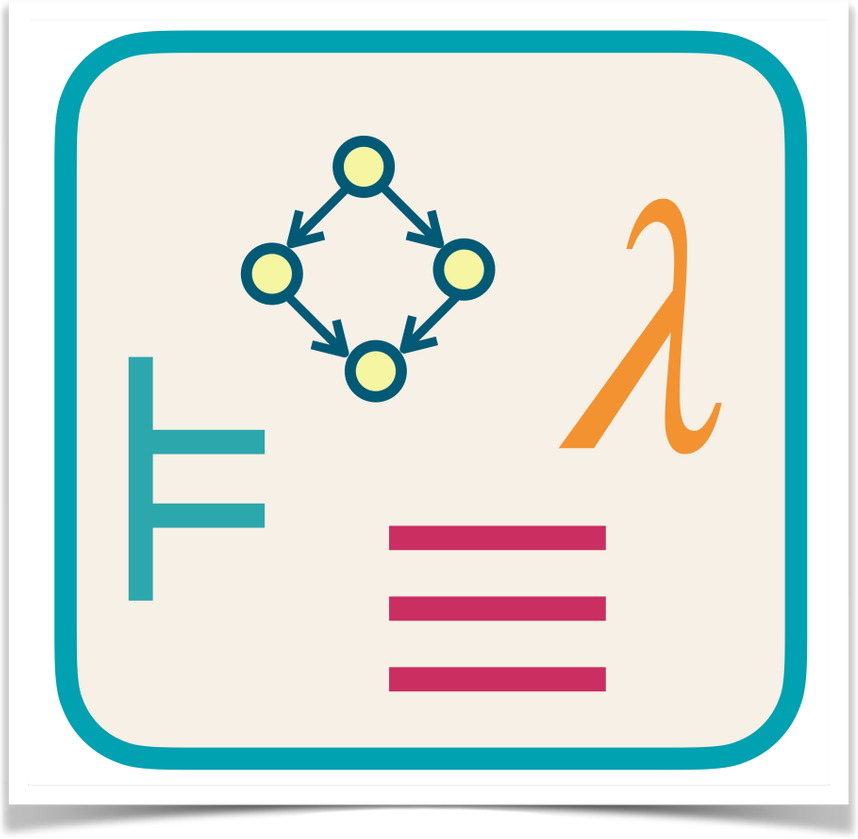


LPP 2026/27 (063AA, 9CFU)

Programming Languages

Fabio Gadducci

and Laboratory

Vincenzo Ciancia

05b - More Induction

Mathematical induction exercises

Exercise

$\forall n > 0.\ P(n)$

$P(n) \triangleq n^n \geq n!$

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Exercise

$\forall n > 0.\ P(n)$ $\qquad\qquad$ $P(n) \triangleq n^n \geq n!$

$P(1)$

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$\forall n > 0.\ P(n)$ $\qquad$ $P(n) \triangleq n^n \geq n!$

$P(1)$ $\qquad$ $1^1 = 1 \geq 1 = 1!$

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$P(1)$ $\qquad$ $1^1 = 1 \geq 1 = 1!$

$\forall n.\ P(n) \Rightarrow P(n+1)$

Exercise

$\forall n > 0.\ P(n)$ $\qquad$ $P(n) \triangleq n^n \geq n!$

$P(1)$ $\qquad$ $1^1 = 1 \geq 1 = 1!$

$\forall n.\ P(n) \Rightarrow P(n+1)$ $\qquad$ Take a generic n

Exercise

$\forall n > 0.\ P(n)$ $\qquad$ $P(n) \triangleq n^n \geq n!$

$P(1)$ $\qquad$ $1^1 = 1 \geq 1 = 1!$

$\forall n.\ P(n) \Rightarrow P(n+1)$ $\qquad$ Take a generic n

Assume $P(n) \triangleq n^n \geq n!$

Exercise

$\forall n > 0.\ P(n)$ $\qquad$ $P(n) \triangleq n^n \geq n!$

$P(1)$ $\qquad$ $1^1 = 1 \geq 1 = 1!$

$\forall n.\ P(n) \Rightarrow P(n+1)$ $\qquad$ Take a generic n

Assume $P(n) \triangleq n^n \geq n!$

We want to prove $P(n+1) \triangleq (n+1)^{n+1} \geq (n+1)!$

Exercise

$\forall n > 0.\ P(n)$ $\qquad$ $P(n) \triangleq n^n \geq n!$

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Assume $P(n) \triangleq n^n \geq n!$

We want to prove $P(n+1) \triangleq (n+1)^{n+1} \geq (n+1)!$

$$(n+1)^{n+1} = (n+1) \cdot (n+1)^n \geq (n+1) \cdot n^n \geq (n+1) \cdot n! = (n+1)!$$

Exercise

$$a_0 \triangleq 0 \qquad a_{n+1} \triangleq 2a_n + n$$

$$\forall n \in \mathbb{N}.\ P(n) \qquad P(n) \triangleq a_n = 2^n - n - 1$$

Exercise

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$$a_0 = 0 = 1 - 0 - 1 = 2^0 - 0 - 1$$

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$$a_0 = 0 = 1 - 0 - 1 = 2^0 - 0 - 1$$

$\forall n.\ P(n) \Rightarrow P(n+1)$ Take a generic n

Assume $P(n) \triangleq a_n = 2^n - n - 1$

We want to prove $P(n+1) \triangleq a_{n+1} = 2^{n+1} - (n+1) - 1$

Exercise

$$a_0 \triangleq 0 \qquad a_{n+1} \triangleq 2a_n + n$$

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We want to prove $P(n+1) \triangleq a_{n+1} = 2^{n+1} - (n+1) - 1$

$$a_{n+1} = 2a_n + n = 2(2^n - n - 1) + n = 2^{n+1} - 2n - 2 + n$$

Exercise

$$a_0 \triangleq 0 \qquad a_{n+1} \triangleq 2a_n + n$$

$$\forall n \in \mathbb{N}.\ P(n) \qquad P(n) \triangleq a_n = 2^n - n - 1$$

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We want to prove $P(n+1) \triangleq a_{n+1} = 2^{n+1} - (n+1) - 1$

$$a_{n+1} = 2a_n + n = 2(2^n - n - 1) + n = 2^{n+1} - 2n - 2 + n$$

$$= 2^{n+1} - n - 2 = 2^{n+1} - (n+1) - 1$$

Exercise: Fibonacci

$$F_1 \triangleq 1 \qquad F_2 \triangleq 1 \qquad F_{n+2} \triangleq F_n + F_{n+1}$$

$$\forall n > 0.\ P(n) \qquad P(n) \triangleq \sum_{i=1}^{n} F_i = F_{n+2} - 1$$

Exercise: Fibonacci

$$F_1 \triangleq 1 \qquad F_2 \triangleq 1 \qquad F_{n+2} \triangleq F_n + F_{n+1}$$

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$$\forall n > 0.\ P(n) \qquad P(n) \triangleq \sum_{i=1}^{n} F_i = F_{n+2} - 1$$

$P(1)$ $\quad \sum_{i=1}^{1} F_i = F_1 = 1$

Exercise: Fibonacci

$$F_1 \triangleq 1 \qquad F_2 \triangleq 1 \qquad F_{n+2} \triangleq F_n + F_{n+1}$$

$$\forall n > 0.\ P(n) \qquad P(n) \triangleq \sum_{i=1}^{n} F_i = F_{n+2} - 1$$

$P(1)$

$$\sum_{i=1}^{1} F_i = F_1 = 1$$

$$F_{1+2} - 1 = F_3 - 1 = F_1 + F_2 - 1 = 1 + 1 - 1 = 1$$

Exercise: Fibonacci

$$F_1 \triangleq 1 \qquad F_2 \triangleq 1 \qquad F_{n+2} \triangleq F_n + F_{n+1}$$

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$P(1)$

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$$F_{1+2} - 1 = F_3 - 1 = F_1 + F_2 - 1 = 1 + 1 - 1 = 1$$

$P(2)$

Exercise: Fibonacci

$$F_1 \triangleq 1 \qquad F_2 \triangleq 1 \qquad F_{n+2} \triangleq F_n + F_{n+1}$$

$$\forall n > 0.\ P(n) \qquad P(n) \triangleq \sum_{i=1}^{n} F_i = F_{n+2} - 1$$

$P(1)$

$$\sum_{i=1}^{1} F_i = F_1 = 1$$

$$F_{1+2} - 1 = F_3 - 1 = F_1 + F_2 - 1 = 1 + 1 - 1 = 1$$

$P(2)$

$$\sum_{i=1}^{2} F_i = F_1 + F_2 = F_3$$

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$$F_1 \triangleq 1 \qquad F_2 \triangleq 1 \qquad F_{n+2} \triangleq F_n + F_{n+1}$$

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$$F_{1+2} - 1 = F_3 - 1 = F_1 + F_2 - 1 = 1 + 1 - 1 = 1$$

$P(2)$

$$\sum_{i=1}^{2} F_i = F_1 + F_2 = F_3$$

$$F_{2+2} - 1 = F_4 - 1 = F_2 + F_3 - 1 = 1 + F_3 - 1 = F_3$$

Exercise: Fibonacci

$$F_1 \triangleq 1 \qquad F_2 \triangleq 1 \qquad F_{n+2} \triangleq F_n + F_{n+1}$$

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$\forall n.\ P(n) \Rightarrow P(n+1)$

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$\forall n.\ P(n) \Rightarrow P(n+1)$ Take a generic n

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$\forall n.\ P(n) \Rightarrow P(n+1)$ Take a generic n

Assume $P(n) \triangleq \sum_{i=1}^{n} F_i = F_{n+2} - 1$

We want to prove $P(n+1) \triangleq \sum_{i=1}^{n+1} F_i = F_{(n+1)+2} - 1$

Exercise: Fibonacci

$$F_1 \triangleq 1 \qquad F_2 \triangleq 1 \qquad F_{n+2} \triangleq F_n + F_{n+1}$$

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Assume $P(n) \triangleq \sum_{i=1}^{n} F_i = F_{n+2} - 1$

We want to prove $P(n+1) \triangleq \sum_{i=1}^{n+1} F_i = F_{(n+1)+2} - 1$

$$\sum_{i=1}^{n+1} F_i = F_{n+1} + \sum_{i=1}^{n} F_i = F_{n+1} + F_{n+2} - 1 = F_{n+3} - 1 = F_{(n+1)+2} - 1$$

