

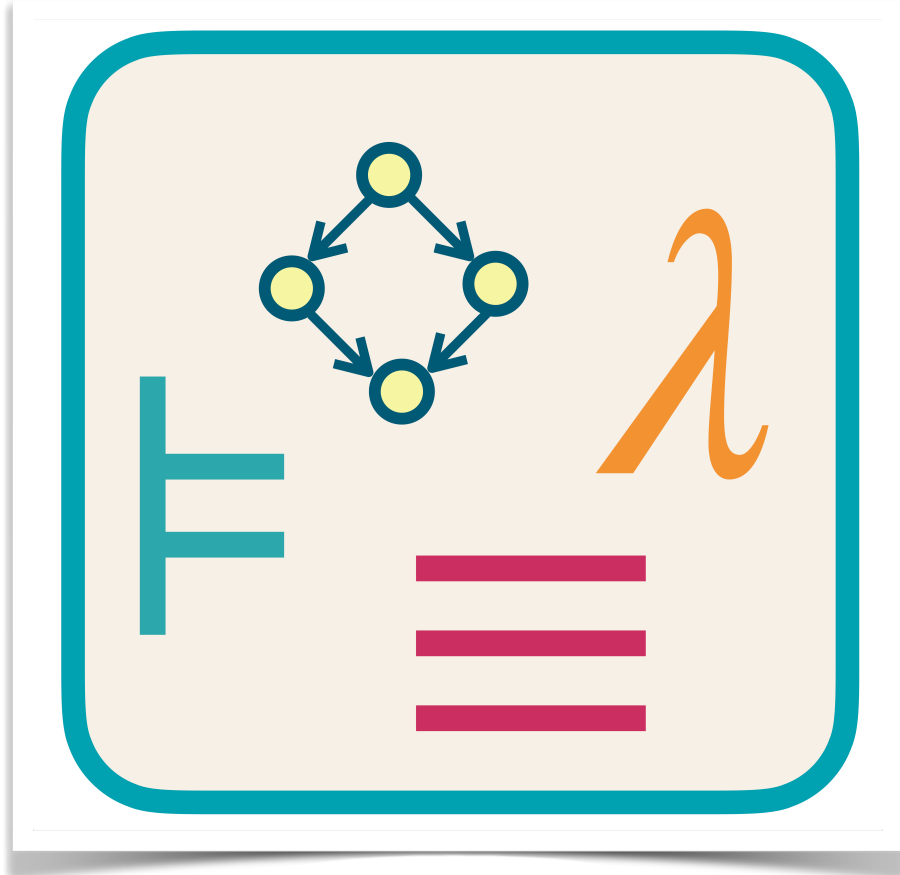
**LPP 2026/27** (063AA, 9CFU)

Programming Languages

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and Laboratory

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## 06 - Operational equivalence

# Operational equivalence

# Program equivalence

Symbolic manipulation of programs is possible  
if we can guarantee that their semantics is preserved

Useful to study program transformations  
(interpreters, compilers, code optimisation, refactoring...)

but... when are two commands equivalent?

# Operational equivalence

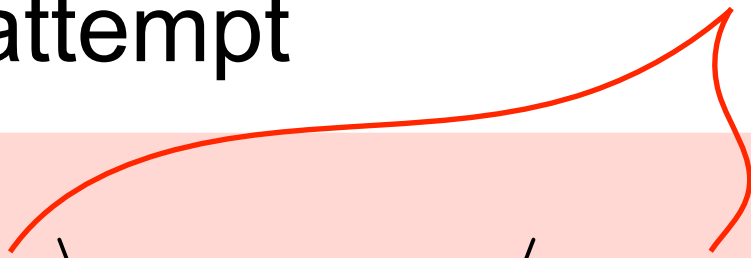
A first attempt

$$c_1 \sim_o c_2 \quad \triangleq \quad \forall \sigma. ( \langle c_1, \sigma_0 \rangle \longrightarrow \sigma \Leftrightarrow \langle c_2, \sigma_0 \rangle \longrightarrow \sigma )$$

# Operational equivalence

A first attempt

default initial memory


$$c_1 \sim_o c_2 \triangleq \forall \sigma. ( \langle c_1, \sigma_0 \rangle \longrightarrow \sigma \Leftrightarrow \langle c_2, \sigma_0 \rangle \longrightarrow \sigma )$$

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 $\forall x. \sigma_0(x) = 0$

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What's wrong with this definition?

