Note Title 2021-05-04

$$A = \left[\begin{array}{cccc|c} 1 & & & & v_1 \\ & 1 & & & v_2 \\ & & \ddots & & \vdots \\ & & & 1 & v_n \\ \hline v_1 & v_2 & \cdots & v_n & 1 \end{array}\right] \qquad M = \left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline v^T & 1 \end{array}\right]$$

$$N = M - A = \left[\begin{array}{c|c} 0 & -v \\ \hline 0 & 0 \end{array}\right]$$

1) $$\underbrace{\left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline -v^T & 1 \end{array}\right]}_{K} \underbrace{\left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline v^T & 1 \end{array}\right]}_{M} = I \Rightarrow K \text{ è l'inverse di } M$$

$$KM = \left[\begin{array}{c|c} I_{n-1} \cdot I_{n-1} + 0 \cdot v^T & I_{n-1} \cdot 0 + 0 \cdot 1 \\ \hline -v^T I_{n-1} + 1 \cdot v^T & -v^T \cdot 0 + 1 \cdot 1 \end{array}\right] = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

2) $$G = M^{-1} N = \begin{array}{c} \\ n-1 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} n-1 & 1 \end{array} \\ \left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline -v^T & 1 \end{array}\right] \end{array} \cdot \begin{array}{c} \begin{array}{cc} n-1 & 1 \end{array} \\ \left[\begin{array}{c|c} 0 & -v \\ \hline 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ n-1 \\ 1 \end{array} = \left[\begin{array}{c|c} 0 & -v \\ \hline 0 & -v^T \cdot (-v) \end{array}\right]$$

$$= \begin{array}{c} \begin{array}{cc} n-1 & 1 \end{array} \\ \left[\begin{array}{c|c} 0 & -v \\ \hline 0 & \|v\|_2^2 \end{array}\right] \end{array} \begin{array}{c} \\ n-1 \\ 1 \end{array}$$

$$v^T \cdot v = [v_1 \ \cdots \ v_{n-1}] \begin{bmatrix} v_1 \\ \vdots \\ v_{n-1} \end{bmatrix} = v_1^2 + v_2^2 + \dots + v_{n-1}^2 = \|v\|_2^2$$

Il metodo di G-S converge <u>se e solo se</u> $\rho(G) < 1$

Gli autoval. di G sono 0 (molt. $n-1$) e $\|v\|_2^2$ (molt. 1).

$$\rho(G) = \max_{\lambda \in \Lambda(G)} (|\lambda|) = \|v\|_2^2$$

G.S. converge $\iff \|v\|_2^2 < 1 \iff \|v\|_2 < 1$

3) È vero che $\|v\|_2 < 1 \Rightarrow A$ pred. diag?

A pred. diag. vuol dire $\forall i = 1, 2, \dots n$ $\quad |A_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |A_{ij}|$

$$A = \begin{bmatrix} I_{n-1} & v \\ v^T & 1 \end{bmatrix} :$$

per le righe $1:n-1$, ho $\quad |1| > |v_i| \quad i = 1, 2, \dots n-1$

per la riga n, ho $\quad |1| > |v_1| + |v_2| + \dots + |v_{n-1}|$

L'ultima implica le altre: $\quad 1 > |v_1| + |v_2| + \dots + |v_n| > |v_i| \ \forall i$

A pred. diag. se e solo se $\quad |v_1| + |v_2| + \dots + |v_{n-1}| < 1$

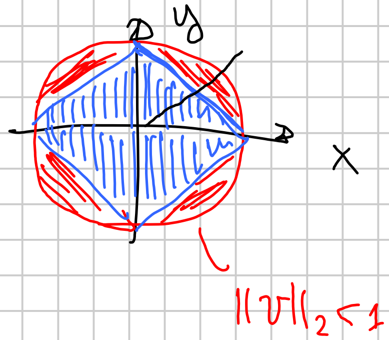
$$\|v\|_1 < 1$$

È vero che $\|v\|_2 < 1$ implica $\|v\|_1 < 1$

$$C_1 \|v\|_1 \leq \|v\|_2 \leq C_2 \|v\|_1$$

Per v di lunghezza $n-1 = 2$, $\quad v = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\{\|v\|_2 < 1\} = \{\sqrt{x^2 + y^2} < 1\} =$$ 



$\{\|v\|_1<1\} = \{|x|+|y|<1\} =$

4 casi, a seconda dei segni di x, y

se $x, y > 0 \Rightarrow x+y<1$

Ci sono punti nel cerchio ma non nel quadrato

$\Rightarrow$ punti t.c. $\|v\|_2<1$ ma $\|v\|_1>1$

Ad es., sulla bisettrice $(x=y)$

$\sqrt{x^2+x^2}<1 \quad \sqrt{2}|x|<1 \quad -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}=0.707\ldots$