Determinacy by structural induction

Termination

A result can always be returned

Determinacy

Any two results are the same

Termination vs determinacy

	termination?	determinacy?
$x + y$		
x^y		
$rand()$		
$\pm sqrt(x)$		

Termination vs determinacy

	termination?	determinacy?
$x + y$	✓	
x^y		
$rand()$		
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Termination vs determinacy

	termination?	determinacy?
$x + y$	✓	✓
x^y		
$rand()$		
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Termination vs determinacy

	termination?	determinacy?
$x + y$	✓	✓
x^y	✗	
$rand()$		
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Termination vs determinacy

	termination?	determinacy?
$x + y$	✓	✓
x^y	✗	✓
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Termination vs determinacy

	termination?	determinacy?
$x + y$	✓	✓
x^y	✗	✓
$rand()$	✓	
$\pm sqrt(x)$		

Termination vs determinacy

	termination?	determinacy?
$x + y$	✓	✓
x^y	✗	✓
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Termination vs determinacy

	termination?	determinacy?
$x + y$	✓	✓
x^y	✗	✓
$rand()$	✓	✗
$\pm sqrt(x)$	✗	

Termination vs determinacy

	termination?	determinacy?
$x + y$	✓	✓
x^y	✗	✓
$rand()$	✓	✗
$\pm sqrt(x)$	✗	✗

Determinacy of arithmetic expressions

$$a ::= x \mid n \mid a \ \mathsf{op} \ a$$

$$x \in \mathsf{Ide} \qquad \mathsf{op} \in \{+, \times, -\}$$

$$n \in \mathbb{Z} \qquad \mathbb{M} \triangleq \{\sigma \mid \sigma : \mathsf{Ide} \to \mathbb{Z}\}$$

$$\frac{}{\langle x, \sigma \rangle \longrightarrow \sigma(x)} \qquad \frac{}{\langle n, \sigma \rangle \longrightarrow n} \qquad \frac{\langle a_0, \sigma \rangle \longrightarrow n_0 \quad \langle a_1, \sigma \rangle \longrightarrow n_1}{\langle a_0 \ \mathsf{op} \ a_1, \sigma \rangle \longrightarrow n_0 \ op \ n_1}$$

Determinacy of arithmetic expressions

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$$P(a) \triangleq \forall \sigma \in \mathbb{M}.\ \forall m, m' \in \mathbb{Z}.\ \langle a, \sigma \rangle \longrightarrow m \wedge \langle a, \sigma \rangle \longrightarrow m' \Rightarrow m = m'$$

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$$\forall a.\ P(a)\ ?$$

Structural induction principle

$$\frac{\forall a_0, a_1.\ P(a_0) \wedge P(a_1) \Rightarrow P(a_0 \ \mathsf{op}\ a_1)}{\forall a.\ P(a)}$$

$\forall x \in \mathsf{Ide}.\ P(x)$ $\qquad$ $\forall n \in \mathbb{Z}.\ P(n)$

Base case

$\forall x \in \mathsf{Ide}.\ P(x)$

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Take a generic $x \in \mathsf{Ide}$

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Take a generic $x \in \mathsf{Ide}$

We want to prove

$$P(x) \triangleq \forall \sigma, m, m'.\ \langle x, \sigma \rangle \longrightarrow m \wedge \langle x, \sigma \rangle \longrightarrow m' \Rightarrow m = m'$$

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Take generic σ, m, m' s.t. $\langle x, \sigma \rangle \longrightarrow m$ and $\langle x, \sigma \rangle \longrightarrow m'$

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We want to prove $m = m'$

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Take generic σ, m, m' s.t. $\langle x, \sigma \rangle \longrightarrow m$ and $\langle x, \sigma \rangle \longrightarrow m'$

We want to prove $m = m'$

Consider the goal $\langle x, \sigma \rangle \longrightarrow m$

Base case

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Take a generic $x \in \mathsf{Ide}$

We want to prove

$$P(x) \triangleq \forall \sigma, m, m'.\ \langle x, \sigma \rangle \longrightarrow m \wedge \langle x, \sigma \rangle \longrightarrow m' \Rightarrow m = m'$$

Take generic σ, m, m' s.t. $\langle x, \sigma \rangle \longrightarrow m$ and $\langle x, \sigma \rangle \longrightarrow m'$

We want to prove $m = m'$

Consider the goal $\langle x, \sigma \rangle \longrightarrow m$

Only the rule $\dfrac{}{\langle x, \sigma \rangle \longrightarrow \sigma(x)}$ is applicable, hence $m = \sigma(x)$

Base case

$\forall x \in \mathsf{Ide}.\ P(x)$

Take a generic $x \in \mathsf{Ide}$

We want to prove

$$P(x) \triangleq \forall \sigma, m, m'.\ \langle x, \sigma \rangle \longrightarrow m \wedge \langle x, \sigma \rangle \longrightarrow m' \Rightarrow m = m'$$

Take generic σ, m, m' s.t. $\langle x, \sigma \rangle \longrightarrow m$ and $\langle x, \sigma \rangle \longrightarrow m'$

We want to prove $m = m'$

Consider the goal $\langle x, \sigma \rangle \longrightarrow m$

Only the rule $\dfrac{}{\langle x, \sigma \rangle \longrightarrow \sigma(x)}$ is applicable, hence $m = \sigma(x)$

Similarly, since $\langle x, \sigma \rangle \longrightarrow m'$ it must be $m' = \sigma(x)$

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Take a generic $x \in \mathsf{Ide}$

We want to prove

$$P(x) \triangleq \forall \sigma, m, m'.\ \langle x, \sigma \rangle \longrightarrow m \wedge \langle x, \sigma \rangle \longrightarrow m' \Rightarrow m = m'$$

Take generic σ, m, m' s.t. $\langle x, \sigma \rangle \longrightarrow m$ and $\langle x, \sigma \rangle \longrightarrow m'$

We want to prove $m = m'$

Consider the goal $\langle x, \sigma \rangle \longrightarrow m$

Only the rule $\dfrac{}{\langle x, \sigma \rangle \longrightarrow \sigma(x)}$ is applicable, hence $m = \sigma(x)$

Similarly, since $\langle x, \sigma \rangle \longrightarrow m'$ it must be $m' = \sigma(x)$

and thus we conclude $m = m'$

Base case

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We want to prove $m = m'$

Consider the goal $\langle n, \sigma \rangle \longrightarrow m$

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$\forall n \in \mathbb{Z}.\ P(n)$

Take a generic $n \in \mathbb{Z}$

We want to prove

$$P(n) \triangleq \forall \sigma, m, m'.\ \langle n, \sigma \rangle \longrightarrow m \wedge \langle n, \sigma \rangle \longrightarrow m' \Rightarrow m = m'$$

Take generic σ, m, m' s.t. $\langle n, \sigma \rangle \longrightarrow m$ and $\langle n, \sigma \rangle \longrightarrow m'$

We want to prove $m = m'$

Consider the goal $\langle n, \sigma \rangle \longrightarrow m$

Only the rule $\dfrac{}{\langle n, \sigma \rangle \longrightarrow n}$ is applicable, hence $m = n$

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Take a generic $n \in \mathbb{Z}$

We want to prove

$$P(n) \triangleq \forall \sigma, m, m'.\ \langle n, \sigma \rangle \longrightarrow m \wedge \langle n, \sigma \rangle \longrightarrow m' \Rightarrow m = m'$$

Take generic σ, m, m' s.t. $\langle n, \sigma \rangle \longrightarrow m$ and $\langle n, \sigma \rangle \longrightarrow m'$

We want to prove $m = m'$

Consider the goal $\langle n, \sigma \rangle \longrightarrow m$

Only the rule $\dfrac{}{\langle n, \sigma \rangle \longrightarrow n}$ is applicable, hence $m = n$

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Take a generic $n \in \mathbb{Z}$

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We want to prove $m = m'$

Consider the goal $\langle n, \sigma \rangle \longrightarrow m$

Only the rule $\dfrac{}{\langle n, \sigma \rangle \longrightarrow n}$ is applicable, hence $m = n$

Similarly, since $\langle n, \sigma \rangle \longrightarrow m'$ it must be $m' = n$

and thus we conclude $m = m'$

Inductive case

$$\forall a_0, a_1.\ P(a_0) \wedge P(a_1) \Rightarrow P(a_0 \text{ op } a_1)$$

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Take generic a_0, a_1

Inductive case

$\forall a_0, a_1.\ P(a_0) \wedge P(a_1) \Rightarrow P(a_0 \text{ op } a_1)$ Take generic a_0, a_1

We assume

$$P(a_i) \triangleq \forall \sigma, m_i, m_i'.\ \langle a_i, \sigma \rangle \longrightarrow m_i \wedge \langle a_i, \sigma \rangle \longrightarrow m_i' \Rightarrow m_i = m_i'$$

Inductive case

$\forall a_0, a_1.\ P(a_0) \wedge P(a_1) \Rightarrow P(a_0 \text{ op } a_1)$ Take generic a_0, a_1

We assume (inductive hypotheses)

$$P(a_i) \triangleq \forall \sigma, m_i, m_i'.\ \langle a_i, \sigma \rangle \longrightarrow m_i \wedge \langle a_i, \sigma \rangle \longrightarrow m_i' \Rightarrow m_i = m_i'$$

Inductive case

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We want to prove

$$P(a_0 \text{ op } a_1) \triangleq \forall \sigma, m, m'.\ \langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m \wedge \langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m' \Rightarrow m = m'$$

Inductive case

$\forall a_0, a_1.\ P(a_0) \wedge P(a_1) \Rightarrow P(a_0 \text{ op } a_1)$ Take generic a_0, a_1

We assume (inductive hypotheses)

$$P(a_i) \triangleq \forall \sigma, m_i, m_i'.\ \langle a_i, \sigma \rangle \longrightarrow m_i \wedge \langle a_i, \sigma \rangle \longrightarrow m_i' \Rightarrow m_i = m_i'$$

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Take generic σ, m, m' such that $\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m$ and $\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m'$

Inductive case

$\forall a_0, a_1.\ P(a_0) \wedge P(a_1) \Rightarrow P(a_0 \text{ op } a_1)$ Take generic a_0, a_1

We assume (inductive hypotheses)

$$P(a_i) \triangleq \forall \sigma, m_i, m_i'.\ \langle a_i, \sigma \rangle \longrightarrow m_i \wedge \langle a_i, \sigma \rangle \longrightarrow m_i' \Rightarrow m_i = m_i'$$

We want to prove

$$P(a_0 \text{ op } a_1) \triangleq \forall \sigma, m, m'.\ \langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m \wedge \langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m' \Rightarrow m = m'$$

Take generic σ, m, m' such that $\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m$ and $\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m'$

We want to prove $m = m'$

Inductive case (cont.)

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Consider the goal $\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m$

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Only the rule $\dfrac{\langle a_0, \sigma \rangle \longrightarrow n_0 \quad \langle a_1, \sigma \rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow n_0 \ op \ n_1}$ is applicable

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hence $m = n_0 \; op \; n_1$ with $\langle a_0, \sigma\rangle \longrightarrow n_0$ and $\langle a_1, \sigma\rangle \longrightarrow n_1$

Inductive case (cont.)

Consider the goal $\langle a_0 \text{ op } a_1, \sigma\rangle \longrightarrow m$

Only the rule $\dfrac{\langle a_0, \sigma\rangle \longrightarrow n_0 \quad \langle a_1, \sigma\rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1, \sigma\rangle \longrightarrow n_0 \; op \; n_1}$ is applicable

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Similarly, since $\langle a_0 \text{ op } a_1, \sigma\rangle \longrightarrow m'$

Inductive case (cont.)

