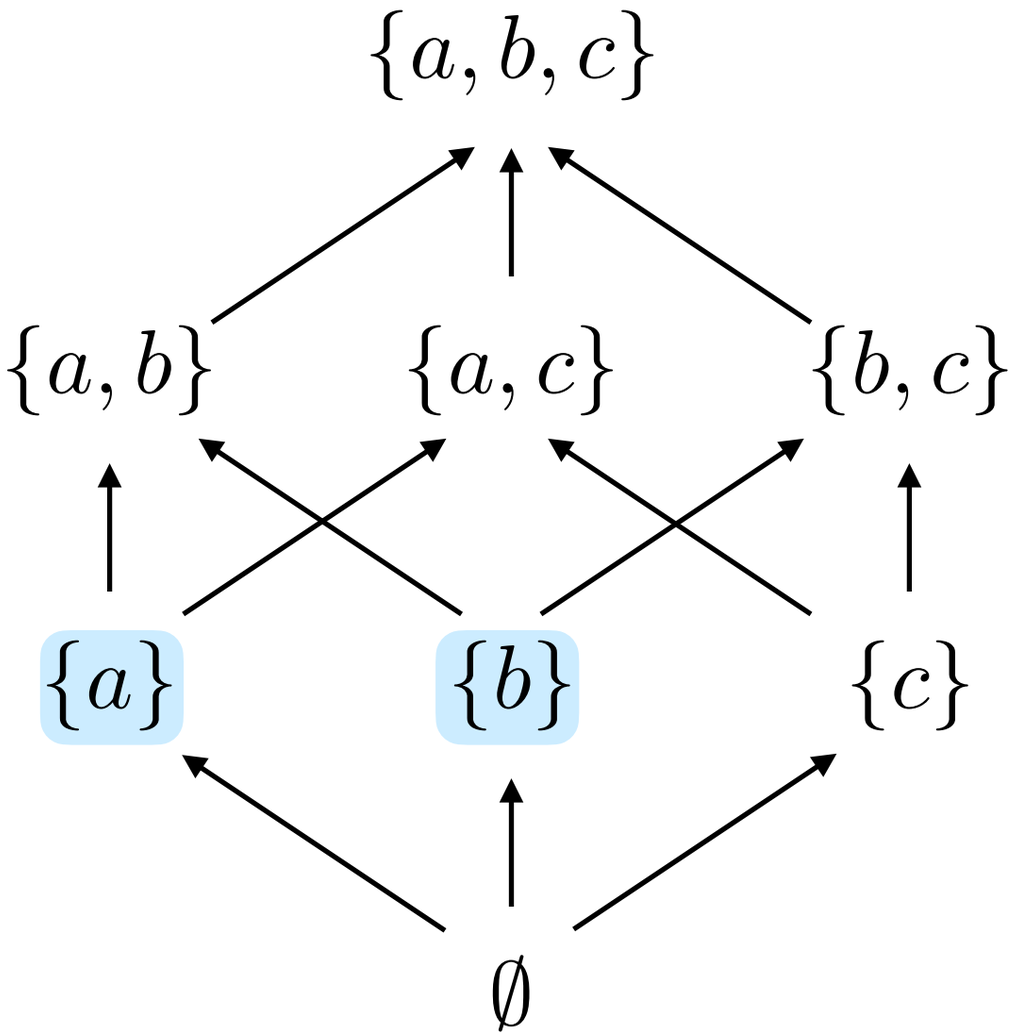
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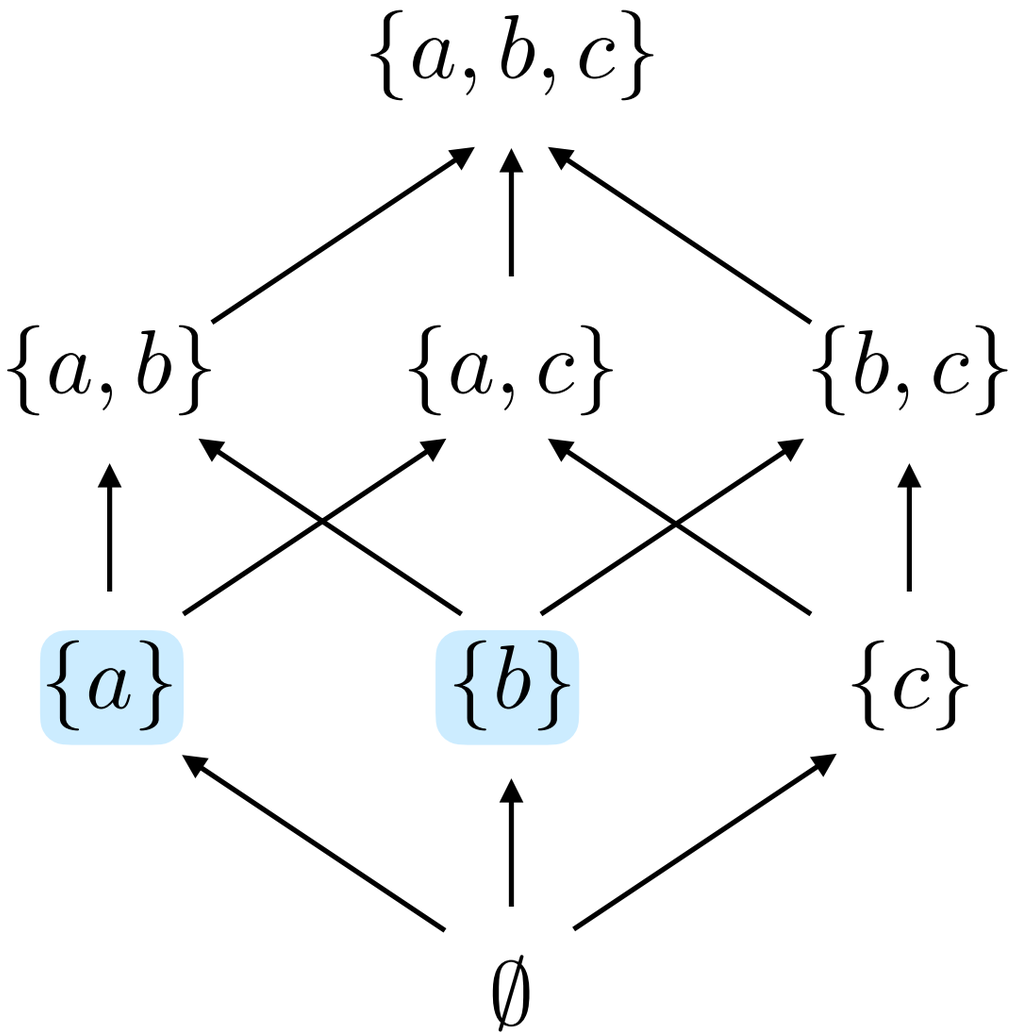
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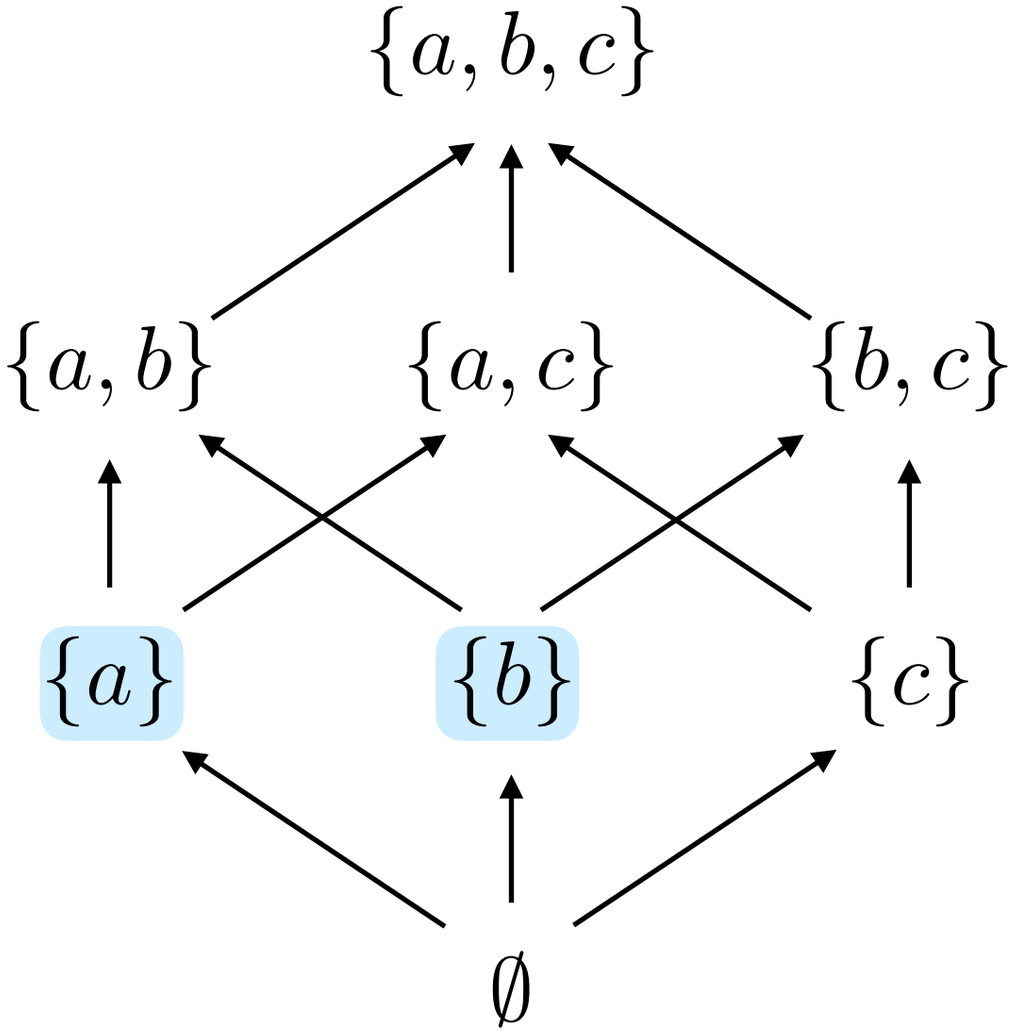
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$$lub\ \{\{a\}, \{b\}\} = \{a, b\}$$

