Note Title 2025-05-29

Algebraic Riccati equations

$$\min_u \int_0^\infty \left(\underline{x(t)^* Q x(t)} + \underline{u(t)^* u(t)} \right) dt \qquad Q \succeq 0$$

Solution obtained from

$$Q + A^* X + XA - XGX = 0 \qquad G = bb^*$$

$$Ax(t) + bu(t) = (A - bb^* X)\, x(t) = (A - GX)\, x(t)$$

$$\boxed{u(t) = -b^* X x(t)} \qquad \text{"feedback control"}$$

Proof:
Datta, Ch. 10

Magic factorization:

$$\boxed{\frac{d}{dt} x^* X x} = \underbrace{(u + b^* X x)^* (u + b^* X x)}_{\geq 0} \boxed{- x^* Q x - u^* u}$$

$$\min_{u(t)} \int x^* Q x + u^* u \quad \text{s.t.} \quad \dot{x} = Ax + Bu$$

[Mehrmann '91]

Discrete-time analogue "the ugly equation":

$$X = Q + A^* X \underline{(I + GX)^{-1}} A \qquad G = bb^*$$

Various forms: $\rho(GX) < 1$

$$(I + GX)^{-1} = I - GX + GXGX - \dots$$

$$X(I + GX)^{-1} = X - XGX + XGXGX - \dots = (I + XG)^{-1} X$$

$$= \boxed{X - X(I - GX)^{-1}}$$

$$= X - Xb(1 - b^* X b - b^* X b\, b^* X b - \dots) b^*$$

$$\boxed{X = Q + A^* X A + \text{other terms}}$$

Many different forms!

CARE $\quad A^* X + XA + Q - XGX = 0$

$$\Longleftrightarrow \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} 1 \\ X \end{bmatrix} = \begin{bmatrix} 1 \\ X \end{bmatrix} K \qquad K = A - GX$$

$$\begin{cases} A - GX = K \\ -Q - A^* X = XK = X(A - GX) \end{cases}$$

X solves CARE $\Longleftrightarrow$ $\underset{2n \times 2n}{\begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix}} \underset{2n \times n}{\begin{bmatrix} 1 \\ X \end{bmatrix}} = \underset{2n \times n}{\begin{bmatrix} 1 \\ X \end{bmatrix}} \underset{n \times n}{K}$

Related to: $\underset{2n\times 2n}{H}\, \underset{2n\times 1}{w} = \underset{2n\times 1}{w}\, \underset{1\times 1}{\lambda}$

$Kv = v\lambda \Rightarrow \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} 1 \\ X \end{bmatrix} v = \begin{bmatrix} 1 \\ X \end{bmatrix} v\lambda \qquad \left(\begin{bmatrix} 1 \\ X \end{bmatrix} v, \lambda \right)$ is an eigenvector/value pair for H

$(v_1, \dots v_n)$ eigenvectors of $K \Rightarrow \begin{bmatrix} 1 \\ X \end{bmatrix} v_1, \dots \begin{bmatrix} 1 \\ X \end{bmatrix} v_n$ are n of the $2n$ eigenvector/value pairs of H

Algorithm (unstable):

1) Form H
2) Compute eigenvectors of H
3) Take n out of those $2n$ eigenvectors $U = [w_1\ w_2 \dots w_n]$

$U \in \mathbb{C}^{2n \times n}$ $\qquad \operatorname{span} U = \operatorname{span}\left(\begin{bmatrix} 1 \\ X \end{bmatrix} v_1, \dots \begin{bmatrix} 1 \\ X \end{bmatrix} v_n \right) = \operatorname{Im} \begin{bmatrix} 1 \\ X \end{bmatrix}$

("invariant subspace")

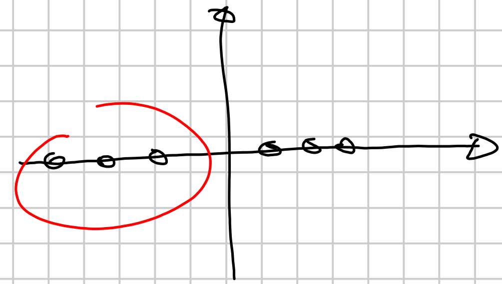
$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ $\qquad UU_1^{-1} = \begin{bmatrix} 1 \\ X \end{bmatrix}$.

Take those in the LHP!

