Basic Data

Title: Teoria dei giochi (Game theory)

Timetable: Mondays 2-4 pm - Tuesdays 9-11 am
room N *room P1*

Instructor: Giancarlo Bigi (giancarlo.bigi@unipi.it)

Webpage: pages.di.unipi.it/bigi/dida/tdg.html

Classroom: https://classroom.google.com/c/NzQ3NDc2OTUzOTI4
(course code: s3rdctl)

Topics: cooperative and noncooperative games, equilibria

Prerequisites: linear algebra, basics of topology, multivariate calculus, basics of probability, (a bit of) optimization

Material: lecture notes (draft) + detailed references + videolectures

Exam: interview or seminar&report

Teoria dei giochi - Game theory

Laurea Magistrale in Matematica

Laurea Magistrale in Fisica

2024/25

Lecture 1

Game theory is not [mainly] about games

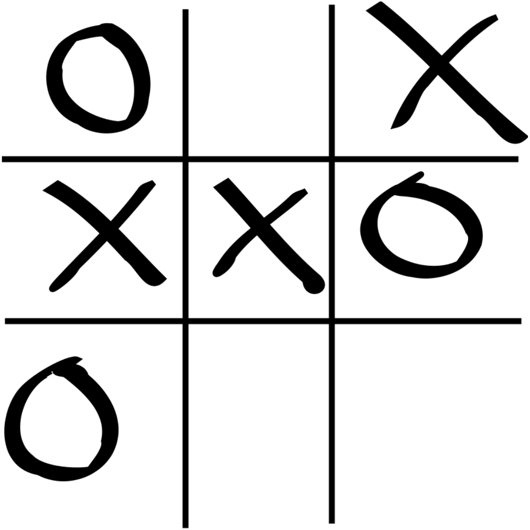
though dealing with games as well

(Zermelo's theorem, 1913)

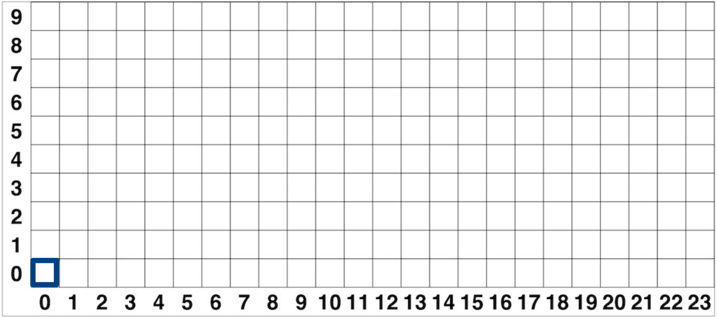
In the game of chess one and only one of the following holds:

- *the white player can enforce a win;*
- *the black player can enforce a win;*
- *both players can enforce a draw.*

tic tac toe: enforcing a draw



another board game: chomp



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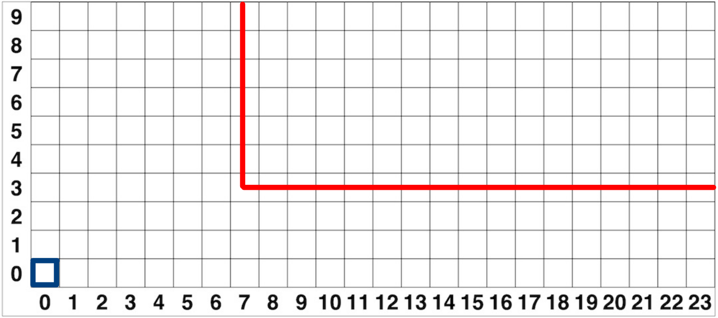

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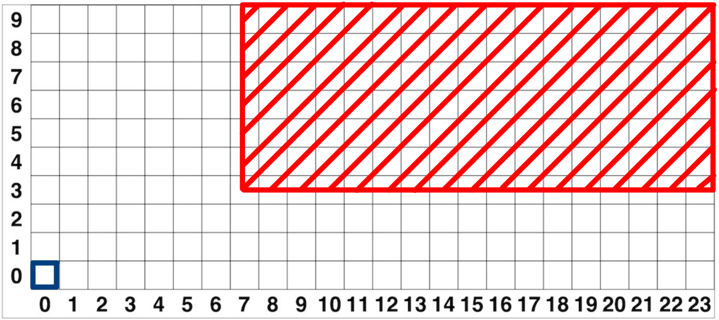