Exercise

$(\wp(S), \subseteq)$ $\quad Q \subseteq \wp(S)$ $\quad$ lub? $\qquad\qquad lub\ Q = \bigcup_{T \in Q} T$



$lub\ \{\{a\}, \{b\}\} = \{a, b\}$

Complete partial orders (CPO)

Completeness: the idea

D a domain

$\sqsubseteq$ a way to compare element

$x \sqsubseteq y$ x is a (less precise) approximation of y

} PO

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x and y are consistent,
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first approximation

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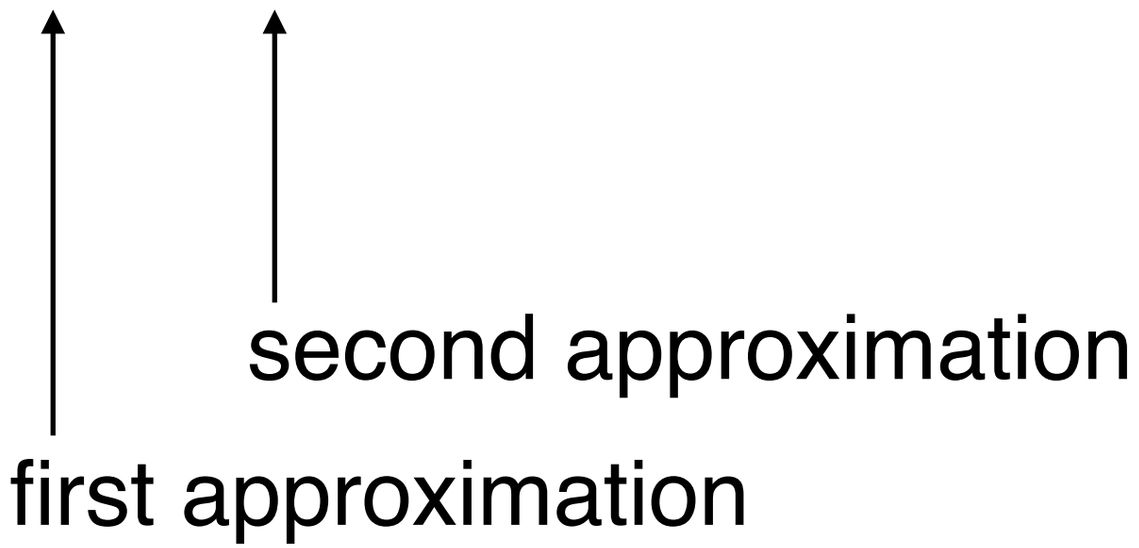
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(n+1)th approximation

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first approximation

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third approximation

(n+1)th approximation

does any sequence of approximations tend to some limit?

