Teoria dei giochi - Game theory

Laurea Magistrale in Matematica

Laurea Magistrale in Fisica

2024/25

Lecture 6

Nash equilibria as fixed points

Consider $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ a strategic game:

$$R_i(x_{-i}) = \arg\max\{u_i(x_i, x_{-i}) \ : \ x_i \in S_i\} \text{ set of best replies (for } i \text{ to } x_{-i})$$

$$R(x) = R_1(x_{-1}) \times \cdots \times R_n(x_{-n}) \subseteq S_1 \times \cdots \times S_n = S$$

Nash equilibria are the fixed points of the set-valued map $R : S \rightrightarrows S$.

$$x^* \text{ Nash equilibrium} \iff x^* \in R(x^*)$$

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Fixed point theorem (Kakutani 1941)

Suppose $K \subseteq \mathbb{R}^m$ is a convex and compact set. If $T : K \rightrightarrows K$ satisfies

(i) $T(x)$ is nonempty, closed and convex for all $x \in K$

(ii) $\mathrm{graph}T = \{(x, y) \in K \times K \ : \ y \in T(x)\}$ is closed (T is closed)

then there exists $x^* \in K$ such that $x^* \in T(x^*)$.

existence of equilibria: apply Kakutani's theorem with $T = R$

which assumptions on the game are needed?

Existence of Nash equilibria

Theorem (Nikaido-Isoda 1955)

Let $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic game. If each $i \in N$ satisfies

(i) $S_i \subseteq \mathbb{R}^{m_i}$ *is convex and compact*

(ii) u_i *is upper semicontinuous*

(iii) $u_i(x_i, \cdot)$ *is lower semicontinuous for all* $x_i \in S_i$

(iv) $R_i(x_{-i})$ *is convex for all* $x_{-i} \in S_{-i}$

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Corollary

Every finite game has at least one Nash equilibrium in mixed strategies.

Strengthening concavity

Let $S \subseteq \mathbb{R}^m$ be convex. $f : \mathbb{R}^m \to \mathbb{R}$ is

- concave on S if
$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$$
holds for all $x, y \in S$, $\lambda \in [0,1]$

- strictly concave on S if
$$f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y)$$
holds for all $x, y \in S$ with $x \neq y$, $\lambda \in]0,1[$

- strongly concave on S with modulus τ if
$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y) + \frac{\tau}{2}\lambda(1-\lambda)\|x-y\|_2^2$$
holds for all $x, y \in S$, $\lambda \in [0,1]$

f strongly concave on S if and only if $f + \frac{\tau}{2}\|\cdot\|_2^2$ is concave on S

Concavity and optimization

Let $S \subseteq \mathbb{R}^m$ be convex and $f : \mathbb{R}^m \to \mathbb{R}$.

$$(P) \qquad \max\{f(x) \ : \ x \in S\}$$

- If f is concave on S, then any local maximum point of (P) is also a global maximum point. Moreover, the set of all the maxima is a convex set.
- If f is strictly concave on S, there exists at most one maximum point of (P).
- If f is strongly concave on S and S is closed, there exists exactly one maximum point of (P).

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Optimality conditions

Suppose f is concave and differentiable on S. Then, $x^* \in S$ is a (global) maximum point of (P) if and only if it satisfies the variational inequality

$$\nabla f(x^*)^T (x - x^*) \leq 0 \qquad \forall x \in S$$

Strict concavity is not enough for uniqueness (in games)

$$N = \{1, 2\},\ \ m_1 = m_2 = 1,\ \ S_1 = S_2 = [0, 1]$$

$$u_1(x_1, x_2) = -{x_1}^2 + 2x_1x_2, \quad u_2(x_1, x_2) = -{x_2}^2 + 3x_1x_2$$

($u_i(\cdot, x_{-i})$ strongly concave for any x_{-i})

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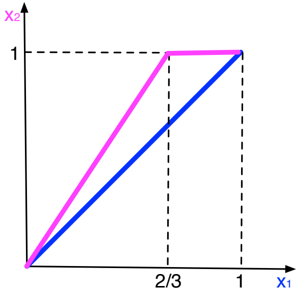
$$R_1(x_2) = \{x_2\} \qquad R_2(x_1) = \begin{cases} 3x_1/2 & \text{if } x_1 \le 2/3 \\ 1 & \text{if } x_1 > 2/3 \end{cases}$$



$(x_1^*, x_2^*) = (0, 0), (1, 1)$ are both Nash equilibria

Concavity and differentiability

Let $S \subseteq \mathbb{R}^m$ be convex and $f : \mathbb{R}^m \to \mathbb{R}$ differentiable on S.

(i) f is (strictly) concave on S if and only if

$$\big(\nabla f(x) - \nabla f(y)\big)^T (y - x) \geq 0 \qquad (>)$$

holds for all $x, y \in S$ (with $x \neq y$).