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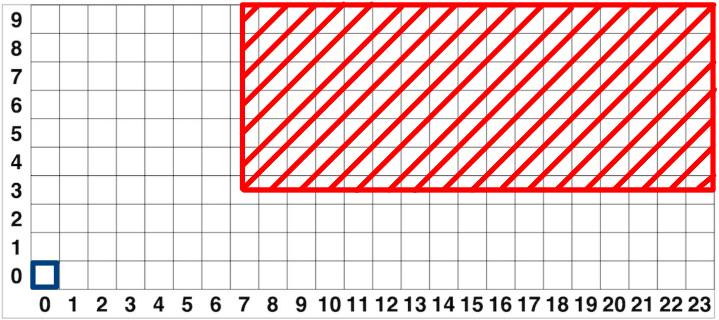


no draw is possible: which player can enforce a win?

Goals of game theory

phenomena/systems with interactions between multiple decision-makers

(decision-makers may be individuals or groups, nature, abstract entities, etc.)

- analyse situations in which their goals may conflict

  (the outcome for each one depends also on the choices of the others)

- understand inner mechanisms of
  - competition and cooperation
  - threats and promises

- forecast the behaviour of decision-makers [players]

- design mechanisms to steer systems towards desired objectives

Basic assumptions: players are rational and reason strategically

game = [math] description of the strategic interactions between players

Classification of games

– Cooperative games

agreements between players are allowed

which coalition(s) will be formed?

how will the outcome be split?

– Noncooperative games

agreements between players are not allowed

players aiming at their own best individually

- Strategic games

  "one shot": actions taken simultaneously at the beginning

  complete/incomplete information: whether or not all data are common knowledge

- Extensive games

  ordered events: actions taken sequentially (games with moves)

  perfect/imperfect information: whether or not all the past moves are disclosed

- Repeated games

  number of repetitions of some base (strategic or extensive) game

Breakthrough work

E.Zermelo, Uber eine Anwendung der Mengenlehre auf die Theorie des Schachspiels, in E.W.Hobson and A.E.H.Love (eds.) *Proceedings of the Fifth International Congress of Mathematicians, Volume II*, Cambridge University Press, 1913, pp. 501-504

J.von Neumann, Zur Theorie der Gesellschaftsspiele, *Mathematische Annalen* 100 (1928) 295-320.

J.von Neumann, O.Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, 1944.

J.F.Nash, Equilibrium Points in N-Person Games, *Proceedings of the National Academy of Sciences of the United States of America* 36 (1950) 48-49.

J.F.Nash, Non-Cooperative Games, *Annals of Mathematics* 54 (1951) 286-295.

L.S.Shapley, A Value for n-Person Games, in H.W. Kuhn and A.W. Tucker (eds.) *Contributions to the Theory of Games, Volume II*, Princeton University Press, 1953, pp. 307-317

Historical curiosity: Talmud Bavlì and game theory

Oldest known example of game theory (500 A.D.)

A man dies leaving debts larger than his estate.
How to divide the estate between the creditors?

The Talmud Bavlì provides rules for some cases with 3 creditors

R.J.Aumann, M.Maschler, Game Theoretic Analysis of a Bankruptcy Problem from the Talmud, *Journal of Economic Theory* 36 (1985) 195-213

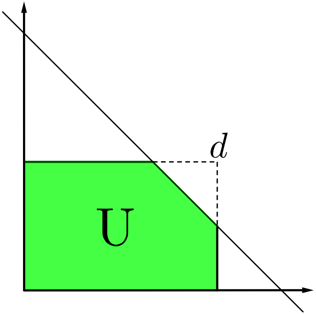
Bankruptcy

- business failure for a company
- asset of the company 4 m€
- n creditors - d_i m€ the liability towards creditor i
- $d_1 + \cdots + d_n > 4$

how to divide the asset between the creditors?

$$U = \{u \in \mathbb{R}^n_+ \ : \ u_1 + \cdots + u_n \leq 4, \ u_i \leq d_i \ i = 1, \ldots n\}$$

(feasible outcomes for an agreement/arbitration)



Some forerunners in economics

A.A.Cournot, *Recherches sur les Principes Mathematiques de la Theorie des Richesses*, Hachette, 1838

Competition between producers: duopoly (foreseeing Nash equilibria)

F.Y.Edgeworth, *Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences*, Kegan Paul, 1881.