Chain

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$(\mathbb{N}, \leq)$

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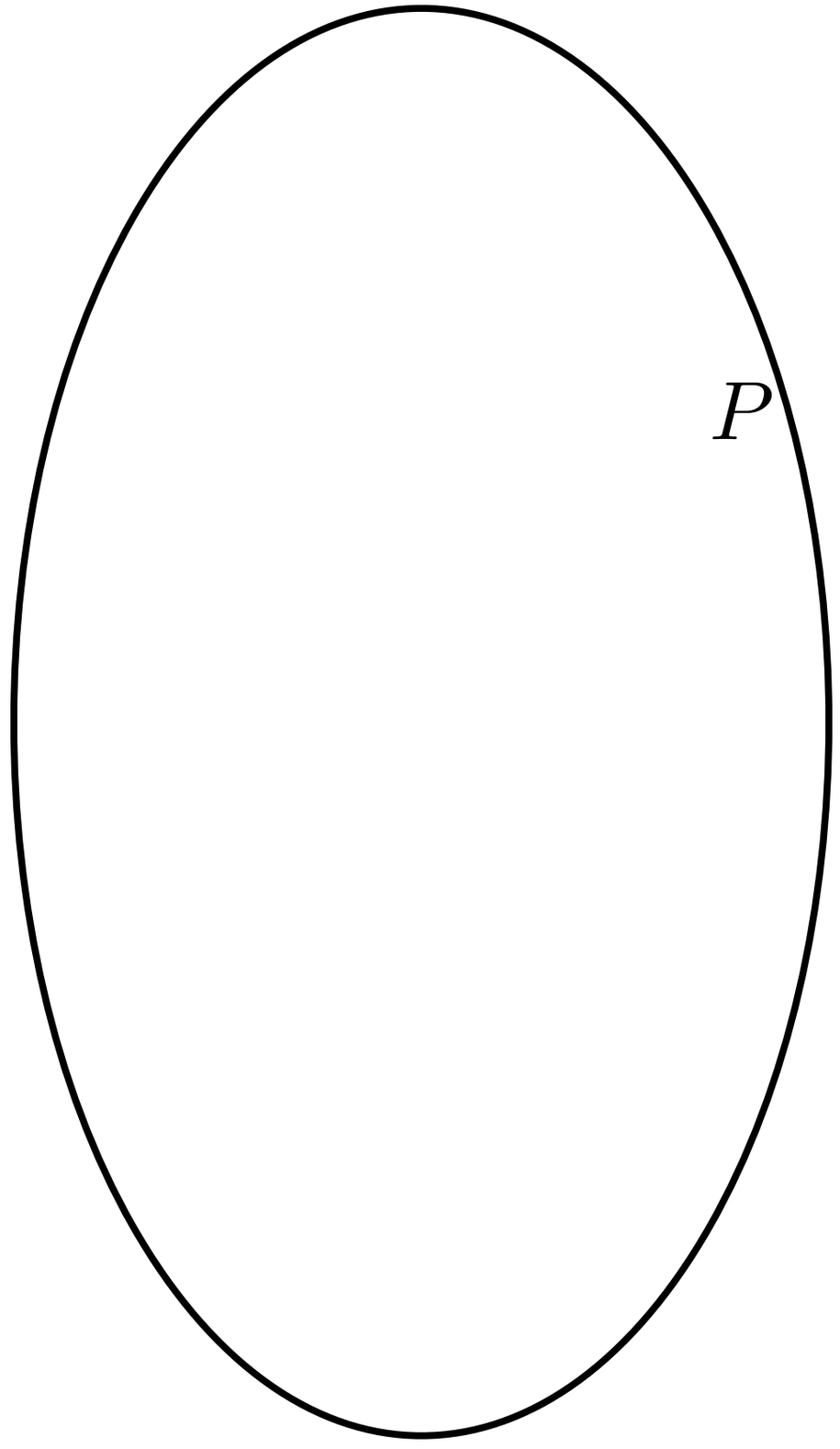
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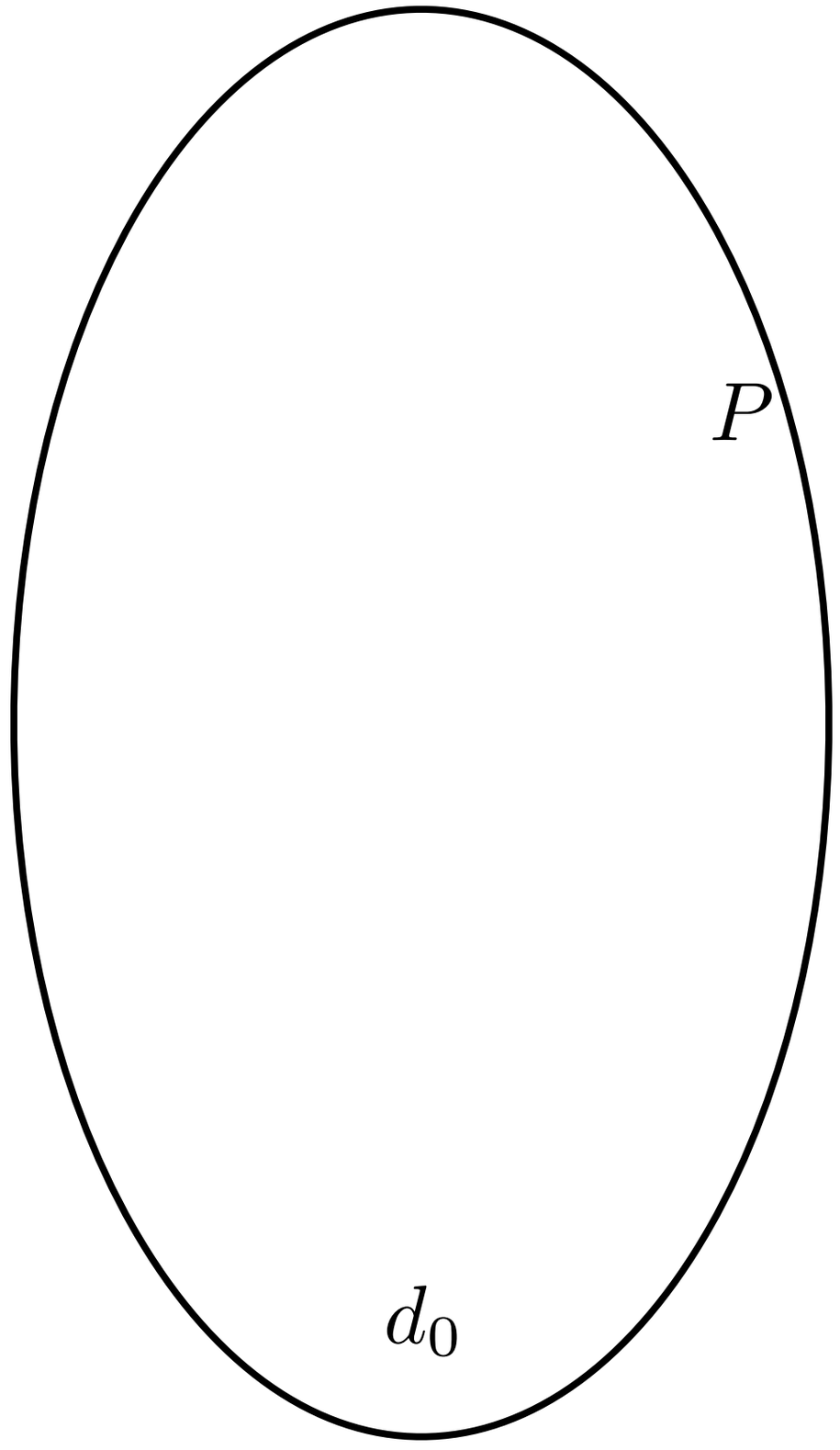
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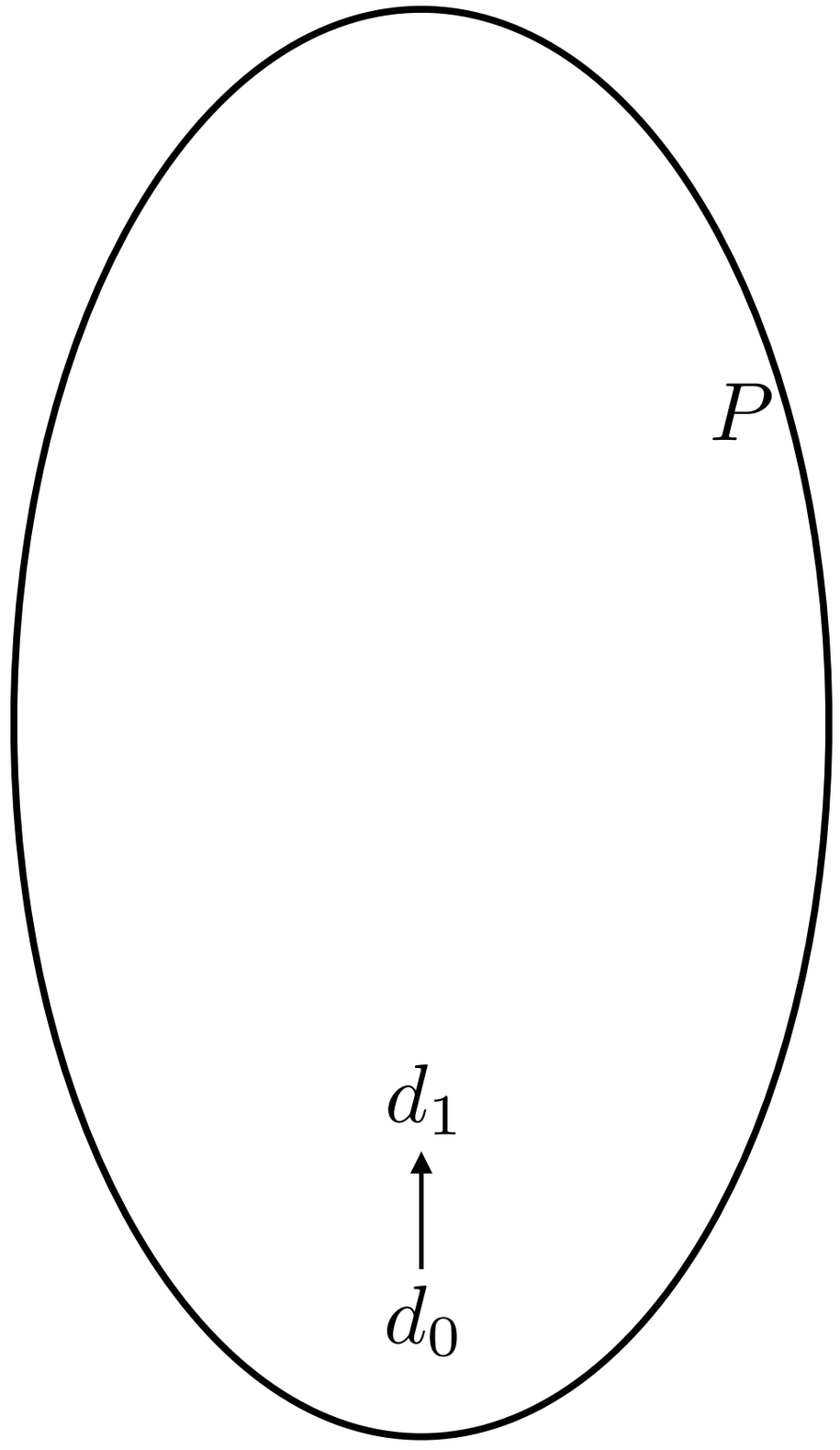
Limit illustrated



