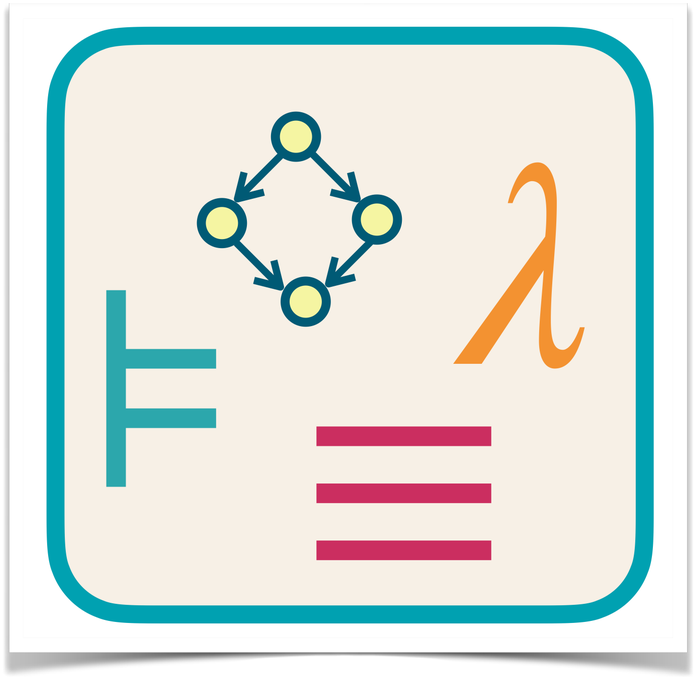


LPL 2026/27 (063AA, 9CFU)

Programming Languages

Fabio Gadducci

and Laboratory

Vincenzo Ciancia

Ole - Exercises on CPOs

[Exercise] Consider the $\mathrm{CPO}_\perp$ $(\wp(\mathbb{N}), \subseteq)$. Prove that for any set $S \subseteq \mathbb{N}$:

1. the function $f_S : \wp(\mathbb{N}) \to \wp(\mathbb{N})$ such that $f_S(X) = X \cap S$ is continuous.
2. the function $g_S : \wp(\mathbb{N}) \to \wp(\mathbb{N})$ such that $g_S(X) = X \cup S$ is continuous.

we omit to check monotonicity

Continuity #1

$$f_S(X) = X \cap S$$

Take a chain $\{X_i\}_{i \in \mathbb{N}}$

We need to prove $f_S\left(\bigcup_{i \in \mathbb{N}} X_i\right) = \bigcup_{i \in \mathbb{N}} f_S(X_i)$

$$f_S\left(\bigcup_{i \in \mathbb{N}} X_i\right) = \left(\bigcup_{i \in \mathbb{N}} X_i\right) \cap S = \bigcup_{i \in \mathbb{N}} (X_i \cap S) = \bigcup_{i \in \mathbb{N}} f_S(X_i)$$

by def

by distributivity

by def

Continuity #2

$$g_S(X) = X \cup S$$

Take a chain $\{X_i\}_{i\in\mathbb{N}}$

We need to prove $g_S\left(\bigcup_{i\in\mathbb{N}} X_i\right) = \bigcup_{i\in\mathbb{N}} g_S(X_i)$

$$g_S\left(\bigcup_{i\in\mathbb{N}} X_i\right) = \left(\bigcup_{i\in\mathbb{N}} X_i\right) \cup S = \bigcup_{i\in\mathbb{N}} (X_i \cup S) = \bigcup_{i\in\mathbb{N}} g_S(X_i)$$

by def by idempotency by def

[Exercise] Prove that any limit-preserving function is monotone.

Limit preserving

$(D, \sqsubseteq_D)$ CPO $\quad (E, \sqsubseteq_E)$ CPO $\quad f : D \to E$ limit-preserving

$$f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = \bigsqcup_{i \in \mathbb{N}} f(d_i)$$

we want to prove monotonicity

take $d \sqsubseteq_D d'$

we want to prove $\quad f(d) \sqsubseteq_E f(d')$

let $d_i = \begin{cases} d & \text{if } i = 0 \\ d' & \text{otherwise} \end{cases}$ hence $\bigsqcup_{i \in \mathbb{N}} d_i = d'$

then $\bigsqcup_{i \in \mathbb{N}} f(d_i) = f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = f(d')$