Trading between people: allocations of 2 commodities between 2 people

13 Nobel laureates ..... and some more

The Sveriges Riksbank Prize in Economic Sciences
(in memory of Alfred Nobel)

1994 J.F.Nash, J.C.Harsanyi, R.Selten
for their pioneering analysis of equilibria in the theory of non-cooperative games

2005 R.J.Aumann, T.C.Schelling
for having enhanced our understanding of conflict and cooperation through game-theory analysis

2007 L.Hurwicz, E.S.Maskin, R.B.Myerson
for having laid the foundations of mechanism design theory

2012 A.E.Roth, L.S.Shapley
for the theory of stable allocations and the practice of market design

2014 J.Tirole
for his analysis of market power and regulation

2020 P.R.Milgrom, R.B.Wilson
for improvements to auction theory and inventions of new auction formats

TdG contents

Basic topics

- Noncooperative games

  Nash (normal) games
  Multilevel games
  Sequential games
  Algorithms

- Bargaining

  Nash solution to bargaining

  Fair allocations

- Cooperative games

  Transferable utility
  Core and nucleoli
  Shapley value
  power indices

Applications to economics and computer science

- Cournot, Bertrand and Stackelberg oligopolies
- Exchange economies
- Networks: routing and security
- Mining blockchains
- .....

Why should I take TdG?

- truly interdisciplinary

  unique blend of mathematics and computer science with economics, psychology and much more!

- expected and unexpected applications

  ranging from political sciences to engineering, from criminology to biology

- to open up your mind to strategic thinking

- first step towards winning a Nobel prize

- (nice?) instructor who keeps playing despite age

- just to have fun

What is a game?

A description of the strategic interactions between players

– (a finite number of) players

– strategies: the actions a player can take

– outcome: it depends on the strategies selected by all players

– preferences: a player's binary relation between outcomes
(complete, reflexive and transitive [total pre-order])

preferences are often given through an *utility function* [payoff]

The prisoner's dilemma

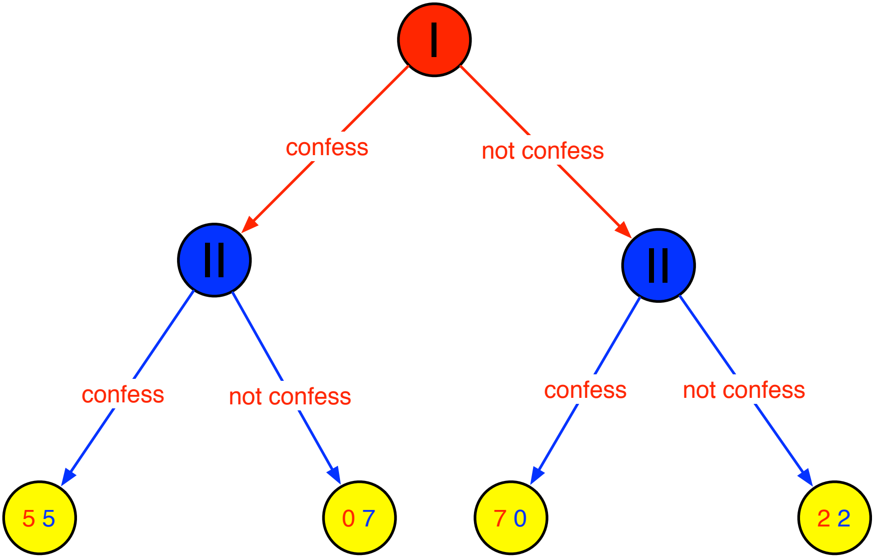
2 prisoners are accused of having committed a felony together