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Similarly, since $\langle a_0 \text{ op } a_1, \sigma\rangle \longrightarrow m'$

it must be $m' = n_0' \; op \; n_1'$ with $\langle a_0, \sigma\rangle \longrightarrow n_0'$ and $\langle a_1, \sigma\rangle \longrightarrow n_1'$

Inductive case (cont.)

Consider the goal $\langle a_0 \text{ op } a_1, \sigma\rangle \longrightarrow m$

Only the rule $\dfrac{\langle a_0, \sigma\rangle \longrightarrow n_0 \quad \langle a_1, \sigma\rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1, \sigma\rangle \longrightarrow n_0 \; op \; n_1}$ is applicable

hence $m = n_0 \; op \; n_1$ with $\langle a_0, \sigma\rangle \longrightarrow n_0$ and $\langle a_1, \sigma\rangle \longrightarrow n_1$

Similarly, since $\langle a_0 \text{ op } a_1, \sigma\rangle \longrightarrow m'$

it must be $m' = n'_0 \; op \; n'_1$ with $\langle a_0, \sigma\rangle \longrightarrow n'_0$ and $\langle a_1, \sigma\rangle \longrightarrow n'_1$

By inductive hypotheses, $n_0 = n'_0$ and $n_1 = n'_1$

Inductive case (cont.)

Consider the goal $\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m$

Only the rule $\dfrac{\langle a_0, \sigma \rangle \longrightarrow n_0 \quad \langle a_1, \sigma \rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow n_0 \; op \; n_1}$ is applicable

hence $m = n_0 \; op \; n_1$ with $\langle a_0, \sigma \rangle \longrightarrow n_0$ and $\langle a_1, \sigma \rangle \longrightarrow n_1$

Similarly, since $\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m'$

it must be $m' = n'_0 \; op \; n'_1$ with $\langle a_0, \sigma \rangle \longrightarrow n'_0$ and $\langle a_1, \sigma \rangle \longrightarrow n'_1$

By inductive hypotheses, $n_0 = n'_0$ and $n_1 = n'_1$

and thus we conclude $m = n_0 \; op \; n_1 = n'_0 \; op \; n'_1 = m'$

Many sorted signatures

Different types of expressions

$$e ::= x \mid n \mid e + e \mid e \leq e \mid e \,\&\, e$$

$$((x + 1) \leq (y - 1)) \,\&\, (x \leq 42)$$

$$((((x + 1) \leq y) - 1) \,\&\, x) \leq 42$$

$$x + (1 \leq (y - (1 \,\&\, (x \leq 42))))$$

Many sorted signatures

$a ::= x \mid n \mid a + a$ arithmetic expressions

$b ::= a \leq a \mid b \,\&\, b$ Boolean expressions

Sorts: identifiers x

numerals n

arithmetic expressions a

Boolean expressions b

Operators:

- $\cdot + \cdot$ takes two arithmetic expressions and returns an arithmetic expression
- $\cdot \leq \cdot$ takes two arithmetic expressions and returns a Boolean expression
- $\cdot \,\&\, \cdot$ takes two Boolean expressions and returns a Boolean expression

Terms over a signature

$S = \{s, ...\}$ a set of sorts

$\Sigma = \{\Sigma_{s_1 \ldots s_n, s}\}_{s_1, \ldots, s_n, s \in S}$ a many sorted signature

$f \in \Sigma_{s_1 \ldots s_n, s}$ $\qquad$ $f : (s_1 \times \cdots \times s_n) \to s$

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it is the least set such that:

- if $c \in \Sigma_{\epsilon, s}$, then $c \in T_{\Sigma, s}$
- if $f \in \Sigma_{s_1 ... s_n, s}$ and $\forall i.\ t_i \in T_{\Sigma, s_i}$, then $f(t_1, ..., t_n) \in T_{\Sigma, s}$

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- if $f \in \Sigma_{s_1 ... s_n, s}$ and $\forall i.\ t_i \in T_{\Sigma, s_i}$, then $f(t_1, ..., t_n) \in T_{\Sigma, s}$

$T_\Sigma = \{T_{\Sigma, s}\}_{s \in S}$ denotes the set of all well-sorted terms

Boolean expressions

$x \in \mathsf{Ide} \qquad n \in \mathbb{Z} \qquad \mathsf{op} \in \{+, \times, -\}$

$$a ::= x \mid n \mid a \ \mathsf{op} \ a$$

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Boolean expressions

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$v \in \mathbb{B}$

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$\Sigma_{\epsilon,\mathsf{Aexp}} \triangleq \mathsf{Ide} \cup \mathbb{Z}$

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$\Sigma_{\mathsf{BexpBexp},\mathsf{Bexp}} \triangleq \{\wedge, \vee\}$

Semantics of expressions

$$a ::= x \mid n \mid a \text{ op } a$$
$$b ::= v \mid a \text{ cmp } a \mid \neg b \mid b \text{ bop } b$$

$$\frac{}{\langle x, \sigma\rangle \longrightarrow \sigma(x)} \qquad \frac{}{\langle n, \sigma\rangle \longrightarrow n} \qquad \frac{\langle a_0, \sigma\rangle \longrightarrow n_0 \quad \langle a_1, \sigma\rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1, \sigma\rangle \longrightarrow n_0 \; op \; n_1}$$

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$$\frac{}{\langle v, \sigma \rangle \longrightarrow v}$$

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Subterms

$$t_i \prec f(t_1, ..., t_n)$$

sometimes, an additional requirement:
sort-preserving relation

$$f \in \Sigma_{s_1 \ldots s_n, s} \ \wedge \ s_i = s$$

One sorted = Unsorted

a special case:

$S = \{*\}$ a singleton set of sorts

$$\Sigma_{\underbrace{*\ldots*}_{n},*} \simeq \Sigma_n$$

Termination of Boolean expressions

Termination of expressions

$$a ::= x \mid n \mid a \text{ op } a$$
$$b ::= v \mid a \text{ cmp } a \mid \neg b \mid b \text{ bop } b$$

$$\frac{}{\langle x, \sigma\rangle \longrightarrow \sigma(x)} \qquad \frac{}{\langle n, \sigma\rangle \longrightarrow n} \qquad \frac{\langle a_0, \sigma\rangle \longrightarrow n_0 \quad \langle a_1, \sigma\rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1, \sigma\rangle \longrightarrow n_0 \; op \; n_1}$$

$$\frac{}{\langle v, \sigma\rangle \longrightarrow v} \qquad \frac{\langle b, \sigma\rangle \longrightarrow v}{\langle \neg b, \sigma\rangle \longrightarrow \neg v} \qquad \frac{\langle a_0, \sigma\rangle \longrightarrow n_0 \quad \langle a_1, \sigma\rangle \longrightarrow n_1}{\langle a_0 \text{ cmp } a_1, \sigma\rangle \longrightarrow n_0 \; cmp \; n_1}$$

$$\frac{\langle b_0, \sigma\rangle \longrightarrow v_0 \quad \langle b_1, \sigma\rangle \longrightarrow v_1}{\langle b_0 \text{ bop } b_1, \sigma\rangle \longrightarrow v_0 \; bop \; v_1}$$

$P(b) \triangleq \forall \sigma \in \mathbb{M}.\ \exists v \in \mathbb{B}.\ \langle b, \sigma\rangle \longrightarrow v$ $\qquad$ $\forall b.\ P(b)$?

Base case

$\forall v \in \mathbb{B}.\ P(v)$

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Take a generic $v \in \mathbb{B}$

Base case

$\forall v \in \mathbb{B}.\ P(v)$

Take a generic $v \in \mathbb{B}$

We want to prove $P(v) \triangleq \forall \sigma.\ \exists u.\ \langle v, \sigma \rangle \longrightarrow u$

Base case

$\forall v \in \mathbb{B}.\ P(v)$ Take a generic $v \in \mathbb{B}$

We want to prove $P(v) \triangleq \forall \sigma.\ \exists u.\ \langle v, \sigma \rangle \longrightarrow u$

Take a generic $\sigma \in \mathbb{M}$ and consider the goal $\langle v, \sigma \rangle \longrightarrow u$

the only variable

Base case

$\forall v \in \mathbb{B}.\ P(v)$ Take a generic $v \in \mathbb{B}$

We want to prove $P(v) \triangleq \forall \sigma.\ \exists u.\ \langle v, \sigma \rangle \longrightarrow u$

Take a generic $\sigma \in \mathbb{M}$ and consider the goal $\langle v, \sigma \rangle \longrightarrow u$

the only variable

By rule $\dfrac{}{\langle v, \sigma \rangle \longrightarrow v}$

Base case

$\forall v \in \mathbb{B}.\ P(v)$ Take a generic $v \in \mathbb{B}$

We want to prove $P(v) \triangleq \forall \sigma.\ \exists u.\ \langle v, \sigma \rangle \longrightarrow u$

Take a generic $\sigma \in \mathbb{M}$ and consider the goal $\langle v, \sigma \rangle \longrightarrow u$ (the only variable)

By rule $\dfrac{}{\langle v, \sigma \rangle \longrightarrow v}$ we have $\langle v, \sigma \rangle \longrightarrow u \nwarrow_{[u=v]} \square$

Base case

$\forall v \in \mathbb{B}.\ P(v)$ Take a generic $v \in \mathbb{B}$

We want to prove $P(v) \triangleq \forall \sigma.\ \exists u.\ \langle v, \sigma \rangle \longrightarrow u$

the only variable

Take a generic $\sigma \in \mathbb{M}$ and consider the goal $\langle v, \sigma \rangle \longrightarrow u$

By rule $\dfrac{}{\langle v, \sigma \rangle \longrightarrow v}$ we have $\langle v, \sigma \rangle \longrightarrow u \nwarrow_{[u=v]} \square$

And we are done (taking $u = v$)

A surprising base case

$\forall a_0, a_1.\ P(a_0 \ \mathsf{cmp}\ a_1)$

A surprising base case

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By termination of arithmetic expressions, such n_0, n_1 exist

And we are done (taking $v = n_0\ cmp\ n_1$)

Proof obligations

try to complete on your own
the missing inductive cases of the proof

$$\forall b.\ P(b) \Rightarrow P(\neg b)$$

$$\forall b_0, b_1.\ (P(b_0) \wedge P(b_1)) \Rightarrow P(b_0 \ \mathsf{bop}\ b_1)$$

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Big-step operational semantics of commands

$\langle a, \sigma \rangle \longrightarrow n$ (the evaluation of a in σ returns the integer n)

$\langle b, \sigma \rangle \longrightarrow v$ (the evaluation of b in σ returns the Boolean v)

$\langle c, \sigma \rangle \longrightarrow \sigma'$ (the evaluation of c in σ returns the memory σ')

Memories

$$\mathbb{M} \triangleq \{\, \sigma : \mathsf{Ide} \to \mathbb{Z} \mid \sigma \text{ has finite support} \,\}$$

$$\{x \in \mathsf{Ide} \mid \sigma(x) \neq 0\} \text{ is finite}$$

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$\sigma_0 \triangleq ()$ is the typical initial memory

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$(n_1/x_1, ..., n_k/x_k) = \sigma_0[n_1/x_1, ..., n_k/x_k]$