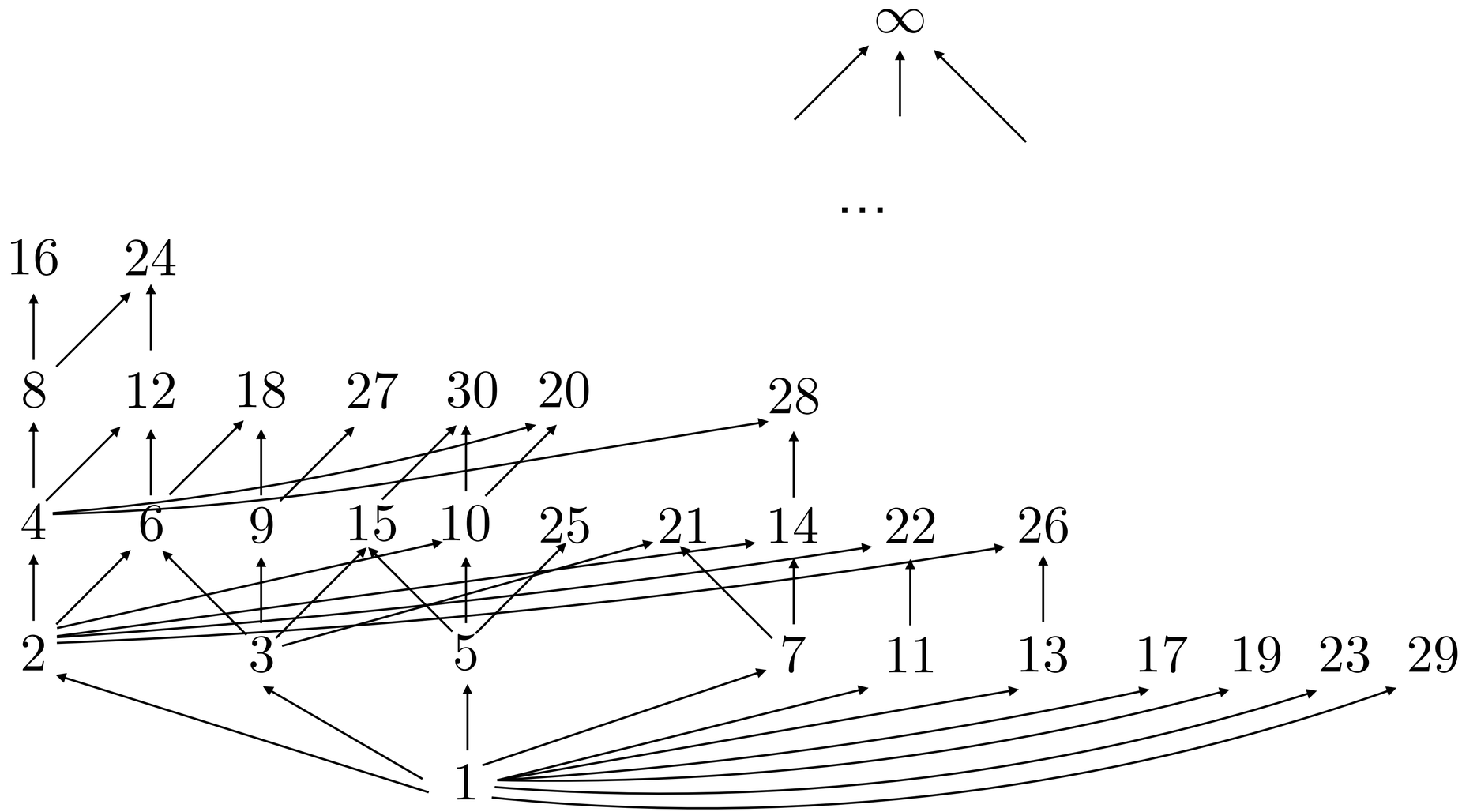
and $f(d) = f(d_0) \sqsubseteq_E f(d')$

[Exercise] Let $D = \{n \in \mathbb{N} \mid n > 0\} \cup \{\infty\}$ and $\sqsubseteq \subseteq (D \times D)$ such that

- for any $n, m \in D \cap \mathbb{N}$, we let $n \sqsubseteq m$ iff n divides m;
- for any $x \in D$, we let $x \sqsubseteq \infty$.

Is $(D, \sqsubseteq)$ a $\mathrm{CPO}_\perp$? Explain.

Divisors


∞
...
16
24
8
12
18
27
30
20
28
4
6
9
15
10
25
21
14
22
26
2
3
5
7
11
13
17
19
23
29
1

Divisors

$\mathrm{CPO}_\perp$?

reflexive?

$d \in D$

- $d \in \mathbb{N}$ — d divides d — $d \sqsubseteq d$
- $d = \infty$ — $\infty \sqsubseteq \infty$

antisymmetric?