Years in jail are decided upon the prisoners' admissions of guilt

I/II	not confess	confess
not confess	(2,2)	(7,0)
confess	(0,7)	(5,5)

A.W.Tucker, A Two-Person Dilemma, memo at Stanford University, 1950

Another description of the prisoner's dilemma



A coordination game: the battle of sexes

A couple's evening out:

she would prefer go dancing, he would prefer the football game


Tonight we could go dancing!
Or... we could go to the football match!
I think Bob will go to the football match. Therefore, I'll go to the match.
I think Amy will expect me to be at the football match. Therefore she will go to the football match. So I shall go to the match.
I think Bob will expect me to be at the dance class. Hence he will be at dance class. So I will go dancing.
I think Amy will be at the dance class. Therefore I will go dancing.
I expected Bob to be at the football match. But he's not here! He must have thought I would go dancing.
I thought Amy would go dancing. But she isn't here! She must have expected me to go to the match.

A coordination game: the battle of sexes

A couple's evening out:

she would prefer go dancing, he would prefer the football game

both wish to go to the same place together rather than going alone

he/she	football	dancing
football	(2,1)	(0,0)
dancing	(0,0)	(1,2)

based on the stag hunt situation by Jean-Jacques Rousseau

Discours sur l'origine et les fondements de l'inégalité parmi les hommes, 1755

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A couple's evening out:

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he/she	football	dancing
football	(A,a)	(C,c)
dancing	(B,b)	(D,d)

$$A > B, \;\; D > C, \;\; a > c, \;\; d > b$$

An anti-coordination game: hawk-dove

(Maynard Smith-Price 1973)

Two animals to contest food:

hawk = aggressive behaviour (physically attack the other)

dove = cooperative behaviour (pacific attitude to share the food)

I/II	hawk	dove
hawk	(-2,-2)	(2,0)
dove	(0,2)	(1,1)

anti-coordination games: hawk-dove, chicken, graph colouring

(brinkmanship in nuclear warfare)

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I/II	hawk	dove
hawk	(A,A)	(B,b)
dove	(b,B)	(D,D)

anti-coordination games: $B > D > b > A$

(brinkmanship in nuclear warfare)

Rock-paper-scissors

paper covers rock - rock crushes scissors - scissors cuts paper

I/II	paper	scissors	rock
paper	(0,0)	(-1,1)	(1,-1)
scissors	(1,-1)	(0,0)	(-1,1)
rock	(-1,1)	(1,-1)	(0,0)

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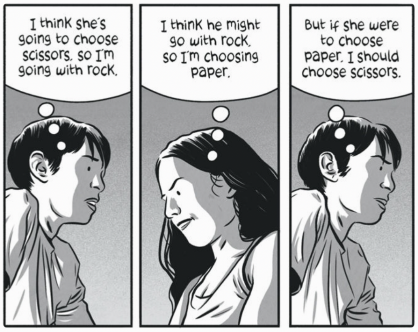
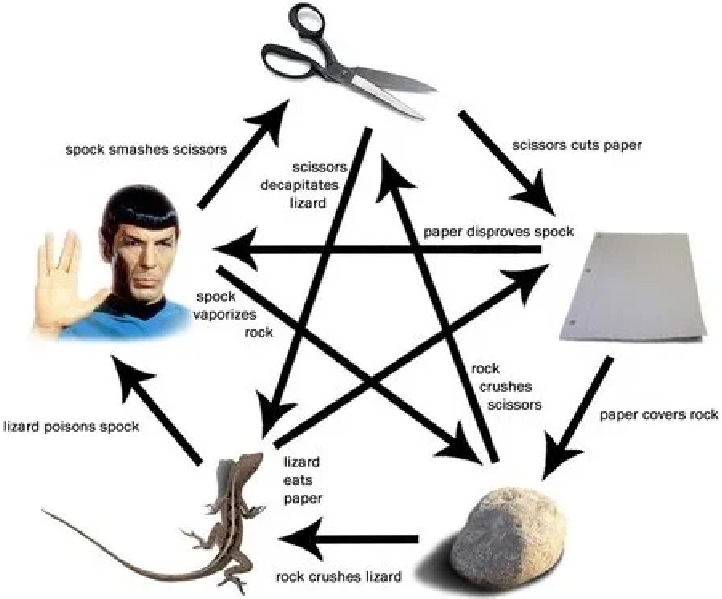