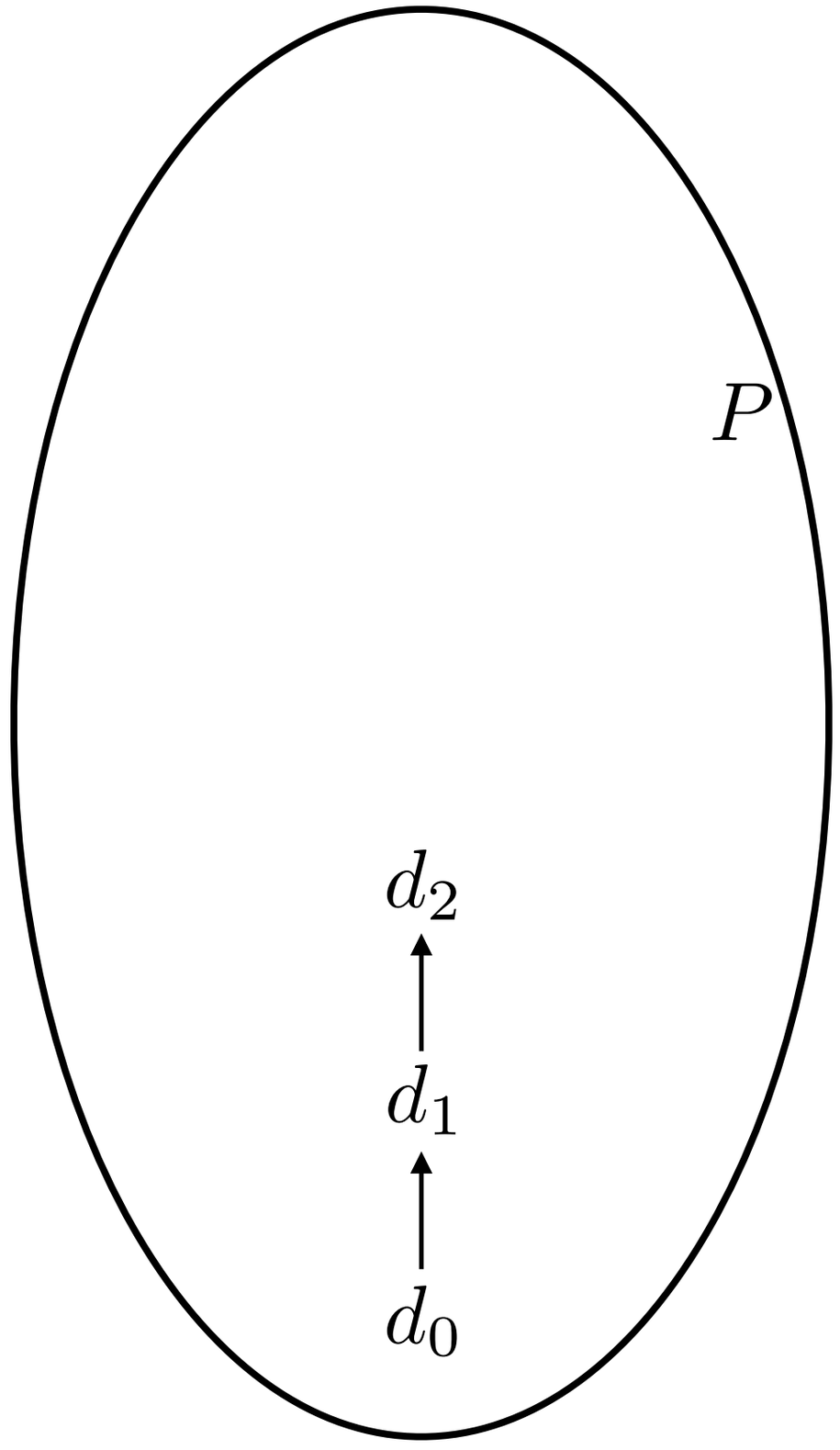
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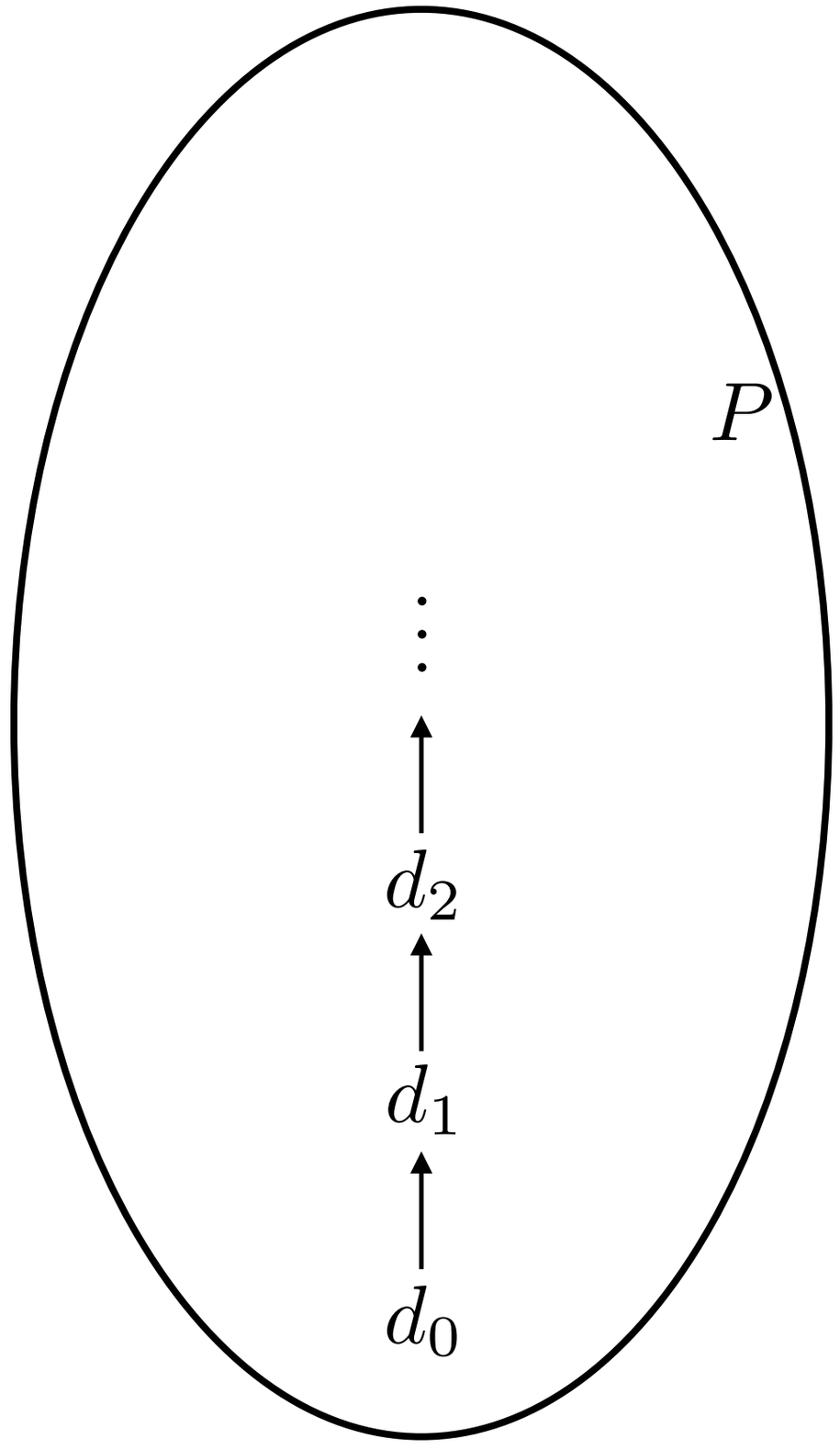
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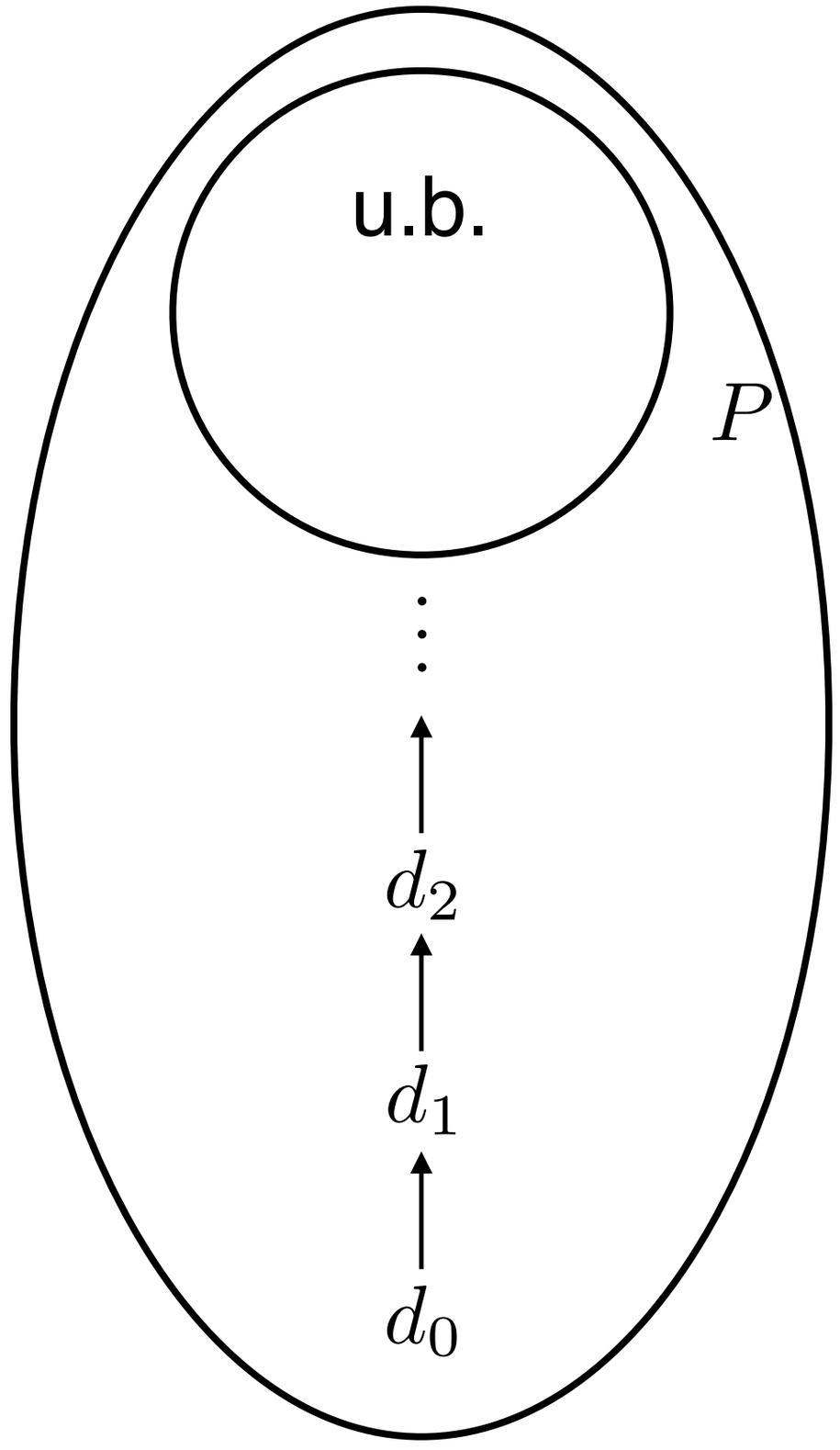