CARE $A^*X + XA + Q - XGX$ has $\binom{2n}{n}$, but only the one that is <u>stabilizing</u> gives the solution to the optimal control problem

$$\Lambda(A - GX) \subset LHP$$

$$\Lambda(A - GX) = \Lambda(H) \cap LHP$$



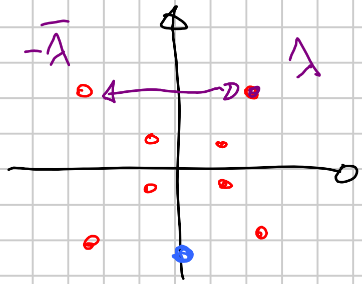
Optimal control $\Leftrightarrow$ X stabilizing solution of CARE

<u>Lemma</u>: suppose we have a matrix $H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \in \mathbb{C}^{2n \times 2n}$

$H_{12} = H_{12}^*$, $H_{21} = H_{21}^*$, $-H_{11}^* = H_{22}$

Then, if $\lambda \in \Lambda(H) \Rightarrow -\bar{\lambda} \in \Lambda(H)$ with the same multiplicity

I.e., $\Lambda(H)$ is symmetric w.r.t. the imaginary axis.



$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \qquad JH = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} = \begin{bmatrix} \boxed{-Q} & -A^* \\ -A & \boxed{G} \end{bmatrix}$$ Hermitian!

$$JH = (JH)^* = H^* J^* \quad \Leftrightarrow \quad JHJ^{-*} = H^*$$

$$JHJ^{-1} = -H^*$$

$\Rightarrow H, -H^*$ are similar, they have the same spectrum!

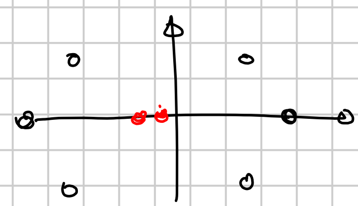
Additional results: (A, b) controllable and (A^*, Q) controllable, then there are no eigenvalues on the imaginary axis, and $X = X^*$ for the matrix such that $\begin{bmatrix} I \\ X \end{bmatrix}$ is a basis of the stable invariant subspace.

<u>Dense equation</u>: Schur algorithm:

1) compute $\mathrm{schur}(H) = U T U^*$
2) <u>reorder</u> Schur form s.t. $\mathrm{diag}(T(1{:}n, 1{:}n))$ = stable eigenvalues
3) $X = U_{21} U_{11}^{-1}$, where $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$ (after reordering)

The computed $\tilde{X}$ (in machine arithmetic) is s.t.
$\begin{bmatrix} I \\ \tilde{X} \end{bmatrix}$ is the invariant subspace of $H + \Delta_H$, $\dfrac{\|\Delta_H\|}{\|H\|}$ of the order of machine precision.

<u>Issues</u>: 1) Schur slow
2) $H + \Delta_H$ does <u>not</u> have the same structure as H (Hamiltonian)



$\Rightarrow$ wrong choice of λ_i's

Iterative algorithms for <u>dense</u> equations

DARE $\quad X = Q + A^* X (I + GX)^{-1} A$

First idea: $\quad X_{k+1} = Q + A^* X_k (I + G X_k)^{-1} A$

One can prove that $0 = X_0 \preceq X_1 \preceq X_2 \preceq \dots \preceq X$

and it converges linearly to the stabilizing solution.

$$\begin{bmatrix} A & 0 \\ Q & 1 \end{bmatrix}\begin{bmatrix} 1 \\ X \end{bmatrix} = \begin{bmatrix} 1 & G \\ 0 & A^* \end{bmatrix}\begin{bmatrix} 1 \\ X \end{bmatrix} K$$

$$\Leftrightarrow \begin{cases} A = (1+GX)K \\ Q+X = A^* X K = A^* X (1+GX)^{-1} A \end{cases}$$

We are looking for the solution such that $\rho\left((1+GX)^{-1}A\right) \subseteq$ open disc

$$Ax_k + bu_k = (1+GX)^{-1} A x_k = (A + b f^*) x_k$$

The analogue of the Hamiltonian is

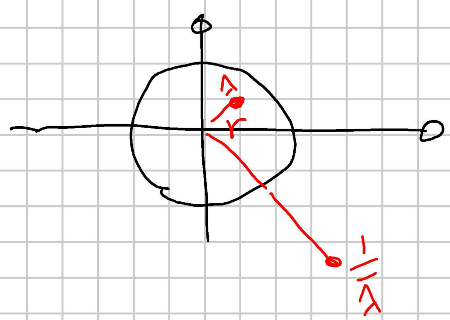
$$S = \begin{bmatrix} I & G \\ 0 & A^* \end{bmatrix}^{-1} \begin{bmatrix} A & 0 \\ Q & I \end{bmatrix}$$