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$$c ::= \textbf{skip} \mid x := a \mid c\,;c \mid \textbf{if } b \textbf{ then } c \textbf{ else } c \mid \textbf{while } b \textbf{ do } c$$

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Multiple rules for the same construct!

$$\frac{\langle b,\sigma\rangle \longrightarrow \mathbf{ff} \quad \langle c_1,\sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1,\sigma\rangle \longrightarrow \sigma'} \qquad \frac{\langle b,\sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c_0,\sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1,\sigma\rangle \longrightarrow \sigma'}$$

$$\frac{}{\langle x,\sigma\rangle \longrightarrow \sigma(x)} \qquad \frac{}{\langle n,\sigma\rangle \longrightarrow n} \qquad \frac{\langle a_0,\sigma\rangle \longrightarrow n_0 \quad \langle a_1,\sigma\rangle \longrightarrow n_1}{\langle a_0\ \mathsf{op}\ a_1,\sigma\rangle \longrightarrow n_0\ op\ n_1}$$

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$$\frac{}{\langle v,\sigma\rangle \longrightarrow v} \qquad \frac{\langle b,\sigma\rangle \longrightarrow v}{\langle \neg b,\sigma\rangle \longrightarrow \neg v} \qquad \frac{\langle a_0,\sigma\rangle \longrightarrow n_0 \quad \langle a_1,\sigma\rangle \longrightarrow n_1}{\langle a_0\ \mathsf{cmp}\ a_1,\sigma\rangle \longrightarrow n_0\ cmp\ n_1}$$

$$\frac{\langle b_0,\sigma\rangle \longrightarrow v_0 \quad \langle b_1,\sigma\rangle \longrightarrow v_1}{\langle b_0\ \mathsf{bop}\ b_1,\sigma\rangle \longrightarrow v_0\ bop\ v_1}$$

Multiple rules for the same construct!

$$\frac{\langle b,\sigma\rangle \longrightarrow \mathbf{ff} \quad \langle c_1,\sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1,\sigma\rangle \longrightarrow \sigma'} \qquad \frac{\langle b,\sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c_0,\sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1,\sigma\rangle \longrightarrow \sigma'}$$

$$\frac{\langle b,\sigma\rangle \longrightarrow \mathbf{ff}}{\langle \mathbf{while}\ b\ \mathbf{do}\ c,\sigma\rangle \longrightarrow \sigma} \qquad \frac{\langle b,\sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c,\sigma\rangle \longrightarrow \sigma'' \quad \langle \mathbf{while}\ b\ \mathbf{do}\ c,\sigma''\rangle \longrightarrow \sigma'}{\langle \mathbf{while}\ b\ \mathbf{do}\ c,\sigma\rangle \longrightarrow \sigma'}$$

$$\frac{}{\langle x,\sigma\rangle \longrightarrow \sigma(x)} \qquad \frac{}{\langle n,\sigma\rangle \longrightarrow n} \qquad \frac{\langle a_0,\sigma\rangle \longrightarrow n_0 \quad \langle a_1,\sigma\rangle \longrightarrow n_1}{\langle a_0\ \mathsf{op}\ a_1,\sigma\rangle \longrightarrow n_0\ op\ n_1}$$

$$\frac{}{\langle \mathbf{skip},\sigma\rangle \longrightarrow \sigma} \qquad \frac{\langle a,\sigma\rangle \longrightarrow n}{\langle x := a,\sigma\rangle \longrightarrow \sigma[n/x]} \qquad \frac{\langle c_0,\sigma\rangle \longrightarrow \sigma'' \quad \langle c_1,\sigma''\rangle \longrightarrow \sigma'}{\langle c_0\,;c_1,\sigma\rangle \longrightarrow \sigma'}$$

$$\frac{}{\langle v,\sigma\rangle \longrightarrow v} \qquad \frac{\langle b,\sigma\rangle \longrightarrow v}{\langle \neg b,\sigma\rangle \longrightarrow \neg v} \qquad \frac{\langle a_0,\sigma\rangle \longrightarrow n_0 \quad \langle a_1,\sigma\rangle \longrightarrow n_1}{\langle a_0\ \mathsf{cmp}\ a_1,\sigma\rangle \longrightarrow n_0\ cmp\ n_1}$$

$$\frac{\langle b_0,\sigma\rangle \longrightarrow v_0 \quad \langle b_1,\sigma\rangle \longrightarrow v_1}{\langle b_0\ \mathsf{bop}\ b_1,\sigma\rangle \longrightarrow v_0\ bop\ v_1}$$

Recursive definition!



one premise is as complex as the conclusion

$$\frac{\langle b,\sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c,\sigma\rangle \longrightarrow \sigma'' \quad \langle \mathbf{while}\ b\ \mathbf{do}\ c,\sigma''\rangle \longrightarrow \sigma'}{\langle \mathbf{while}\ b\ \mathbf{do}\ c,\sigma\rangle \longrightarrow \sigma'}$$

$$\frac{}{\langle x,\sigma\rangle \longrightarrow \sigma(x)} \qquad \frac{}{\langle n,\sigma\rangle \longrightarrow n} \qquad \frac{\langle a_0,\sigma\rangle \longrightarrow n_0 \quad \langle a_1,\sigma\rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1,\sigma\rangle \longrightarrow n_0\ op\ n_1}$$

$$\frac{}{\langle \mathbf{skip},\sigma\rangle \longrightarrow \sigma} \qquad \frac{\langle a,\sigma\rangle \longrightarrow n}{\langle x := a,\sigma\rangle \longrightarrow \sigma[n/x]} \qquad \frac{\langle c_0,\sigma\rangle \longrightarrow \sigma'' \quad \langle c_1,\sigma''\rangle \longrightarrow \sigma'}{\langle c_0\,;c_1,\sigma\rangle \longrightarrow \sigma'}$$

$$\frac{}{\langle v,\sigma\rangle \longrightarrow v} \qquad \frac{\langle b,\sigma\rangle \longrightarrow v}{\langle \neg b,\sigma\rangle \longrightarrow \neg v} \qquad \frac{\langle a_0,\sigma\rangle \longrightarrow n_0 \quad \langle a_1,\sigma\rangle \longrightarrow n_1}{\langle a_0 \text{ cmp } a_1,\sigma\rangle \longrightarrow n_0\ cmp\ n_1}$$

$$\frac{\langle b,\sigma\rangle \longrightarrow \mathbf{ff} \quad \langle c_1,\sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1,\sigma\rangle \longrightarrow \sigma'} \qquad \frac{\langle b,\sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c_0,\sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1,\sigma\rangle \longrightarrow \sigma'}$$

$$\frac{\langle b_0,\sigma\rangle \longrightarrow v_0 \quad \langle b_1,\sigma\rangle \longrightarrow v_1}{\langle b_0 \text{ bop } b_1,\sigma\rangle \longrightarrow v_0\ bop\ v_1}$$

$$\frac{\langle b,\sigma\rangle \longrightarrow \mathbf{ff}}{\langle \mathbf{while}\ b\ \mathbf{do}\ c,\sigma\rangle \longrightarrow \sigma}$$