The Lizard-Spock expansion



Rock-paper-scissors-lizard-Spock

scissors cuts paper - paper covers rock - rock crushes lizard
lizard poisons Spock -Spock smashes scissors - scissors decapitates lizard
lizard eats paper- paper disproves Spock - Spock vaporizes rock
rock crushes scissors

I/II	paper	scissors	rock	lizard	Spock
paper	(0,0)	(-1,1)	(1,-1)	(-1,1)	(1,-1)
scissors	(1,-1)	(0,0)	(-1,1)	(1,-1)	(-1,1)
rock	(-1,1)	(1,-1)	(0,0)	(1,-1)	(-1,1)
lizard	(1,-1)	(-1,1)	(-1,1)	(0,0)	(1,-1)
Spock	(-1,1)	(1,-1)	(1,-1)	(-1,1)	(0,0)

Colonel Blotto game(s)

2 players: colonel Blotto b, enemy e

limited amount of resources: R_b, R_e (> 0)

n battlefields, each with its own value: $w_1, \dots, w_n$ (> 0)

battlefield winner: the player deploying most resources

how to allocate resources between the battlefields?

strategies:

$$x_{b/e} \in \mathbb{R}_+^n \quad \text{s.t.} \quad (x_{b/e})_1 + \cdots + (x_{b/e})_n = R_{b/e}$$

utility functions:

$$u_b(x_b, x_e) = \sum_{i=1}^{n} w_i \operatorname{sign}((x_b)_i - (x_e)_i) = -u_e(x_b, x_e)$$

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Individual decision-making under risk

- a unique decision-maker
- n mutually exclusive events: $A_1, \ldots, A_n$ (exactly one will occur)
- A_i is preferred to A_{i+1}

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Lottery

$$L = [(A_1, p_1), ..., (A_i, p_i), ..., (A_n, p_n)] \quad \textit{with} \quad p_i \geq 0 \quad \textit{s.t.} \sum_{i=1}^{n} p_i = 1$$

p_i probability that A_i occurs

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Preferences over $\mathcal{L} = \{\text{lotteries}\}$ through a binary relation $\geq$ satisfying

- reflexivity: $L \geq L$
- transitivity: $L_1 \geq L_2,\; L_2 \geq L_3 \Longrightarrow L_1 \geq L_3$
- completeness: $L_1 \geq L_2$ or $L_2 \geq L_1$ holds

(antisymmetry not required: $L_1 \geq L_2,\; L_2 \geq L_1 \not\Longrightarrow L_1 = L_2$)

equivalence: $L_1 \sim L_2 \Longleftrightarrow L_1 \geq L_2,\, L_2 \geq L_1$

Preferences versus utility

- Monotonicity

  $$p, q \in [0,1]: \quad [(A_1, p), (A_n, 1-p)] \geq [(A_1, q), (A_n, 1-q)] \iff p \geq q$$

- Continuity

  $$\exists\ \mu_i \in [0,1] \text{ s.t. } [(A_i, 1)] \sim [(A_1, \mu_i), (A_n, 1-\mu_i)]$$

- [De]composition

$$[(A_1, p_1), ..., (A_i, p_i), ..., (A_n, p_n)] \sim [(A_1, p_1 + p_i\mu_i), ..., (A_i, 0), ..., (A_n, p_n + p_i(1-\mu_i))]$$

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Expected utility theorem (von Neumann-Morgenstern 1944)

If the pair $(\mathcal{L}, \geq)$ satisfies the above monotonicity, continuity and [de]composition properties, then there exists $u : \mathcal{L} \to \mathbb{R}$ such that

$$L_1 \geq L_2 \iff u(L_1) \geq u(L_2).$$

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$$L_1 \geq L_2 \quad \iff \quad u(L_1) \geq u(L_2).$$

$$u(L) = \sum_{i=1}^{n} p_i \mu_i \quad \text{for } L = [(A_1, p_1), ..., (A_i, p_i), ..., (A_n, p_n)]$$