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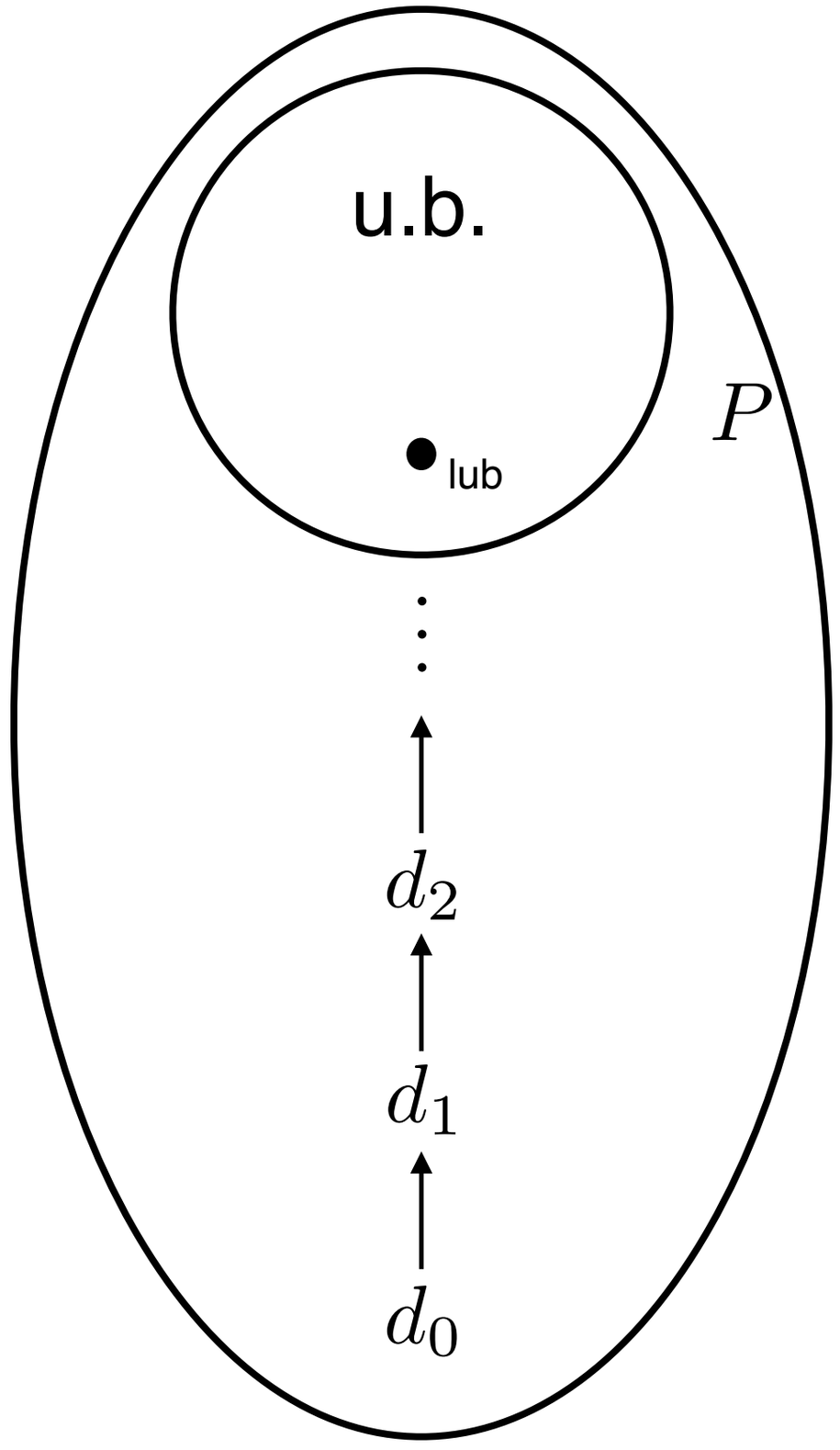
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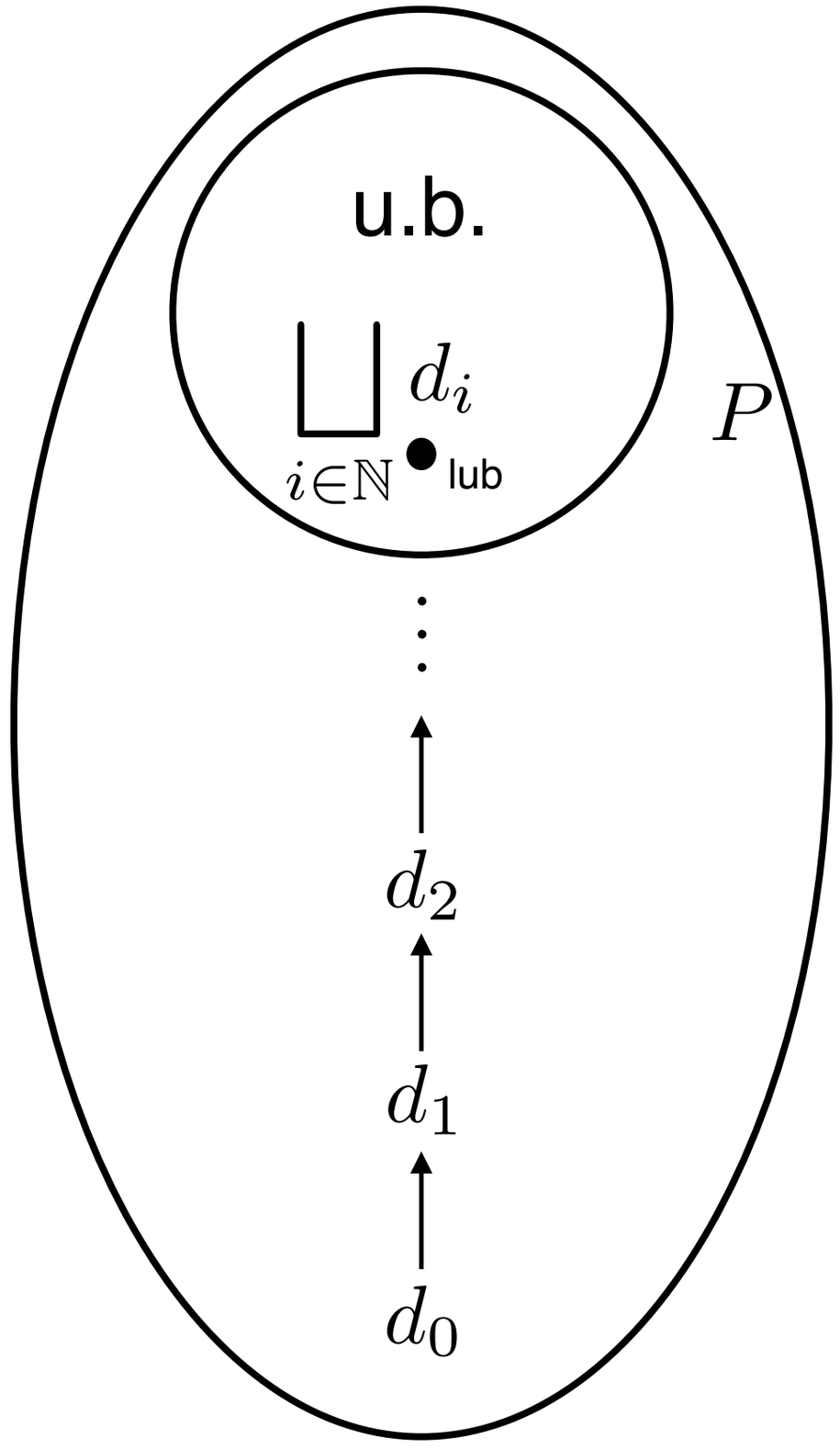
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(which upper bounds?)