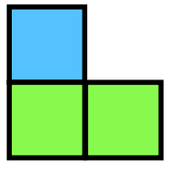
Triangular numbers

T_1  1

$$T_k \triangleq \sum_{i=1}^{k} i$$

Triangular numbers

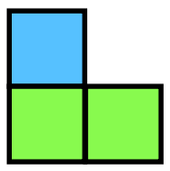
T_1  1

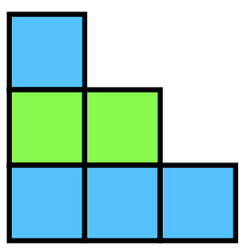
T_2  3

$$T_k \triangleq \sum_{i=1}^{k} i$$

Triangular numbers

T_1  1

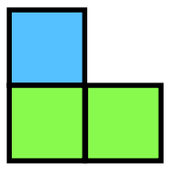
T_2  3

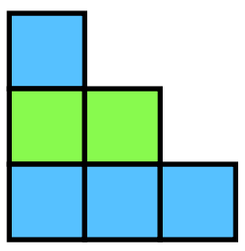
T_3  6

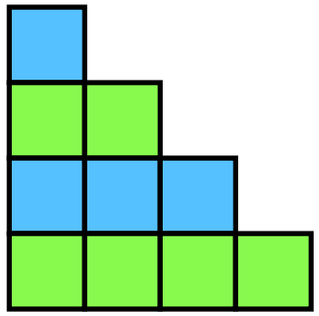
$$T_k \triangleq \sum_{i=1}^{k} i$$

Triangular numbers

T_1  1

T_2  3

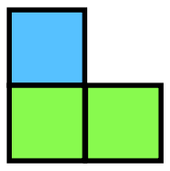
T_3  6

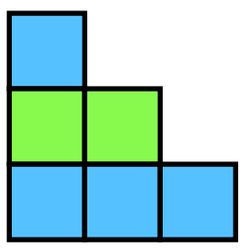
T_4  10

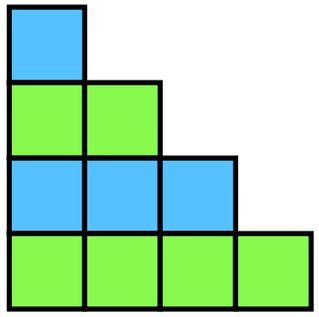
$$T_k \triangleq \sum_{i=1}^{k} i$$

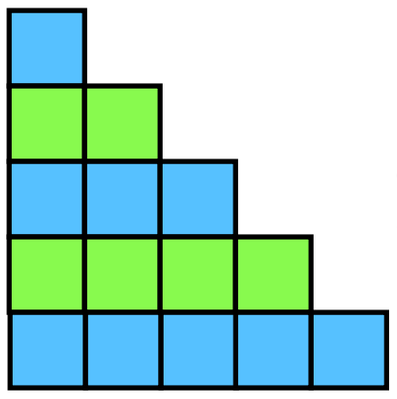
Triangular numbers

T_1  1

T_2  3

T_3  6

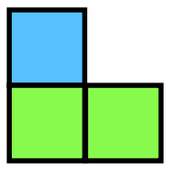
T_4  10

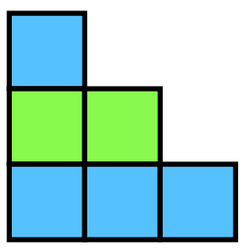
T_5  15

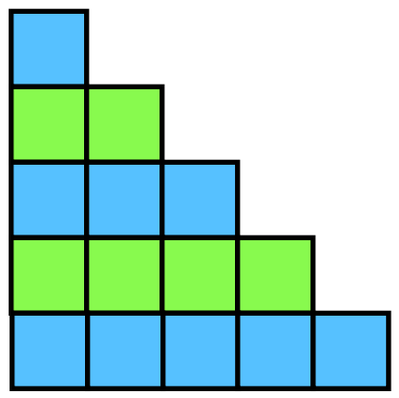
$$T_k \triangleq \sum_{i=1}^{k} i$$

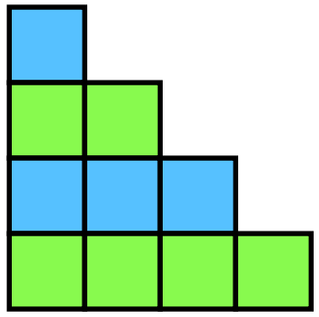
Triangular numbers

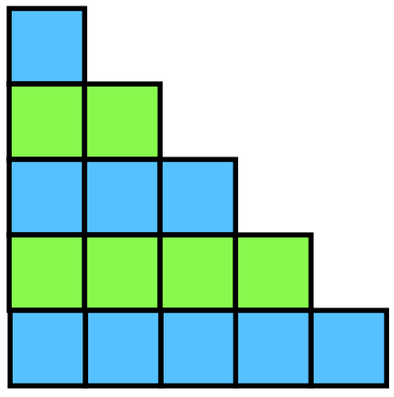
T_1  1

T_2  3

T_3  6



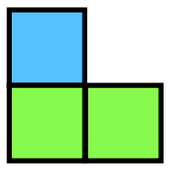
T_4  10

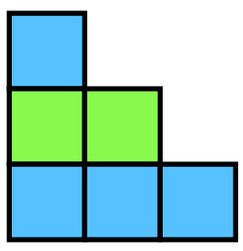
T_5  15

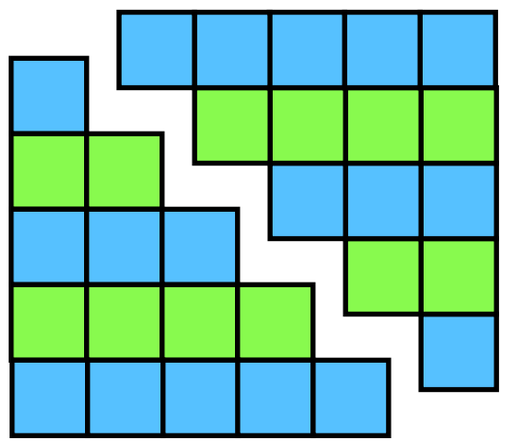
$$T_k \triangleq \sum_{i=1}^{k} i$$

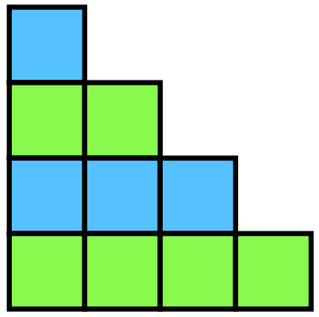
Triangular numbers

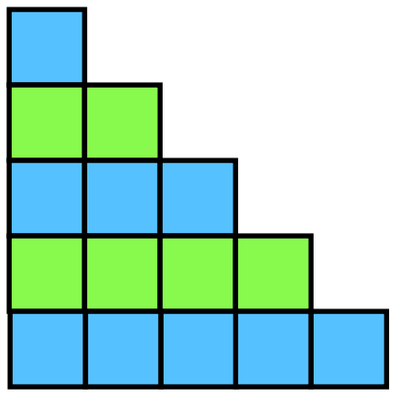
T_1  1

T_2  3

T_3  6



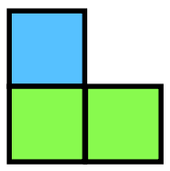
T_4  10

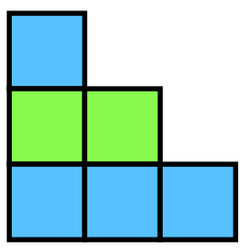
T_5  15

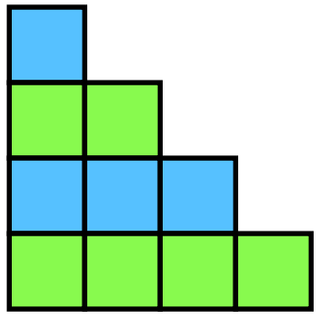
$$T_k \triangleq \sum_{i=1}^{k} i$$

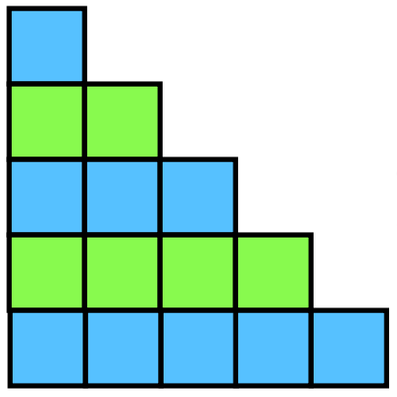
Triangular numbers

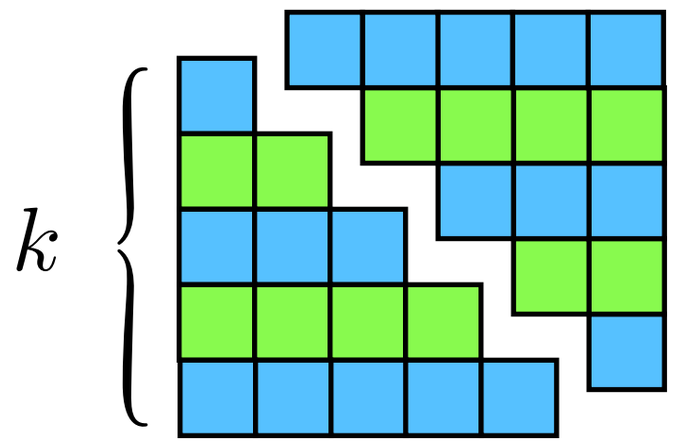
T_1  1

T_2  3

T_3  6

T_4  10

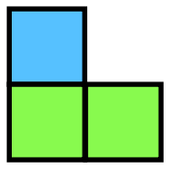
T_5  15

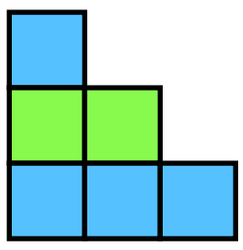


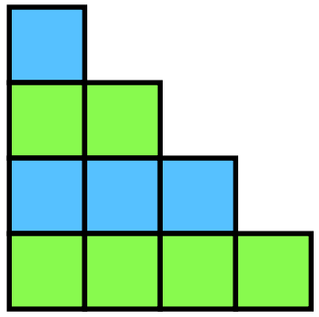
$$T_k \triangleq \sum_{i=1}^{k} i$$

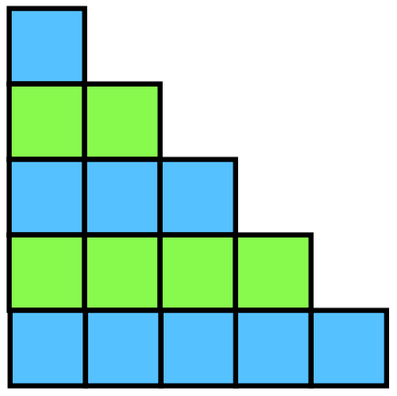
Triangular numbers

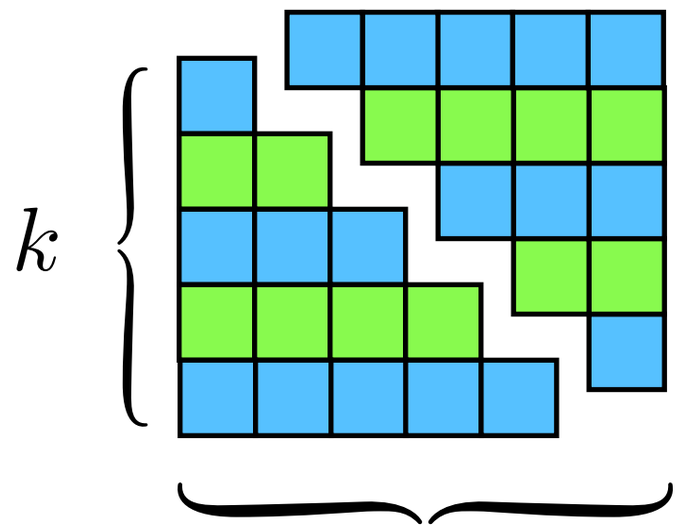
T_1  1

T_2  3

T_3  6

T_4  10

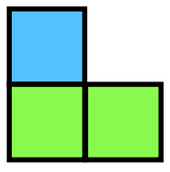
T_5  15

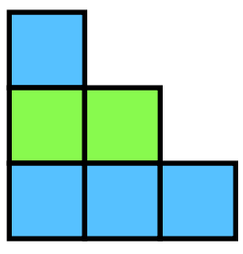


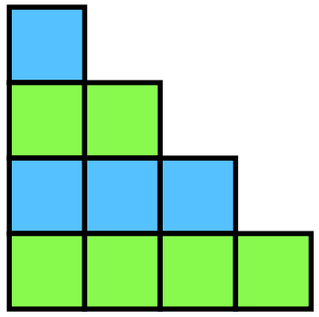
$$T_k \triangleq \sum_{i=1}^{k} i$$

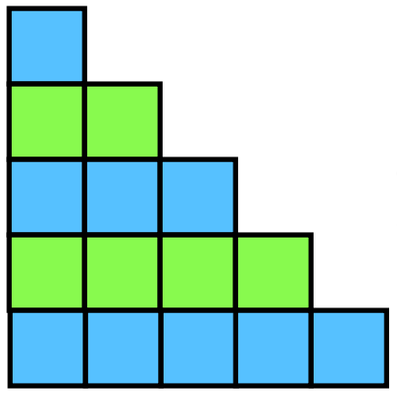
Triangular numbers

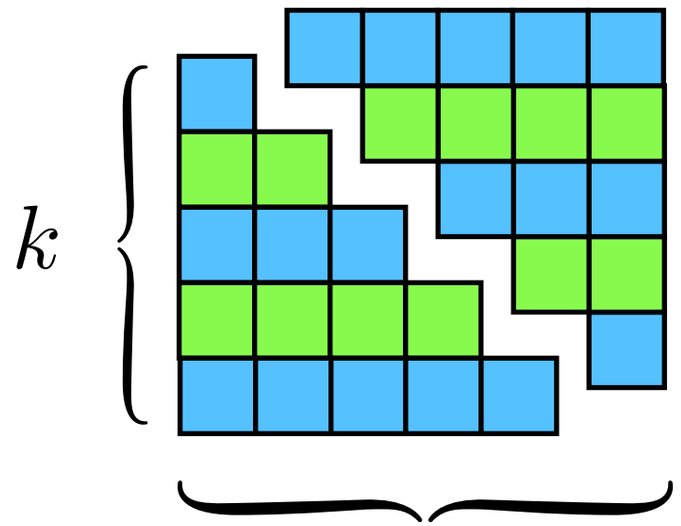
T_1  1

T_2  3

T_3  6

T_4  10

T_5  15



$$T_k \triangleq \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Compute T_k without div

$t := 0;$
$i := 1;$
while $i \leq k$ **do** (
$\quad t := t + i;$
$\quad i := i + 1$)

Compute T_k without div

c_0 {

$t := 0;$
$i := 1;$
while $i \leq k$ **do** (
$t := t + i;$
$i := i + 1$)