$|x|+|x|<1 \quad 2|x|<1 \quad -0.5<x<0.5$

In part., se prendo $v=[0.6, 0.6]$

ho $\|v\|_2<1$, $\|v\|_1>1$

$\begin{bmatrix}1&0\\0&1\end{bmatrix} \quad [0.6 \;\; 0.6]$

$\begin{bmatrix}0.6\\0.6\end{bmatrix} \qquad 1$

3. No, perché per esempio se $v=\begin{bmatrix}0.6\\0.6\end{bmatrix}$

$\|v\|_2<1$ ma $\begin{bmatrix}1&&0.6\\&1&0.6\\0.6&0.6&1\end{bmatrix}$ non è pred. diag.

in quanto $|1|<|0.6|+|0.6|$ (3ª riga)

L'unica disuguaglianza che posso scrivere è

(Cauchy-Schwarz tra i vettori $x = [|v_1|, |v_2|, \dots |v_{n-1}|]$)
e $y = [1 \; 1 \; 1 \dots 1]$, ottenete $|x \cdot y| \leq \|x\|_2 \cdot \|y\|_2$

$$|v_1| + |v_2| + \dots + |v_{n-1}| \leq \sqrt{|v_1|^2 + |v_2|^2 + \dots + |v_{n-1}|^2} \cdot \sqrt{1+1+\dots+1}$$

$$\|v\|_1 \leq \|v\|_2 \cdot \sqrt{n-1}$$

che è più debole di $\|v\|_1 \leq \|v\|_2 < 1$ (falso)

$$M x^{(k+1)} = N x^{(k)} + b$$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \\ v_1 & v_2 & \cdots & v_{n-1} & 1 \end{bmatrix} \begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \\ \vdots \\ x_{n-1}^{k+1} \\ x_n^{k+1} \end{bmatrix} = \begin{bmatrix} & -v_1 \\ O & -v_2 \\ & \vdots \\ & -v_{n-1} \\ O & O \end{bmatrix} \begin{bmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_n^k \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

$$\boxed{\begin{matrix} x_1^{k+1} \\ x_2^{k+1} \\ \vdots \\ x_{n-1}^{k+1} \end{matrix}} = xnew(1:n-1)$$

$$\begin{aligned} x_1^{k+1} &= -v_1 x_n^k + b_1 \\ x_2^{k+1} &= -v_2 x_n^k + b_2 \\ &\vdots \\ x_{n-1}^{k+1} &= -v_{n-1} x_n^k + b_{n-1} \end{aligned}$$

for $i = 1:n-1$

$$v_1 x_1^{k+1} + v_2 x_2^{k+1} \dots + v_{n-1} x_{n-1}^{k+1} + x_n^{k+1} = O + b_n$$

$$x_n^{k+1} = b_n - v_1 x_1^{k+1} - \dots - v_{n-1} x_{n-1}^{k+1}$$

Costo comp.:

per ogni iterazione,

2 ops. $\times$ $n-1$

2 ops. $\times$ $n-1$

1 op

n ops per xnew - xold = y

n ops per $y_1^2, \dots y_n^2$

n-1 ops per $y_1^2 + \dots + y_n^2$

1 op per $\sqrt{}$

$$2(n-1) + 2(n-1) + 2n + n - 1 + 1 = \underbrace{7n + O(1)}_{O(n)}$$

$$S = v_1 x_1^k + v_2 x_2^k + \dots + v_{n-1} x_{n-1}^k = v^T \cdot \begin{bmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_{n-1}^k \end{bmatrix}$$

Alternativamente:

$$x^{(k+1)} = P x^{(k)} + q$$

Ricorda: $P = G = \left[\begin{array}{c|c} 0 & -v \\ \hline 0 & \|v\|_2^2 \end{array}\right]$

$$q = M^{-1} b = \begin{bmatrix} I & 0 \\ -v^T & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_{n-1} \\ b_n - v_1 b_1 - v_2 b_2 \dots v_{n-1} b_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} x_1^{k+1} \\ \vdots \\ \\ x_n^{k+1} \end{bmatrix} = \left[\begin{array}{c|c} 0 & -v \\ \hline 0 & \|v\|_2^2 \end{array}\right] \begin{bmatrix} x_1^k \\ \vdots \\ x_n^k \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n - v^T b(1:n-1) \end{bmatrix}$$

$$\begin{bmatrix} x_1^{k+1} \\ \vdots \\ \hline x_n^{k+1} \end{bmatrix} = \begin{bmatrix} -v_1 x_n^k \\ \vdots \\ -v_{n-1} x_n^k \\ \hline \|v\|_2^2 \, x_n^k \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ \hline b_n - v^T b(1:n-1) \end{bmatrix}$$