Lemma on finite chains

Lemma (any finite chain has a limit)

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take u such that $\forall i.\ d_i \sqsubseteq u$ then $d_k \sqsubseteq u$

Prefix independence

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   for $j \geq k$ it is obvious
   if $j < k$ then $d_j \sqsubseteq d_k \sqsubseteq u$ because $d_k \in \{d_{i+k}\}_{i\in\mathbb{N}}$

Complete partial order

$(P, \sqsubseteq)$ PO P is complete if each chain has a limit (lub)

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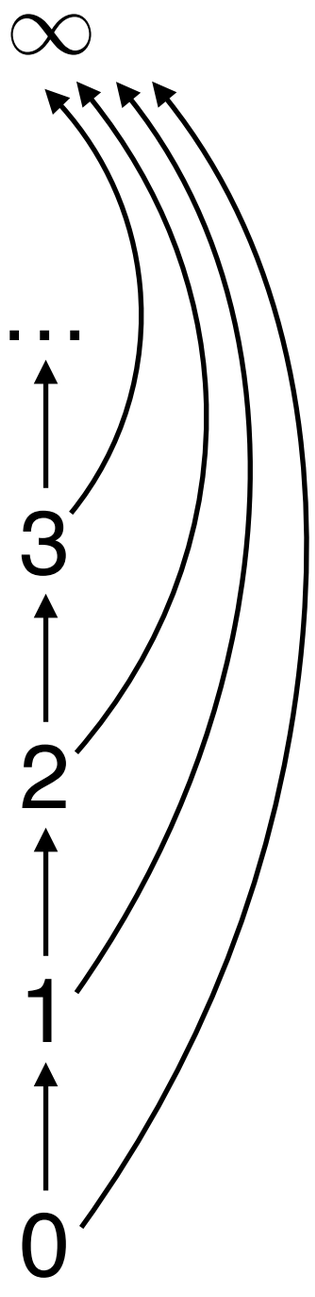
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has no lub
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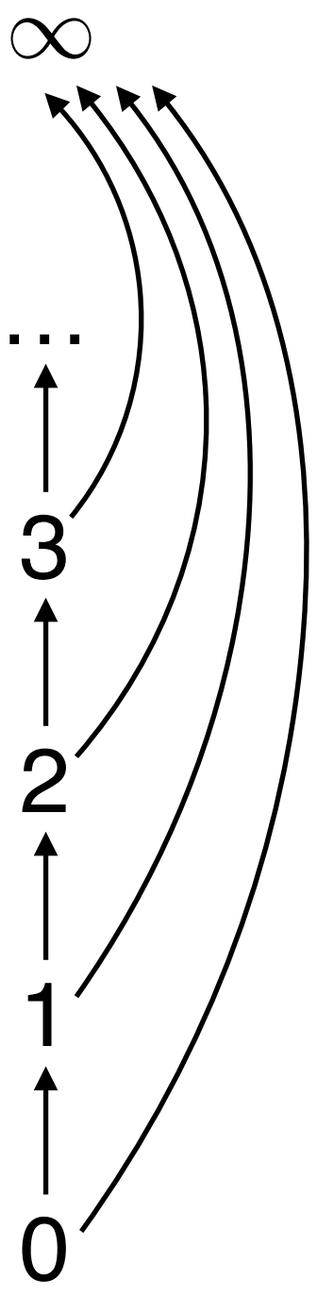
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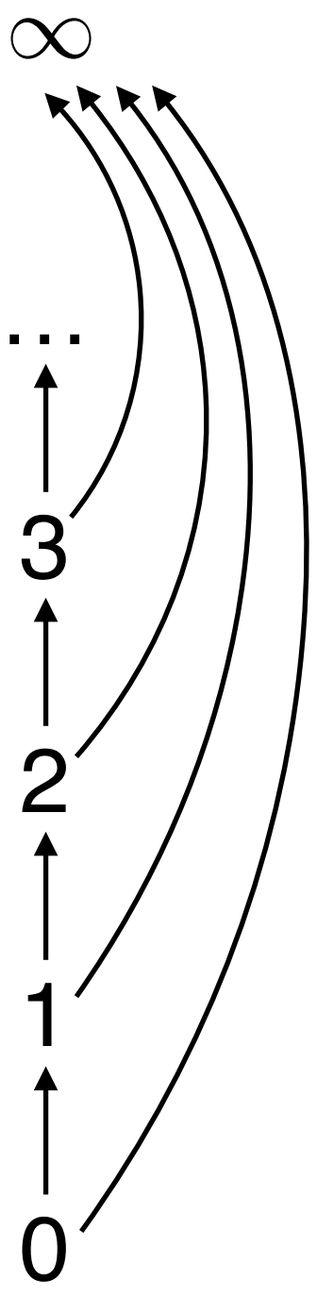


any infinite chain has limit ∞
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$(\wp(S), \subseteq)$ complete?

$\{S_i\}_{i \in \mathbb{N}}$

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$\{S_i\}_{i \in \mathbb{N}}$ $\bigsqcup_{i \in \mathbb{N}} S_i = \bigcup_{i \in \mathbb{N}} S_i = \{x \mid \exists k \in \mathbb{N}. x \in S_k\}$