Compute T_k without div

c_0 { $t := 0;$
c_1 { $i := 1;$
while $i \leq k$ **do** (
$t := t + i;$
$i := i + 1$)

Compute T_k without div

$$
\begin{array}{ll}
c_0 & \{ \\
c_1 & \{ \\
w & \left\{ \begin{array}{l}
t := 0; \\
i := 1; \\
\textbf{while } i \leq k \textbf{ do } (\\
\qquad\qquad t := t + i; \\
\qquad\qquad i := i + 1 \;)
\end{array} \right.
\end{array}
$$

Compute T_k without div

$$
\begin{array}{ll}
c_0 & \{ \quad t := 0; \\
c_1 & \{ \quad i := 1; \\
w & \left\{ \quad \mathbf{while} \overbrace{i \leq k}^{b} \mathbf{do} \; (\right. \\
 & \qquad\qquad t := t + i; \\
 & \qquad\qquad \left. i := i + 1 \;) \right.
\end{array}
$$

Compute T_k without div

$$
\begin{array}{l}
c_0 \; \{ \\
c_1 \; \{ \\
w \; \left\{ \right.
\end{array}
\begin{array}{l}
t := 0; \\
i := 1; \\
\textbf{while } \overbrace{i \leq k}^{b} \textbf{ do } (\\
\qquad t := t + i; \\
\qquad i := i + 1 \;)
\end{array}
\quad \} \; c_2
$$

Compute T_k without div

$$
\begin{array}{ll}
c_0 \{ & t := 0; \\
c_1 \{ & i := 1; \\
w \left\{ \right. & \textbf{while } \overbrace{i \leq k}^{b} \textbf{ do } (\\
& \qquad t := t + i; \quad \} c_2 \\
& \qquad i := i + 1 \;) \quad \} c_3
\end{array}
$$

Compute T_k without div

$$
\begin{array}{ll}
c_0 \{ & t := 0; \\
c_1 \{ & i := 1; \\
w \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & \textbf{while}\ \overbrace{i \leq k}^{b}\ \textbf{do}\ (\\
& \qquad t := t + i; \quad \} c_2 \\
& \qquad i := i + 1\ ) \quad \} c_3
\end{array}
\quad \left.\begin{array}{l} c_2 \\ c_3 \end{array}\right\} c
$$

Compute T_k without div

$$
\begin{array}{l}
c_0 \; \{ \\
c_1 \; \{ \\
w \; \left\{ \begin{array}{l}
\\
\\
\mathbf{while}\ \overbrace{i \leq k}^{b}\ \mathbf{do}\ (\\
\qquad t := t + i; \\
\qquad i := i + 1\ )
\end{array} \right.
\end{array}
\begin{array}{l}
\\
\\
\\
\} \, c_2 \\
\} \, c_3
\end{array}
\left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} c
$$

initial memory

let's find σ such that $\langle (c_0; c_1); w, (2/k) \rangle \longrightarrow \sigma$

Compute T_k without div

$$c_0 \left\{ \begin{array}{l} t := 0; \\ c_1 \left\{ \begin{array}{l} i := 1; \\ w \left\{ \begin{array}{l} \textbf{while } \overbrace{i \leq k}^{b} \textbf{ do } (\\ \qquad t := t + i; \quad \} c_2 \\ \qquad i := i + 1 \;) \quad \} c_3 \end{array} \right. c \end{array} \right. \end{array} \right.$$

initial memory

let's find σ such that $\langle (c_0; c_1); w, (2/k) \rangle \longrightarrow \sigma$

let's find σ', σ such that $\left\{ \begin{array}{l} \langle c_0; c_1, (2/k) \rangle \longrightarrow \sigma' \\ \langle w, \sigma' \rangle \longrightarrow \sigma \end{array} \right.$

Let's derive

let's find σ' such that $\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1;\ \overbrace{\ }^{b} \\
w\ \left\{\begin{array}{l} \textbf{while}\ \overbrace{i \leq k}\ \textbf{do}\ (\\ \qquad t := t + i; \\ \qquad i := i + 1\ ) \end{array}\right.
\end{array}
\begin{array}{l} \\ \\ \left.\begin{array}{l} \} c_2 \\ \} c_3 \end{array}\right\} c \end{array}
$$

$$\sigma \triangleq (2/n)$$

Let's derive

let's find σ' such that $\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1;\ \overbrace{}^{b} \\
w\ \left\{\begin{array}{l} \textbf{while}\ \overbrace{i \le k}^{b}\ \textbf{do}\ (\\ \qquad t := t + i; \\ \qquad i := i + 1\ ) \end{array}\right.
\end{array}
\begin{array}{l} \\ \\ \left.\begin{array}{l} \} c_2 \\ \} c_3 \end{array}\right\} c \end{array}
$$

$$\sigma \triangleq (2/n)$$

$\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

Let's derive

let's find σ' such that $\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$$\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$$

$$\nwarrow \langle t := 0, (2/k)\rangle \longrightarrow \sigma_1, \langle i := 1, \sigma_1\rangle \longrightarrow \sigma'$$

$$\begin{array}{l} c_0\ \{ \\ c_1\ \{ \\ w\ \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. \end{array} \begin{array}{l} t := 0; \\ i := 1; \quad \overbrace{}^{b} \\ \textbf{while}\ \overbrace{i \le k}\ \textbf{do}\ (\\ \qquad t := t + i; \\ \qquad i := i + 1\ ) \end{array} \begin{array}{l} \\ \\ \\ \end{array} \left. \begin{array}{l} \\ \} c_2 \\ \} c_3 \end{array} \right\} c \end{array}$$

$$\sigma \triangleq (2/n)$$

Let's derive

let's find σ' such that $\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$$c_0\ \{\ t := 0;$$
$$c_1\ \{\ i := 1;\ \overbrace{i \leq k}^{b}$$
$$w\ \left\{\ \textbf{while}\ i \leq k\ \textbf{do}\ (\right.$$
$$\left.\begin{array}{r} t := t + i; \\ i := i + 1\ ) \end{array}\right.\ \left.\begin{array}{l} \}\ c_2 \\ \}\ c_3 \end{array}\right\}\ c$$

$$\sigma \triangleq (2/n)$$

$\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$\nwarrow \langle t := 0, (2/k)\rangle \longrightarrow \sigma_1, \langle i := 1, \sigma_1\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma_1 = (2/k)[n_1/t]} \langle 0, (2/k)\rangle \longrightarrow n_1, \langle i := 1, (2/k, n_1/t)\rangle \longrightarrow \sigma'$

Let's derive

let's find σ' such that $\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1; \quad \overbrace{i \leq k}^{b} \\
w\ \left\{\ \textbf{while}\ i \leq k\ \textbf{do}\ (\right. \\
\qquad\qquad t := t + i; \qquad \left.\right\}\ c_2 \\
\qquad\qquad i := i + 1\ ) \qquad \left.\right\}\ c_3
\end{array}
\qquad \sigma \triangleq (2/n)
$$

$\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$\nwarrow \langle t := 0, (2/k)\rangle \longrightarrow \sigma_1, \langle i := 1, \sigma_1\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma_1=(2/k)[n_1/t]} \langle 0, (2/k)\rangle \longrightarrow n_1, \langle i := 1, (2/k, n_1/t)\rangle \longrightarrow \sigma'$

$\nwarrow_{n_1=0} \langle i := 1, (2/k, 0/t)\rangle \longrightarrow \sigma'$

Let's derive

let's find σ' such that $\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1; \quad \overbrace{i \leq k}^{b} \\
w\ \left\{\ \textbf{while}\ i \leq k\ \textbf{do}\ (\right. \\
\qquad\qquad t := t + i; \qquad \left.\right\}\ c_2 \\
\qquad\qquad i := i + 1\ ) \qquad \left.\right\}\ c_3
\end{array}
\qquad \sigma \triangleq (2/n)
$$

$\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$\nwarrow \langle t := 0, (2/k)\rangle \longrightarrow \sigma_1, \langle i := 1, \sigma_1\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma_1=(2/k)[n_1/t]} \langle 0, (2/k)\rangle \longrightarrow n_1, \langle i := 1, (2/k, n_1/t)\rangle \longrightarrow \sigma'$

$\nwarrow_{n_1=0} \langle i := 1, (2/k, 0/t)\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma'=(2/k,0/t)[n_2/i]} \langle 1, (2/k, 0/t)\rangle \longrightarrow n_2$

Let's derive

let's find σ' such that $\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1; \quad \overbrace{i \leq k}^{b} \\
w\ \left\{\ \textbf{while}\ i \leq k\ \textbf{do}\ (\right. \\
\qquad\qquad t := t + i; \qquad \left.\right\}\ c_2 \\
\qquad\qquad i := i + 1\ ) \qquad \left.\right\}\ c_3
\end{array}
\qquad \sigma \triangleq (2/n)
$$

$\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$\nwarrow \langle t := 0, (2/k)\rangle \longrightarrow \sigma_1, \langle i := 1, \sigma_1\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma_1=(2/k)[n_1/t]} \langle 0, (2/k)\rangle \longrightarrow n_1, \langle i := 1, (2/k, n_1/t)\rangle \longrightarrow \sigma'$

$\nwarrow_{n_1=0} \langle i := 1, (2/k, 0/t)\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma'=(2/k,0/t)[n_2/i]} \langle 1, (2/k, 0/t)\rangle \longrightarrow n_2$

$\nwarrow_{n_2=1} \square$

Let's derive

let's find σ' such that $\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1; \quad \overbrace{i \leq k}^{b} \\
w\ \left\{\ \textbf{while}\ i \leq k\ \textbf{do}\ (\right. \\
\qquad\qquad t := t + i; \qquad \left.\right\}\ c_2 \\
\qquad\qquad i := i + 1\ ) \qquad \left.\right\}\ c_3
\end{array}
\qquad \sigma \triangleq (2/n)
$$

$\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$\nwarrow \langle t := 0, (2/k)\rangle \longrightarrow \sigma_1, \langle i := 1, \sigma_1\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma_1=(2/k)[n_1/t]} \langle 0, (2/k)\rangle \longrightarrow n_1, \langle i := 1, (2/k, n_1/t)\rangle \longrightarrow \sigma'$

$\nwarrow_{n_1=0} \langle i := 1, (2/k, 0/t)\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma'=(2/k,0/t)[n_2/i]} \langle 1, (2/k, 0/t)\rangle \longrightarrow n_2$

$\nwarrow_{n_2=1} \square$

Let's derive

let's find σ' such that $\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$$c_0\ \{\ t := 0;$$
$$c_1\ \{\ i := 1;\ \overbrace{i \leq k}^{b}$$
$$w\ \left\{\ \textbf{while}\ i \leq k\ \textbf{do}\ (\right.$$
$$\left.\begin{array}{r} t := t + i; \\ i := i + 1\ ) \end{array}\right.\ \left.\begin{array}{l} \}\ c_2 \\ \}\ c_3 \end{array}\right\}\ c$$

$$\sigma \triangleq (2/n)$$

$\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$\nwarrow \langle t := 0, (2/k)\rangle \longrightarrow \sigma_1, \langle i := 1, \sigma_1\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma_1=(2/k)[n_1/t]} \langle 0, (2/k)\rangle \longrightarrow n_1, \langle i := 1, (2/k, n_1/t)\rangle \longrightarrow \sigma'$