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What's wrong with this definition?

$$x := y + 1 \sim_o x := 1$$

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$$C[\bullet] \triangleq y := 1; [\bullet]$$

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$$x := y + 1 \sim_o x := 1$$

$$C[\bullet] \triangleq y := 1; [\bullet] \qquad C[x := y + 1] \triangleq y := 1; x := y + 1$$

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$$C[x := y + 1] \triangleq y := 1; x := y + 1$$

$$C[x := 1] \triangleq y := 1; x := 1$$

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$$C[x := y + 1] \triangleq y := 1; x := y + 1$$

$$C[x := 1] \triangleq y := 1; x := 1$$

$$C[x := y + 1] \not\sim_o C[x := 1]$$

# Operational equivalence

$$c_1 \sim_o c_2 \quad \triangleq \quad \forall \sigma, \sigma'. ( \langle c_1, \sigma \rangle \longrightarrow \sigma' \Leftrightarrow \langle c_2, \sigma \rangle \longrightarrow \sigma' )$$

# Operational equivalence

any initial memory

$$c_1 \sim_o c_2 \quad \triangleq \quad \forall \sigma, \sigma'. \left( \langle c_1, \sigma \rangle \longrightarrow \sigma' \Leftrightarrow \langle c_2, \sigma \rangle \longrightarrow \sigma' \right)$$

# Operational equivalence

any initial memory

$$c_1 \sim_o c_2 \quad \triangleq \quad \forall \sigma, \sigma'. \left( \langle c_1, \sigma \rangle \longrightarrow \sigma' \Leftrightarrow \langle c_2, \sigma \rangle \longrightarrow \sigma' \right)$$

much better:  
it is an equivalence relation  
and also a congruence  
(but we will see this later)

# Concrete equivalence

$c \stackrel{?}{\sim}_o w$

$w \triangleq \text{while } x > 0 \text{ do } x := x - 1$

$c \triangleq \text{if } x > 0 \text{ then } x := 0 \text{ else skip}$

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Take a generic  $\sigma$

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Take a generic  $\sigma$  if  $\sigma(x) \leq 0$ :

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$\langle c, \sigma \rangle \longrightarrow \sigma'$

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$\forall n > 0. P(n) \quad P(n) \triangleq \forall \sigma. \sigma(x) = n \Rightarrow \langle w, \sigma \rangle \longrightarrow \sigma[0/x]$

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take a generic  $\sigma$  and suppose  $\sigma(x) = 1$

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$$\swarrow^* \square$$

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$$\forall b, c, \sigma, \sigma'. \langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma' \Rightarrow \langle b, \sigma' \rangle \longrightarrow \mathbf{ff}$$

# Example (ctd)

$$w \stackrel{?}{\sim}_o ww \qquad \begin{array}{l} w \triangleq \text{while } b \text{ do } c \\ ww \triangleq \text{while } b \text{ do } (\overbrace{\text{while } b \text{ do } c}^w) \end{array}$$

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$w_1 \triangleq \text{while true do skip}$