$d_1, d_2 \in D$ — $d_1 \sqsubseteq d_2$, $d_2 \sqsubseteq d_1$ — $d_1 \stackrel{?}{=} d_2$

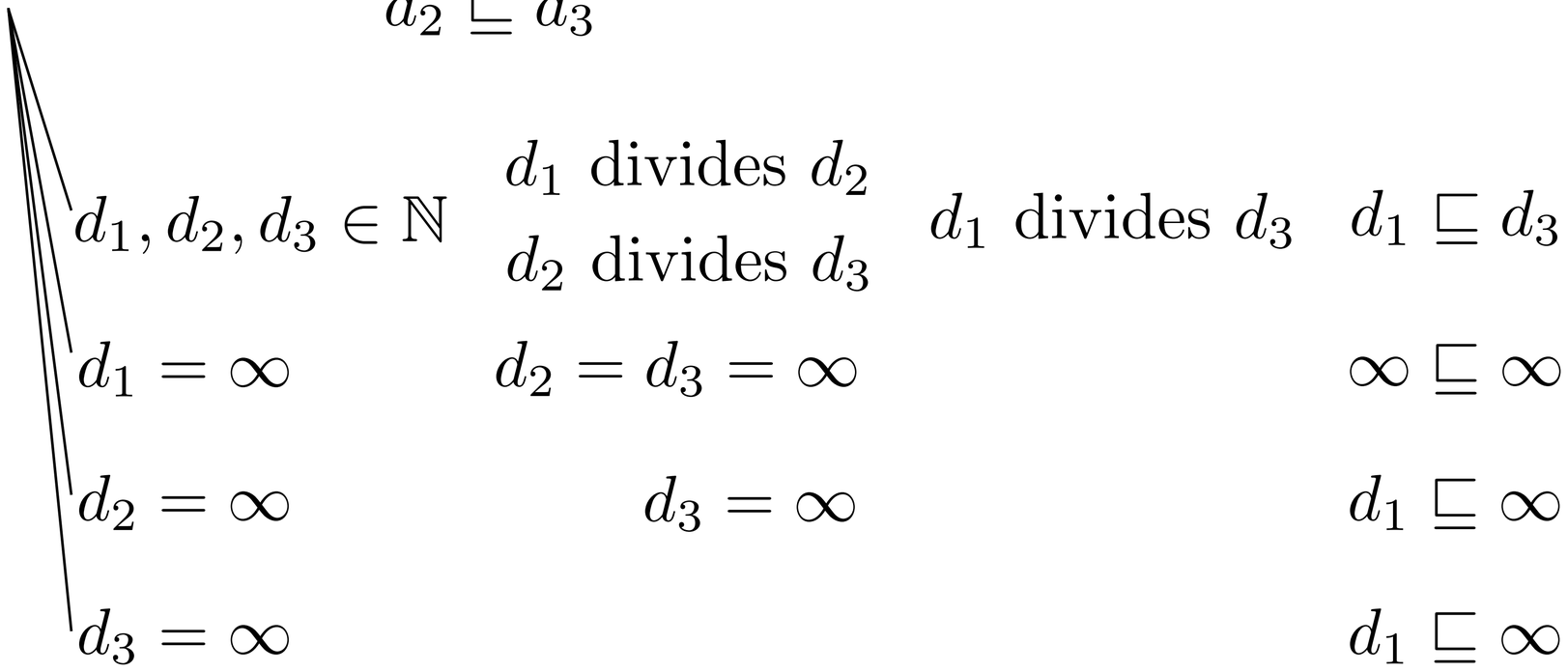
- $d_1, d_2 \in \mathbb{N}$ — d_1 divides d_2, d_2 divides d_1 — $d_1 \leq d_2$, $d_2 \leq d_1$ — $d_1 = d_2$
- $d_1 = \infty$ — $\infty \sqsubseteq d_2$ — $d_2 = \infty$ — $d_2 = \infty = d_1$
- $d_2 = \infty$ — $\infty \sqsubseteq d_1$ — $d_1 = \infty$ — $d_1 = \infty = d_2$

Divisors

$\mathrm{CPO}_\perp$?

transitive?

$d_1, d_2, d_3 \in D \quad d_1 \sqsubseteq d_2 \quad d_2 \sqsubseteq d_3 \quad d_1 \overset{?}{\sqsubseteq} d_3$



bottom? 1 divides every number and $1 \sqsubseteq \infty$

Divisors

$\mathrm{CPO}_\perp$?

complete?

- every finite chain has a limit
- infinite chains can only contain increasing natural numbers, then the limit is ∞

[Exercise] Define two functions $f_i : D_i \to D_i$ over two suitable CPOs D_i for $i \in [1, 2]$ (not necessarily with bottom) such that

1. f_1 is continuous, has fixpoints but not a least fixpoint;
2. f_2 is continuous and has no fixpoint;

Fixpoints #1

1. f_1 is continuous, has fixpoints but not a least fixpoint;

Let us try to find a minimal example
How many elements do we need at least?
How should they be ordered?