$\nwarrow_{n_1=0} \langle i := 1, (2/k, 0/t)\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma'=(2/k,0/t)[n_2/i]} \langle 1, (2/k, 0/t)\rangle \longrightarrow n_2$

$\nwarrow_{n_2=1} \square$

Let's derive

let's find σ' such that $\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$$c_0\ \{\ t := 0;$$
$$c_1\ \{\ i := 1;\ \overbrace{i \leq k}^{b}$$
$$w\ \left\{\ \textbf{while}\ i \leq k\ \textbf{do}\ (\right.$$
$$\left.\begin{array}{r} t := t + i; \\ i := i + 1\ ) \end{array}\right.\ \left.\begin{array}{l} \}\ c_2 \\ \}\ c_3 \end{array}\right\}\ c$$

$$\sigma \triangleq (2/n)$$

$\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$\nwarrow \langle t := 0, (2/k)\rangle \longrightarrow \sigma_1, \langle i := 1, \sigma_1\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma_1=(2/k)[n_1/t]} \langle 0, (2/k)\rangle \longrightarrow n_1, \langle i := 1, (2/k, n_1/t)\rangle \longrightarrow \sigma'$

$\nwarrow_{n_1=0} \langle i := 1, (2/k, 0/t)\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma'=(2/k,0/t)[n_2/i]} \langle 1, (2/k, 0/t)\rangle \longrightarrow n_2$

$\nwarrow_{n_2=1} \square$

$$\sigma' = (2/k, 0/t, 1/i)$$

Let's derive

let's find σ' such that $\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1; \quad \overbrace{i \leq k}^{b} \\
w\ \left\{\ \textbf{while}\ i \leq k\ \textbf{do}\ (\right. \\
\qquad\qquad t := t + i; \qquad \left.\right\}\ c_2 \\
\qquad\qquad i := i + 1\ ) \qquad \left.\right\}\ c_3
\end{array}
\qquad \sigma \triangleq (2/n)
$$

$\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$\nwarrow \langle t := 0, (2/k)\rangle \longrightarrow \sigma_1, \langle i := 1, \sigma_1\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma_1=(2/k)[n_1/t]} \langle 0, (2/k)\rangle \longrightarrow n_1, \langle i := 1, (2/k, n_1/t)\rangle \longrightarrow \sigma'$

$\nwarrow_{n_1=0} \langle i := 1, (2/k, 0/t)\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma'=(2/k,0/t)[n_2/i]} \langle 1, (2/k, 0/t)\rangle \longrightarrow n_2$

$\nwarrow_{n_2=1} \square$

$$\sigma' = (2/k, 0/t, 1/i)$$

more concisely: $\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$\nwarrow^{*}_{\sigma'=(2/k,0/t,1/i)} \square$

Let's derive

let's find σ such that $\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1;\ \overbrace{i \leq k}^{b}\ \textbf{do}\ (\\
w\ \left\{\begin{array}{rl} & t := t + i; \\ & i := i + 1\ ) \end{array}\right.
\end{array}
\begin{array}{l} \\ \\ \left.\begin{array}{l} \} c_2 \\ \} c_3 \end{array}\right\} c \end{array}
$$

$$\sigma \triangleq (2/n)$$

Let's derive

let's find σ such that $\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1;\ \overbrace{i \leq k}^{b} \\
w\ \left\{\ \begin{array}{l} \textbf{while}\ i \leq k\ \textbf{do}\ (\\ \qquad t := t + i; \\ \qquad i := i + 1\ ) \end{array}\ \right.
\end{array}
\begin{array}{l} \\ \\ \\ \left.\begin{array}{l} \} c_2 \\ \} c_3 \end{array}\right\} c \end{array}
$$

$$\sigma \triangleq (2/n)$$

$\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

Let's derive

let's find σ such that $\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1; \ \overbrace{\ \ }^{b} \\
w\ \left\{\ \begin{array}{l} \textbf{while}\ \overbrace{i \leq k}\ \textbf{do}\ (\\ \qquad t := t + i; \\ \qquad i := i + 1\ ) \end{array}\ \right.
\end{array}
\begin{array}{l} \\ \\ \\ \} c_2 \\ \} c_3 \end{array} \Big\} c
$$

$\sigma \triangleq (2/n)$

$\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 0/t, 1/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

Let's derive

let's find σ such that $\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1;\ \overbrace{i \leq k}^{b} \\
w\ \left\{\ \textbf{while}\ i \leq k\ \textbf{do}\ (\right. \\
\qquad\qquad t := t + i; \quad \left.\right\}\ c_2 \\
\qquad\qquad i := i + 1\ ) \quad \left.\right\}\ c_3
\end{array}
\qquad \sigma \triangleq (2/n)
$$

$\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 0/t, 1/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^* \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

Let's derive

let's find σ such that $\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1; \\
w\ \left\{\ \textbf{while}\ \overbrace{i \leq k}^{b}\ \textbf{do}\ (\right. \\
\qquad\qquad t := t + i; \quad \}\ c_2 \\
\qquad\qquad i := i + 1\ ) \quad \left.\}\ c_3\right\}\ c
\end{array}
$$

$\sigma \triangleq (2/n)$

$\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 0/t, 1/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^* \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^*_{\sigma_1 = (2/k, 0/t, 1/i)[1/t, 2/i]} \langle w, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma$

Let's derive

let's find σ such that $\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1; \\
w\ \left\{\ \begin{array}{l} \textbf{while}\ \overbrace{i \leq k}^{b}\ \textbf{do}\ (\\ \qquad t := t + i; \\ \qquad i := i + 1\ ) \end{array}\ \begin{array}{l} \} c_2 \\ \} c_3 \end{array}\right\} c
\end{array}
$$

$\sigma \triangleq (2/n)$

$\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 0/t, 1/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^* \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^*_{\sigma_1 = (2/k, 0/t, 1/i)[1/t, 2/i]} \langle w, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 1/t, 2/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma_2, \langle w, \sigma_2\rangle \longrightarrow \sigma$

Let's derive

let's find σ such that $\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1; \\
w\ \left\{\ \begin{array}{l} \textbf{while}\ \overbrace{i \leq k}^{b}\ \textbf{do}\ (\\ \qquad t := t + i; \\ \qquad i := i + 1\ ) \end{array}\ \begin{array}{l} \}\ c_2 \\ \}\ c_3 \end{array}\right\}\ c
\end{array}
$$

$\sigma \triangleq (2/n)$

$\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 0/t, 1/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^* \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^*_{\sigma_1 = (2/k, 0/t, 1/i)[1/t, 2/i]} \langle w, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 1/t, 2/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma_2, \langle w, \sigma_2\rangle \longrightarrow \sigma$

$\nwarrow^* \langle c, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma_2, \langle w, \sigma_2\rangle \longrightarrow \sigma$

Let's derive

let's find σ such that $\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1; \\
w\ \left\{\ \begin{array}{l} \textbf{while}\ \overbrace{i \leq k}^{b}\ \textbf{do}\ (\\ \qquad t := t + i; \\ \qquad i := i + 1\ ) \end{array}\ \begin{array}{l} \}\ c_2 \\ \}\ c_3 \end{array}\right\}\ c
\end{array}
$$

$\sigma \triangleq (2/n)$

$\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 0/t, 1/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^* \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^*_{\sigma_1 = (2/k, 0/t, 1/i)[1/t, 2/i]} \langle w, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 1/t, 2/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma_2, \langle w, \sigma_2\rangle \longrightarrow \sigma$

$\nwarrow^* \langle c, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma_2, \langle w, \sigma_2\rangle \longrightarrow \sigma$

$\nwarrow^*_{\sigma_2 = (2/k, 1/t, 2/i)[3/t, 3/i]} \langle w, (2/k, 3/t, 3/i)\rangle \longrightarrow \sigma$

Let's derive

let's find σ such that $\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1; \ \overbrace{\ \ }^{b} \\
w\ \left\{\ \textbf{while}\ \overbrace{i \leq k}\ \textbf{do}\ (\right. \\
\qquad\qquad t := t + i; \qquad \left.\} c_2 \right\} c \\
\qquad\qquad i := i + 1\ ) \qquad \left.\} c_3 \right\} c
\end{array}
$$

$\sigma \triangleq (2/n)$

$\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 0/t, 1/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^* \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^*_{\sigma_1=(2/k,0/t,1/i)[1/t,2/i]} \langle w, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 1/t, 2/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma_2, \langle w, \sigma_2\rangle \longrightarrow \sigma$

$\nwarrow^* \langle c, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma_2, \langle w, \sigma_2\rangle \longrightarrow \sigma$

$\nwarrow^*_{\sigma_2=(2/k,1/t,2/i)[3/t,3/i]} \langle w, (2/k, 3/t, 3/i)\rangle \longrightarrow \sigma$

$\nwarrow_{\sigma=(2/k,3/t,3/i)} \langle b, (2/k, 3/t, 3/i)\rangle \longrightarrow \mathbf{ff}$

Let's derive

let's find σ such that $\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$$c_0\ \{\ t := 0;$$
$$c_1\ \{\ i := 1;\ \overbrace{i \leq k}^{b}$$
$$w\ \left\{\ \textbf{while}\ i \leq k\ \textbf{do}\ (\right.$$
$$\left.\begin{array}{r} t := t + i; \\ i := i + 1\ ) \end{array}\right.\ \left.\begin{array}{l} \}\ c_2 \\ \}\ c_3 \end{array}\right\}\ c$$

$$\sigma \triangleq (2/n)$$

$\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 0/t, 1/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^* \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^*_{\sigma_1=(2/k,0/t,1/i)[1/t,2/i]} \langle w, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 1/t, 2/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma_2, \langle w, \sigma_2\rangle \longrightarrow \sigma$

$\nwarrow^* \langle c, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma_2, \langle w, \sigma_2\rangle \longrightarrow \sigma$

$\nwarrow^*_{\sigma_2=(2/k,1/t,2/i)[3/t,3/i]} \langle w, (2/k, 3/t, 3/i)\rangle \longrightarrow \sigma$

$\nwarrow_{\sigma=(2/k,3/t,3/i)} \langle b, (2/k, 3/t, 3/i)\rangle \longrightarrow \mathbf{ff}$

$\nwarrow^* \square$

Let's derive

let's find σ such that $\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$$c_0\ \{\ t := 0;$$
$$c_1\ \{\ i := 1;\ \overbrace{i \leq k}^{b}$$
$$w\ \left\{\ \textbf{while}\ i \leq k\ \textbf{do}\ (\right.$$
$$\left.\begin{array}{r} t := t + i; \\ i := i + 1\ ) \end{array}\right.\ \left.\begin{array}{l} \}\ c_2 \\ \}\ c_3 \end{array}\right\}\ c$$

$\sigma \triangleq (2/n)$

$\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 0/t, 1/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^* \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^*_{\sigma_1=(2/k,0/t,1/i)[1/t,2/i]} \langle w, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 1/t, 2/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma_2, \langle w, \sigma_2\rangle \longrightarrow \sigma$