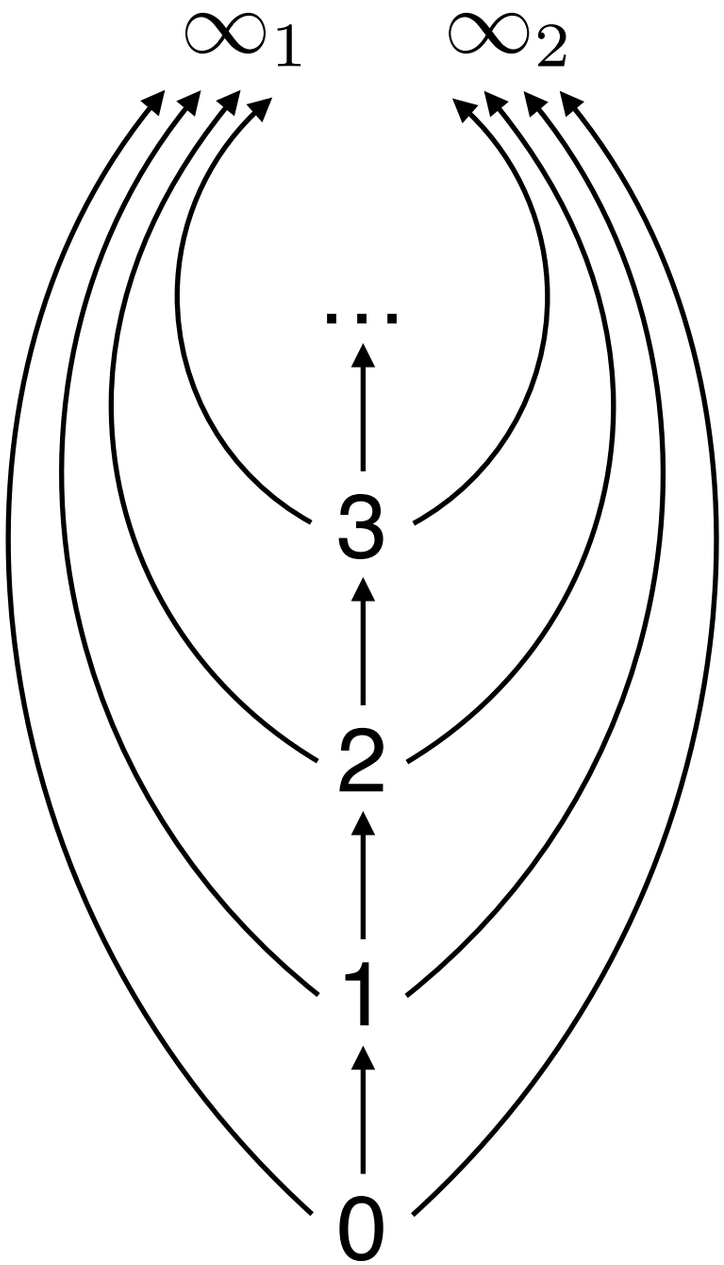
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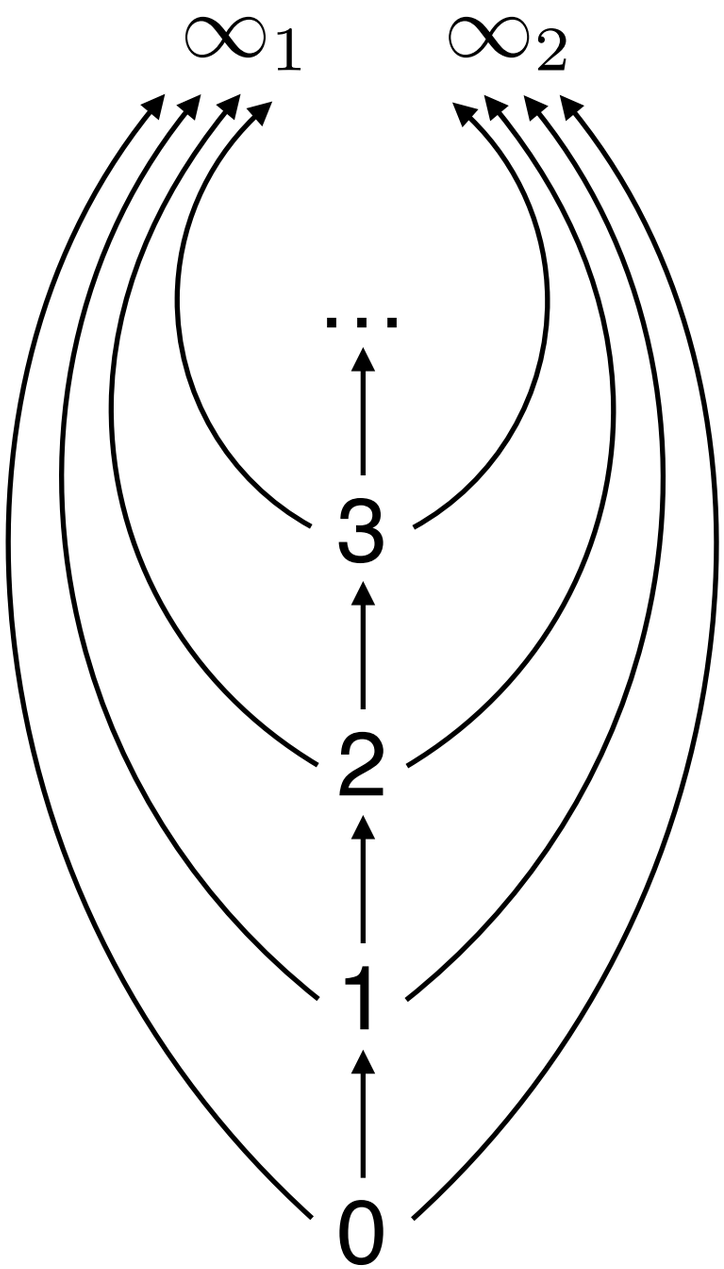
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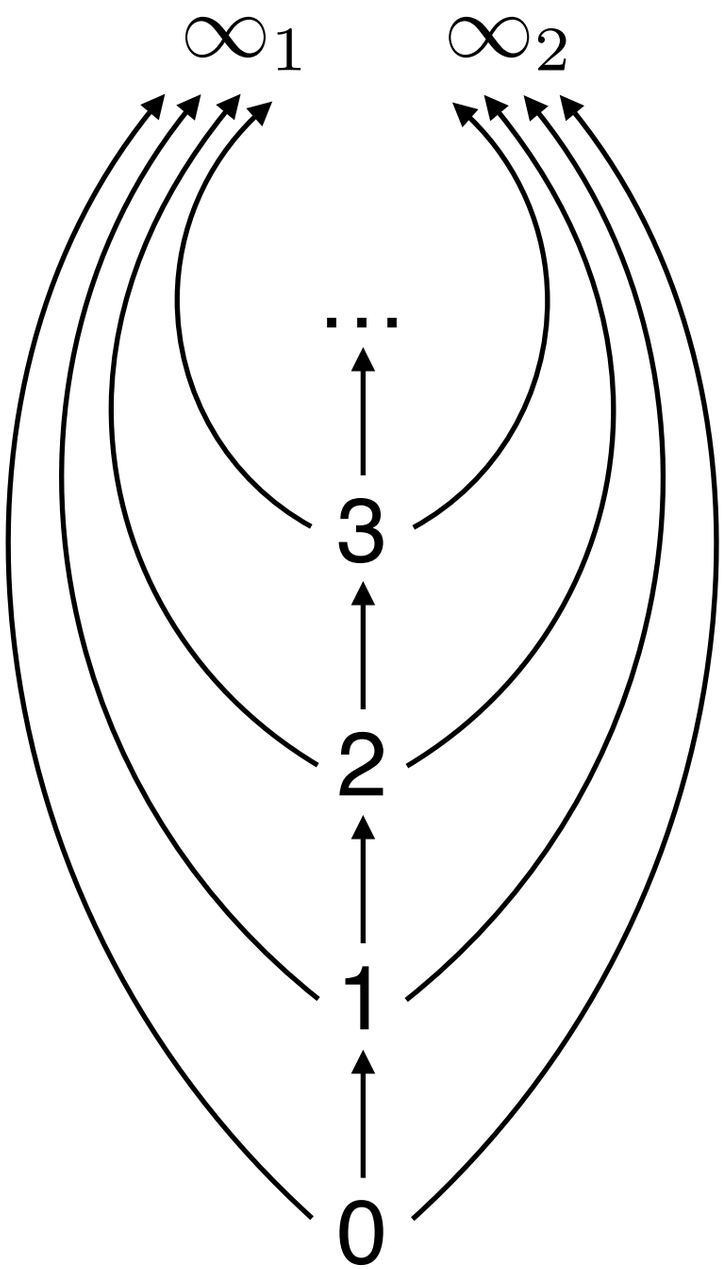
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(difficult) exercise

Let D be a CPO

let $\{d_i\}_{i \in \mathbb{N}}$ be a chain in D

let $\{k_j\}_{j \in \mathbb{N}}$ be an infinite chain in $(\mathbb{N}, \leq)$

1. Prove that $\{d_{k_j}\}_{j \in \mathbb{N}}$ is a chain in D
2. Prove or disprove that $\bigsqcup_{j \in \mathbb{N}} d_{k_j} = \bigsqcup_{i \in \mathbb{N}} d_i$

Partial functions

Comparing functions

given two functions $f, g : A \to B$, when can we say $f = g$?