$p \triangleq$   $x := 3;$

$s := 1;$

**while**  $x \neq s$  **do** (

$x := x + 2;$

$s := 1;$

$i := 2;$

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always terminate

when  $x \geq 2$ , at the end,  $s$  takes the sum of all proper divisors of  $x$

```
 $s := 1;$   
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while  $2 \times i \leq x$  do (  
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$$s := \sum_{i \in \text{div}(x)} i$$

where  $\text{div}(x) \triangleq \{1\} \cup \{d \mid 1 < d < x, x \% d = 0\}$

$$i := 1 + (x/2)$$

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$w_1 \triangleq \text{while true do skip}$

$p \triangleq x := 3; \text{ an odd number}$

$w_1 \overset{?}{\sim}_o p$

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$p \triangleq$   $x := 3;$  an odd number  $w_1 \overset{?}{\sim}_o p$

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if it equals the sum of its proper divisors

$$6 = 1 + 2 + 3$$

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$w_1 \triangleq$  while true do skip

$p \triangleq$  terminate when the first odd perfect number is found  $w_1 \overset{?}{\sim}_o p$

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as of 2026

it is not known if any odd perfect number exists

Ochem, Pascal; Rao, Michaël.

Odd perfect numbers are greater than  $10^{1500}$ .

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It is not known if there are infinitely many Mersenne primes

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| prime number $p$ | Mersenne prime $2^p - 1$ | even perfect number $2^{p-1}(2^p - 1)$ |
|------------------|--------------------------|----------------------------------------|
| 2                | 3                        | 6                                      |
| 3                | 7                        | 28                                     |
| 5                | 31                       | 496                                    |
| 7                | 127                      | 8,128                                  |
| 13               | 8,191                    | 33,550,336                             |
| 17               | 131,071                  | 8,589,869,056                          |
| ...              | ...                      | ...                                    |
| 82,589,933       | 24.862.048 digits!       | 49.724.095 digits!                     |

in binary form:  
 $p$  ones followed  
 by  $p - 1$  zeros

136.279.841

41.024.320 digits!

82.048.640 digits!

GIMPS on October 12, 2024

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$w_1 \triangleq \text{while true do skip}$

$p_1 \triangleq$   $x := 2;$

$s := 1;$

**while**  $x \neq s$  **do** (

$x := x + 2;$

$s := 1;$

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$x := 6 \stackrel{?}{\sim}_o p_1$



$x := 6;$

$s := 6;$

$i := 4;$

$\stackrel{?}{\sim}_o p_1$

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# A variant

$w_1 \triangleq \text{while true do skip}$

$p_2 \triangleq$

```
  s := 1;
  while x ≠ s do (
    x := x + 2;
    s := 1;
    i := 2;
    while 2 × i ≤ x do (
      if x % i = 0 then s := s + i
      else skip;
      i := i + 1
    )
  )
```

$w_1 \overset{?}{\sim}_o p_2$

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$w_1 \triangleq \text{while true do skip}$

$p_2 \triangleq$   $s := 1;$   
  **while**  $x \neq s$  **do** (  
     $x := x + 2;$   
     $s := 1;$   
     $i := 2;$   
    **while**  $2 \times i \leq x$  **do** (  
      **if**  $x \% i = 0$  **then**  $s := s + i$   
      **else skip;**  
       $i := i + 1$   
    )  
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$w_1 \overset{?}{\sim}_o p_2$



$x := 6;$   
 $s := 6;$   
 $i := 4;$

$\overset{?}{\sim}_o p_2$

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$w_1 \overset{?}{\sim}_o p_2$



$x := 6;$

$s := 6; \overset{?}{\sim}_o p_2$

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# A last instance

$$p_2 \stackrel{?}{\sim}_o p_3$$

$p_2 \triangleq$   $s := 1;$   
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     $x := x + 2;$   
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    **while**  $2 \times i \leq x$  **do** (  
        **if**  $x \% i = 0$  **then**  $s := s + i$   
        **else skip;**  
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$p_3 \triangleq$   $s := 1;$   
**while**  $x \neq s$  **do** (  
     $x := x + 1;$   
     $s := 1;$   
     $i := 2;$   
    **while**  $2 \times i \leq x$  **do** (  
        **if**  $x \% i = 0$  **then**  $s := s + i$   
        **else skip;**  
         $i := i + 1$   
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# A last instance

$$p_2 \stackrel{?}{\sim}_o p_3$$


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**while**  $x \neq s$  **do** (  
     $x := x + 2;$   
     $s := 1;$   
     $i := 2;$   
    **while**  $2 \times i \leq x$  **do** (  
        **if**  $x \% i = 0$  **then**  $s := s + i$   
        **else skip;**  
         $i := i + 1$   
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$p_3 \triangleq$   $s := 1;$   
**while**  $x \neq s$  **do** (  
     $x := x + 1;$   
     $s := 1;$   
     $i := 2;$   
    **while**  $2 \times i \leq x$  **do** (  
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