Fixpoints #1

1. f_1 is continuous, has fixpoints but not a least fixpoint;

$$D_1 = (\{0, 1\}, =) \qquad\qquad f_1 : D_1 \to D_1$$

Discrete order: CPO

Discrete order: any function is monotone and continuous

$$f_1(0) = 0$$

$$f_1(1) = 1$$

Kleene's theorem is not applicable: why?

Fixpoints #2

2. f_2 is continuous and has no fixpoint;

Let us try to find a minimal example
How many elements do we need at least?
How should they be ordered?

Fixpoints #2

2. f_2 is continuous and has no fixpoint;

$$D_2 = D_1 = (\{0, 1\}, =) \qquad f_2 : D_2 \to D_2$$

Discrete order: CPO

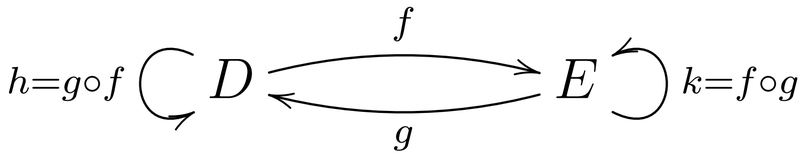
Discrete order: any function is monotone and continuous

$$f_2(0) = 1$$

$$f_2(1) = 0$$

Kleene's theorem is not applicable: why?

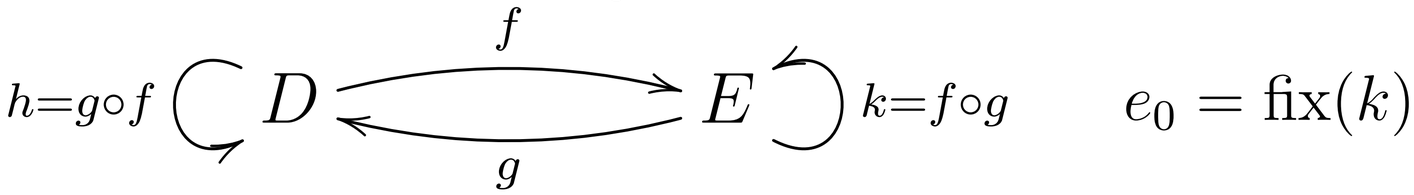
[Exercise] Let D, E be two $\mathrm{CPO}_\perp$s and $f : D \to E$, $g : E \to D$ be two continuous functions between them. Their compositions $h = g \circ f : D \to D$ and $k = f \circ g : E \to E$ are known to be continuous and thus have least fixpoints.



Let $e_0 = \mathrm{fix}(k) \in E$. Prove that $g(e_0) = \mathrm{fix}(h) \in D$ by showing that

1. $g(e_0)$ is a fixpoint for h, and
2. $g(e_0)$ is the least pre-fixpoint for h.

FP via composition



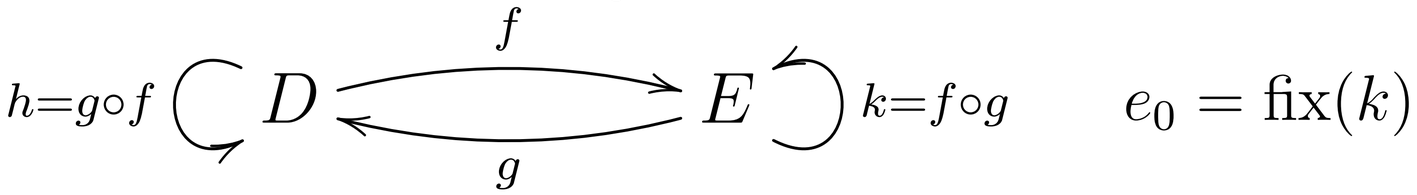
1. $g(e_0)$ is a fixpoint for h

we need to prove $h(g(e_0)) = g(e_0)$

$$h(g(e_0)) = g(f(g(e_0))) = g(k(e_0)) = g(e_0)$$

by def by def $e_0 = \mathrm{fix}(k)$

FP via composition



2. $g(e_0)$ is the least pre-fixpoint for h.

take $d \sqsupseteq_D h(d)$ we want to prove $g(e_0) \sqsubseteq_D d$

$d \sqsupseteq_D h(d) = g(f(d))$ (by def of h)

$f(d) \sqsupseteq_E f(g(f(d))) = k(f(d))$ (by monotonicity f, def. k)

hence $f(d)$ is a pre-fixpoint of k

e_0 is the least pre-fixpoint of k

$\Rightarrow e_0 \sqsubseteq_E f(d)$

$g(e_0) \sqsubseteq_D g(f(d)) = h(d) \sqsubseteq_D d$ (by mon. g, def. h, hyp. d)