$\nwarrow^* \langle c, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma_2, \langle w, \sigma_2\rangle \longrightarrow \sigma$

$\nwarrow^*_{\sigma_2=(2/k,1/t,2/i)[3/t,3/i]} \langle w, (2/k, 3/t, 3/i)\rangle \longrightarrow \sigma$

$\nwarrow_{\sigma=(2/k,3/t,3/i)} \langle b, (2/k, 3/t, 3/i)\rangle \longrightarrow \mathbf{ff}$

$\nwarrow^* \square$

Let's derive

let's find σ such that $\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1; \ \overbrace{\ \ }^{b} \\
w\ \left\{\ \textbf{while}\ \overbrace{i \leq k}\ \textbf{do}\ (\right. \\
\qquad\qquad t := t + i; \qquad \} c_2 \\
\qquad\qquad i := i + 1\ ) \qquad \} c_3
\end{array}
\Bigg\} c
$$

$\sigma \triangleq (2/n)$

$\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 0/t, 1/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^* \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^*_{\sigma_1=(2/k,0/t,1/i)[1/t,2/i]} \langle w, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma$

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$\nwarrow^*_{\sigma_2=(2/k,1/t,2/i)[3/t,3/i]} \langle w, (2/k, 3/t, 3/i)\rangle \longrightarrow \sigma$

$\nwarrow_{\sigma=(2/k,3/t,3/i)} \langle b, (2/k, 3/t, 3/i)\rangle \longrightarrow \mathbf{ff}$

$\nwarrow^* \square$

$\sigma = (2/k, 3/t, 3/i)$

Divergence

Termination of commands?

$$c ::= \mathbf{skip} \mid x := a \mid c\,;c \mid \mathbf{if}\ b\ \mathbf{then}\ c\ \mathbf{else}\ c \mid \mathbf{while}\ b\ \mathbf{do}\ c$$

$$\frac{}{\langle \mathbf{skip}, \sigma\rangle \longrightarrow \sigma} \qquad \frac{\langle a, \sigma\rangle \longrightarrow n}{\langle x := a, \sigma\rangle \longrightarrow \sigma[n/x]} \qquad \frac{\langle c_0, \sigma\rangle \longrightarrow \sigma'' \quad \langle c_1, \sigma''\rangle \longrightarrow \sigma'}{\langle c_0\,; c_1, \sigma\rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma\rangle \longrightarrow \mathbf{ff} \quad \langle c_1, \sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1, \sigma\rangle \longrightarrow \sigma'} \qquad \frac{\langle b, \sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c_0, \sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1, \sigma\rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma\rangle \longrightarrow \mathbf{ff}}{\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma\rangle \longrightarrow \sigma} \qquad \frac{\langle b, \sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c, \sigma\rangle \longrightarrow \sigma'' \quad \langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma''\rangle \longrightarrow \sigma'}{\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma\rangle \longrightarrow \sigma'}$$

$$P(c) \stackrel{\triangle}{=} \forall \sigma \in \mathbb{M}.\ \exists \sigma' \in \mathbb{M}.\ \langle c, \sigma\rangle \longrightarrow \sigma' \qquad \forall c.\ P(c)\ ?$$

Termination?

$P(c) \triangleq \forall \sigma.\ \exists \sigma'.\ \langle c, \sigma \rangle \longrightarrow \sigma'$

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same goal from which we started!
no other options: $\langle \mathbf{tt}, \sigma \rangle \not\rightarrow \mathbf{ff}$
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$\neg P(w)$

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take $w \triangleq$ **while tt do skip**

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$\neg P(w)$ we have found a counterexample to termination

Divergence

$\langle c, \sigma \rangle \not\longrightarrow$ means $\neg \exists \sigma'.\ \langle c, \sigma \rangle \longrightarrow \sigma'$

and we say that c diverges on σ

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sometimes divergence is very difficult to detect
(well… it is undecidable)

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$$\sigma(x) > 0 \Rightarrow \sigma[\sigma(x) + 1/x](x) = \sigma(x) + 1 > 0$$

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note that if $\langle c, \sigma \rangle \not\longrightarrow$, then $\langle c, \sigma \rangle \longrightarrow \sigma' \Rightarrow \sigma' \in S$ trivially holds

Revisiting the example

take $w \triangleq \mathbf{while}\ x > 0\ \mathbf{do}\ x := x + 1$

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$$\sigma(x) > 0 \Rightarrow \sigma[\sigma(x) + 1/x](x) = \sigma(x) + 1 > 0$$

therefore, if $\sigma(x) > 0$, then $\langle w, \sigma \rangle \not\longrightarrow$

Exercise

take $w \triangleq \mathbf{while}\ x \neq 0\ \mathbf{do}\ x := x - 2$

find the set of all and only memories σ such that $\langle w, \sigma \rangle \not\rightarrow$

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find the set of all and only memories σ such that $\langle w, \sigma \rangle \not\longrightarrow$

$S_1 \triangleq \{\sigma \mid \sigma(x) < 0\}$ $\qquad$ $S_2 \triangleq \{\sigma \mid \exists k \in \mathbb{Z}.\ \sigma(x) = 2k + 1\}$

- $\forall \sigma \in S.\ \langle x \neq 0, \sigma \rangle \longrightarrow \mathbf{tt}$
- $\forall \sigma \in S.\ \forall \sigma'.\ \langle x := x - 2, \sigma \rangle \longrightarrow \sigma' \Rightarrow \sigma' \in S$

  $\langle x := x - 2, \sigma \rangle \longrightarrow \sigma' \Rightarrow \sigma' = \sigma[\sigma(x) - 2/x]$

Collatz's conjecture: half or triple plus one

$$w \triangleq \textbf{while } x > 1 \textbf{ do } (\ \textbf{if } x\%2 = 0 \textbf{ then } x := x/2 \textbf{ else } x := (3 \times x) + 1\ )$$

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more precisely: $\forall \sigma.\ \sigma(x) > 1 \Rightarrow \langle w, \sigma \rangle \longrightarrow \sigma[1/x]$

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Experimental evidence:
as of 2025, the conjecture has been verified for all values up to 2^{71}.
Barina, David. "Improved verification limit for the convergence of the Collatz conjecture". The Journal of Supercomputing 81 (2025).

Determinacy?

Determinacy of commands?

$$c ::= \mathbf{skip} \mid x := a \mid c\,;c \mid \mathbf{if}\ b\ \mathbf{then}\ c\ \mathbf{else}\ c \mid \mathbf{while}\ b\ \mathbf{do}\ c$$

$$\frac{}{\langle \mathbf{skip}, \sigma\rangle \longrightarrow \sigma} \qquad \frac{\langle a, \sigma\rangle \longrightarrow n}{\langle x := a, \sigma\rangle \longrightarrow \sigma[n/x]} \qquad \frac{\langle c_0, \sigma\rangle \longrightarrow \sigma'' \quad \langle c_1, \sigma''\rangle \longrightarrow \sigma'}{\langle c_0\,; c_1, \sigma\rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma\rangle \longrightarrow \mathbf{ff} \quad \langle c_1, \sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1, \sigma\rangle \longrightarrow \sigma'} \qquad \frac{\langle b, \sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c_0, \sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1, \sigma\rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma\rangle \longrightarrow \mathbf{ff}}{\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma\rangle \longrightarrow \sigma} \qquad \frac{\langle b, \sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c, \sigma\rangle \longrightarrow \sigma'' \quad \langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma''\rangle \longrightarrow \sigma'}{\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma\rangle \longrightarrow \sigma'}$$

$$P(c) \triangleq \forall \sigma, \sigma_1, \sigma_2.\ \langle c, \sigma\rangle \longrightarrow \sigma_1 \wedge \langle c, \sigma\rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2 \qquad \forall c.\ P(c)\ ?$$

Structural induction principle

$$\frac{}{\forall c \in \mathsf{Com}.\ P(c)}$$

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$$\forall x, a.\ P(x := a)$$

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Structural induction principle

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by determinacy of Aexp we have $n = m$ and thus $\sigma_1 = \sigma_2$

Inductive case

$$\forall c_0, c_1.\ P(c_0) \wedge P(c_1) \Rightarrow P(c_0\,;c_1)$$

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We assume (inductive hypotheses)

$$P(c_i) \triangleq \forall \sigma, \sigma_1, \sigma_2.\ \langle c_i, \sigma\rangle \longrightarrow \sigma_1 \wedge \langle c_i, \sigma\rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2$$

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hence $\sigma_1 = \sigma_1'$ with $\langle c_0,\sigma\rangle \longrightarrow \sigma_1''$ and $\langle c_1,\sigma_1''\rangle \longrightarrow \sigma_1'$

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By inductive hypothesis $P(c_0)$, we have $\sigma_1'' = \sigma_2''$

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and thus $\langle c_1,\sigma_2''\rangle \longrightarrow \sigma_1'$ and $\langle c_1,\sigma_2''\rangle \longrightarrow \sigma_2'$

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By inductive hypothesis $P(c_1)$, we then have $\sigma_1' = \sigma_2'$

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and thus we conclude $\sigma_1 = \sigma_1' = \sigma_2' = \sigma_2$

Inductive case

$\forall b, c.\ P(c) \Rightarrow P(\textbf{while}\ b\ \textbf{do}\ c)$

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Take $\sigma, \sigma_1, \sigma_2$ such that $\langle \textbf{while}\ b\ \textbf{do}\ c, \sigma \rangle \longrightarrow \sigma_1$ and $\langle \textbf{while}\ b\ \textbf{do}\ c, \sigma \rangle \longrightarrow \sigma_2$

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Take $\sigma, \sigma_1, \sigma_2$ such that $\langle \textbf{while}\ b\ \textbf{do}\ c, \sigma \rangle \longrightarrow \sigma_1$ and $\langle \textbf{while}\ b\ \textbf{do}\ c, \sigma \rangle \longrightarrow \sigma_2$

We want to prove $\sigma_1 = \sigma_2$

Inductive case

$\forall b, c.\ P(c) \Rightarrow P(\textbf{while}\ b\ \textbf{do}\ c)$ Take generic b, c

We assume (inductive hypothesis)

$$P(c) \triangleq \forall \sigma, \sigma_1, \sigma_2.\ \langle c, \sigma \rangle \longrightarrow \sigma_1 \wedge \langle c, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2$$

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We want to prove $\sigma_1 = \sigma_2$

By determinacy of boolean expressions, there are two cases

$\langle b, \sigma\rangle \longrightarrow \mathbf{ff}$ $\langle b, \sigma\rangle \longrightarrow \mathbf{tt}$

Inductive case (ctd)

if $\langle b, \sigma \rangle \longrightarrow \mathbf{ff}$

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and thus we conclude $\sigma_1 = \sigma = \sigma_2$

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but there is no inductive hypothesis $P(\mathbf{while}\ b\ \mathbf{do}\ c)$!

Recursive definition!



one premise is as much complex as the conclusion

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to close the proof of determinacy
we need a more convenient induction principle:

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