(ii) f is strongly concave on S with modulus $\tau > 0$ if and only if

$$\big(\nabla f(x) - \nabla f(y)\big)^T (y - x) \geq \tau \|y - x\|_2^2$$

holds for all $x, y \in S$.

(kind of monotonicities of the gradient map)

Diagonal strict concavity

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ with $u_i(\cdot, x_{-i})$ differentiable on S_i for any $x_{-i} \in S_{-i}$

Definition

The game G is diagonally strictly concave on S if

$$\sum_{i \in N} \left(\nabla_i u_i(x_i, x_{-i}) - \nabla_i u_i(x_i', x_{-i}')\right)^T (x_i' - x_i) > 0$$

holds for any $x, x' \in S$ with $x \neq x'$.

(where $\nabla_i u_i(x_i, x_{-i})$ denotes the gradient of $u_i(\cdot, x_{-i})$ at x_i)

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The game G is diagonally strictly concave on S if

$$\sum_{i \in N} \left(\nabla_i u_i(x_i, x_{-i}) - \nabla_i u_i(x'_i, x'_{-i})\right)^T (x'_i - x_i) > 0$$

holds for any $x, x' \in S$ with $x \neq x'$.

(where $\nabla_i u_i(x_i, x_{-i})$ denotes the gradient of $u_i(\cdot, x_{-i})$ at x_i)

$$\text{diagonal strict concavity} \underset{\not\Longleftarrow}{\Longrightarrow} u_i(\cdot, x_{-i}) \text{ strictly concave on } S_i$$

Uniqueness of Nash equilibria (Rosen 1965)

If the game G is diagonally strictly concave on S, then there exists at most one Nash equilibrium for G.

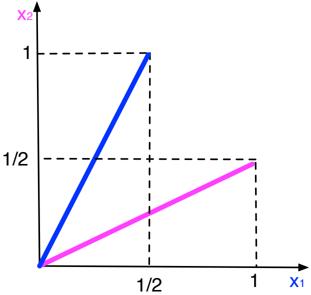
A simple example for uniqueness

$$N = \{1, 2\},\ m_1 = m_2 = 1,\ S_1 = S_2 = [0, 1]$$

$$u_1(x_1, x_2) = -{x_1}^2 + x_1 x_2, \quad u_2(x_1, x_2) = -{x_2}^2 + x_1 x_2$$

- $u_i(\cdot, x_{-i})$ strongly concave with modulus 2 for any x_{-i}
- $\nabla_i u_i(x_i, \cdot)$ Lipschitz with modulus 1 for any x_i
- the game is diagonally strictly concave on $S_1 \times S_2$

$$R_1(x_2) = \{x_2/2\} \quad R_2(x_1) = \{x_1/2\}$$



$(x_1^*, x_2^*) = (0, 0)$ is the unique Nash equilibrium

Criterion for diagonal strict concavity

Definition

The function $F : \mathbb{R}^m \to \mathbb{R}^m$ (with $m = m_1 + ... + m_n$) given by

$$F(x) = \begin{bmatrix} \vdots \\ -\nabla_i u_i(x_i, x_{-i}) \\ \vdots \end{bmatrix}$$

is called the gradient of the game G.

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Proposition

Suppose that u_i is twice continuously differentiable on S for all $i \in N$. If the matrix $J_F(x) + J_F(x)^T$ is positive definite for any $x \in S$, then the game G is diagonally strictly concave.

(where $J_F(x)$ denotes the Jacobian matrix of F)

$J_F(x)$ is the block matrix $(-\nabla_j \nabla_i u_i(x)^T)_{i,j}$

Existence and uniqueness via contractive best responses

Theorem

Suppose $u_i(\cdot, x_{-i})$ is differentiable on S_i for any $x_{-i} \in S_{-i}$. If each $i \in N$ satisfies

(i) $u_i(\cdot, x_{-i})$ *is strongly concave on* S_i *with modulus* τ_i *for any* $x_{-i} \in S_{-i}$

(ii) $\nabla_i u_i(x_i, \cdot)$ *is Lipschitz [cont] on* S_{-i} *with modulus* L_i *for any* $x_i \in S_i$

(iii) $L_i < \tau_i / \sqrt{(n-1)}$

then R is single-valued and it is a contraction, i.e., there exists $\rho \in]0, 1[$ *such that*

$$\|R(x) - R(x')\|_2 \leq \rho \|x - x'\|_2 \qquad \forall\, x, x' \in S.$$

Hence, the game has a unique Nash equilibrium x^* *and the sequence*

$$x^{k+1} = R(x^k) \qquad \textit{(best responses algorithm)}$$

converges to x^* *for any choice* $x^0 \in S$.