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if we see functions as relations

$$\{ (a, f(a)) \mid a \in A \} \subseteq A \times B$$

we can use set equality

Example

$$f(n) = n!$$

$$
\begin{aligned}
f = \{ \quad & (0,1), \\
& (1,1), \\
& (2,2), \\
& (3,6), \\
& \ldots \\
& (k,k!), \\
& \ldots \qquad \}
\end{aligned}
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$$\{ (a, f(a)) \mid a \in A, f(a) \neq \perp \} \subseteq A \times B$$

omit pairs where $f(a)$ is undefined

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if we see partial functions as relations

$$\{(x, f(x)) \mid f(x) \neq \bot\} \subseteq A \times B$$

$f \sqsubseteq g$ means essentially $f \subseteq g$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ \perp & \text{otherwise} \end{cases}$$

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$\emptyset \sqsubseteq$ { (1,6) } $\sqsubseteq$ { (1,6), (2,28) } $\sqsubseteq$ { (1,6), (2,28), (3,496) } $\sqsubseteq$ { (1,6), (2,28), (3,496), (4,8128) } $\sqsubseteq$ ...

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$f(a) \downarrow \triangleq \exists b \in B.\ (a, b) \in f$ function f is defined on a

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$$\mathbf{Pf}(A, B) = \{f \subseteq A \times B \mid \forall a \in A. \forall b_1, b_2 \in B. (a, b_1) \in f \wedge (a, b_2) \in f \Rightarrow b_1 = b_2\}$$

functional property

$f(a)\downarrow \triangleq \exists b \in B.\ (a, b) \in f$ function f is defined on a

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what is bottom?

is it complete?

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the empty relation
(the function always undefined)

Is Pf complete?

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i.e. that $f = \bigcup_{i\in\mathbb{N}} f_i$ satisfies the functional property

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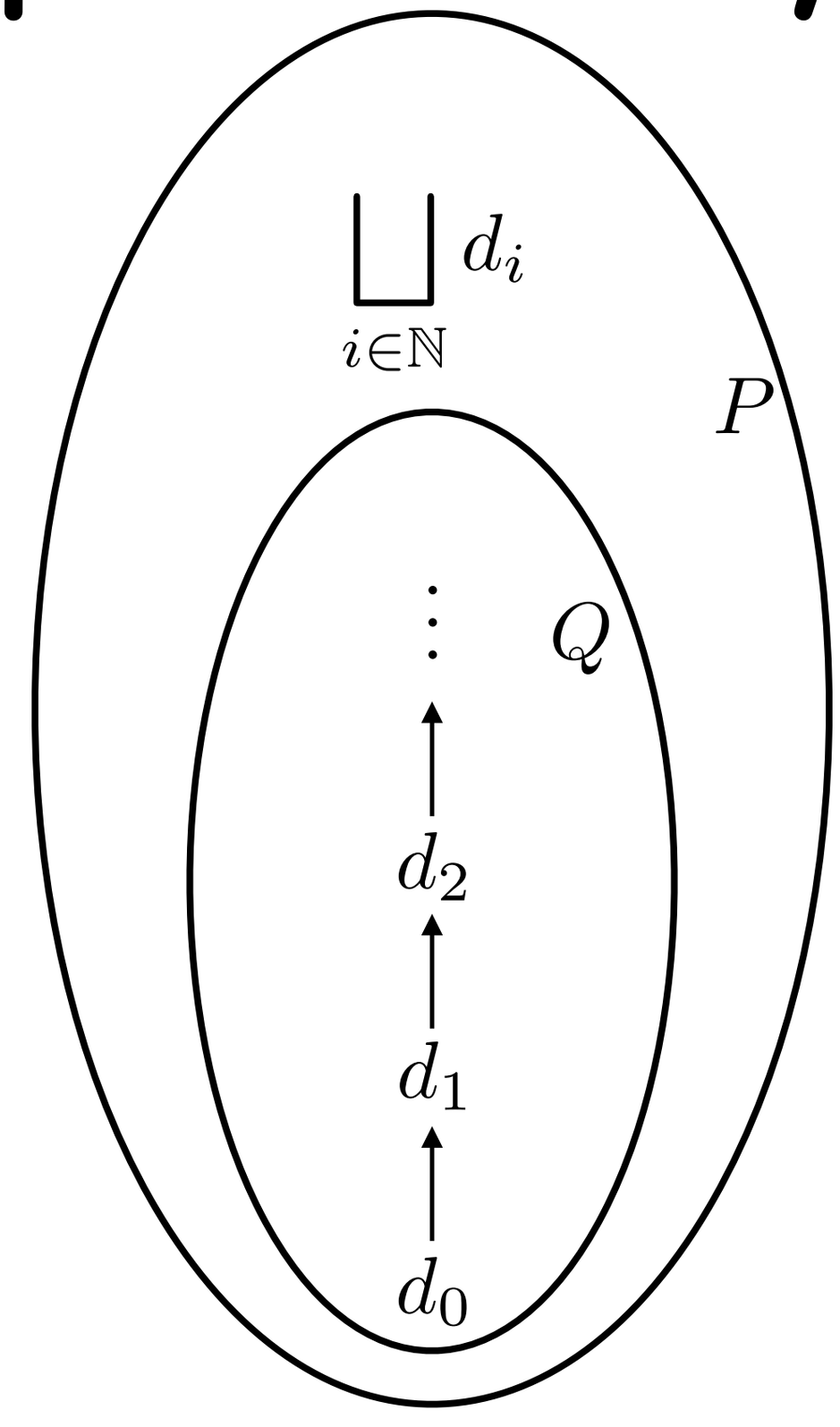
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pictorially



is the limit in Q ?

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Example

$\mathbf{Pf}(\mathbb{N},\mathbb{N})$

$$
\begin{aligned}
\emptyset &\subseteq \{(0,1)\} \\
&\subseteq \{(0,1),(1,1)\} \\
&\subseteq \{(0,1),(1,1),(2,2)\} \\
&\subseteq \{(0,1),(1,1),(2,2),(3,6)\} \\
&\subseteq \{(0,1),(1,1),(2,2),(3,6),(4,24)\} \\
&\subseteq \ldots
\end{aligned}
$$

Example

$$
\begin{array}{lll}
\mathbf{Pf}(\mathbb{N},\mathbb{N}) & f_0\ \emptyset & \\
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note: the limit of partial functions can be a total function

Total functions

$\mathbf{Tf}(A, B) = (A \to B)$ total functions

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$\sqsubseteq_{B_\perp} \triangleq$ flat order

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PO? immediate to check

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$\mathbf{Pf}(A, B) \equiv \mathbf{Tf}(A, B_\bot)$ $B_\bot \triangleq B \uplus \{\bot\}$

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bottom? $f_\perp(a) = \perp$ for any $a \in A$

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bottom? $f_\bot(a) = \bot$ for any $a \in A$

complete? we will prove it later

(as an instance of a more general result)

Total functions

$\mathbf{Tf}(A, B) = (A \to B)$ total functions

$\mathbf{Pf}(A, B) \equiv \mathbf{Tf}(A, B_\bot)$ $B_\bot \triangleq B \uplus \{\bot\}$

$\sqsubseteq_{B_\bot} \triangleq$ flat order

$$f \sqsubseteq g \Leftrightarrow \forall a \in A.\ f(a) \sqsubseteq_{B_\bot} g(a)$$

PO? immediate to check

bottom? $f_\bot(a) = \bot$ for any $a \in A$

complete? we will prove it later

(as an instance of a more general result)

$$(\bigsqcup_{i \in \mathbb{N}} f_i)(a) \triangleq \bigsqcup_{i \in \mathbb{N}} f_i(a)$$

Total functions

$\mathbf{Tf}(A, B) = (A \to B)$ total functions

$\mathbf{Pf}(A, B) \equiv \mathbf{Tf}(A, B_\perp)$ $B_\perp \triangleq B \uplus \{\perp\}$

$\sqsubseteq_{B_\perp} \triangleq$ flat order

$$f \sqsubseteq g \Leftrightarrow \forall a \in A.\ f(a) \sqsubseteq_{B_\perp} g(a)$$

PO? immediate to check

bottom? $f_\perp(a) = \perp$ for any $a \in A$

complete? we will prove it later

(as an instance of a more general result)

$$(\bigsqcup_{i \in \mathbb{N}} f_i)(a) \triangleq \bigsqcup_{i \in \mathbb{N}} f_i(a)$$ (flat order, limit exists)