We are looking for invariant subspaces of this matrix

<u>Lemma</u>: if G, Q symmetric, then any matrix of the form

$$S = \begin{bmatrix} I & G \\ 0 & A^* \end{bmatrix}^{-1} \begin{bmatrix} A & 0 \\ Q & I \end{bmatrix}$$

has eigenvalues that are inversion-symmetric w.r.t. the unit disc:



$$\lambda \in \Lambda(S) \Leftrightarrow \frac{1}{\bar\lambda} \in \Lambda(S)$$

Factorization for the iteration:

$$\begin{bmatrix} A & 0 \\ -Q & I \end{bmatrix}\begin{bmatrix} I \\ X_{k+1} \end{bmatrix} = \begin{bmatrix} I & G \\ 0 & A^* \end{bmatrix}\begin{bmatrix} I \\ X_k \end{bmatrix} K_k$$

$$\Leftrightarrow \begin{cases} A = (I + GX_k)K_k \\ -Q + X_{k+1} = A^* X_k K_k = A^* X_k (I + GX_k)^{-1} A \end{cases}$$

We can write our iteration in the form

$$\begin{bmatrix} I \\ X_{k+1} \end{bmatrix} K_k^{-1} = S^{-1} \begin{bmatrix} I \\ X_k \end{bmatrix}$$

$$v_{k+1} = M v_k$$

$$v_0, \ M v_0, \ M^2 v_0, \ldots$$

Subspace iteration: given $V_0 \in \mathbb{C}^{n \times k}$,

$V_0, MV_0, M^2V_0, M^3V_0, \ldots$ converges to k copies of the leading eigenvector

If we add a normalization,

$V_{k+1} = \mathrm{orth}(MV_k)$ so that V_k orthogonal at each step,

$V_k \to \mathrm{span}(\text{leading } k \text{ eigenvectors})$

Our algorithm

$$\begin{bmatrix} I \\ X_{k+1} \end{bmatrix} \underset{\substack{\uparrow \\ \text{normalization}}}{K_k^{-1}} = S^{-1} \underbrace{\begin{bmatrix} I \\ X_k \end{bmatrix}}_{V_k \in \mathbb{C}^{2n \times n}}$$

Normalization: the first n rows contain I.
This is the subspace iteration on S^{-1}.

It is going to converge to the invariant subspace associated to the n eigenvectors of S inside the unit disc.

Convergence is linear with rate $\left|\frac{\lambda_n}{\lambda_{n+1}}\right|$

where λ_n is the largest eigenvalue in the disc,

$\frac{1}{\overline{\lambda_n}} = \lambda_{n+1}$ is the smallest outside the disc

$$\left|\frac{\lambda_n}{\lambda_{n+1}}\right| = r^2 \qquad r = \rho\left((I+GX)^{-1}A\right).$$

Like the Smith method, we can build a squaring variant

$$\begin{bmatrix} I \\ X_k \end{bmatrix} K_k^{-1} K_{k-1}^{-1} \cdots X_1 = S^{-k} \begin{bmatrix} I \\ X_0 \end{bmatrix}$$

We can speed up the iteration by computing

$$S^{-1}, S^{-2}, S^{-4}, S^{-8}, \ldots$$

by repeated squaring, getting

$X_0, X_1, X_2, X_4, X_8, \ldots$ $\rightsquigarrow$ quadratic convergence

$\sim r^{2^k}$

$10^{-16} - 2^{-52}$

One can compute factorizations iteratively:

$$S^{2^k} = \begin{bmatrix} I & G_k \\ 0 & A_k^* \end{bmatrix} \begin{bmatrix} A_k & 0 \\ -Q_k & I \end{bmatrix}$$

$$\begin{cases} A_{k+1} = A_k(I+G_kQ_k)^{-1}A_k \\ G_{k+1} = G_k + A_kG_k(I+Q_kG_k)^{-1}A_k^* \\ Q_{k+1} = Q_k + A_k^*(I+Q_kG_k)^{-1}Q_kA_k \end{cases} \sim 10n^3$$

Structured doubling algorithm.

Finite horizon control $\qquad \dot{X} = A^*X + XA + G - XQX$

$t_f < \infty$ $\qquad \to \boxed{\text{eigenvalues}}$

M large, dense   <u>many</u> eigenpairs

FEAST algorithms $f(M)$