Initial test for the class of Numerical Methods and Optimization

(1) Find intervals containing solutions to the following equations:

(a) $x - 2^{-x} = 0$;
(b) $3x^3 - e^x = 0$;
(c) $x^3 - 2x^2 - 4x + 1 = 0$.

(2) Find local maxima and minima, if any, of the following functions:

(a) $f(x) = e^{-x^2}$;
(b) $f(x) = 2x - e^{-x}$;
(c) $f(x) = \dfrac{x-2}{x^2+1}$.

(3) Find the Taylor polynomial $p_2(x)$ of degree 2 for the the following functions around $x_0 = 0$, and give a bound to the error $|f(x) - p_2(x)|$ for $|x| \leq 0.5$:

(a) $f(x) = e^x$;
(b) $f(x) = \cos x$;
(c) $f(x) = \sqrt[3]{x+1}$.

(4) Given $x, y \in \mathbb{R}$, find $z \in \mathbb{R}$, $x \leq z \leq y$, satisfying the mean value formula on $[x, y]$ for the following functions and pairs of points:

(a) $f(t) = t^2, \quad x = -1, y = 2$;
(b) $f(t) = \sin t, \quad x = 0, y = \pi/2$;
(c) $f(t) = e^t, \quad x = -1, y = 1$.

(5) The following iterative method is proposed to approximate $\sqrt[3]{2}$:

$$x_{i+1} = x_i - \frac{x_i^3 - 2}{12}.$$

Show that any sequence obtained for an x_0 choosen in the interval $[1, 2]$ converges.

(6) Compute two iterates of the Newton's method applied to the equation $x^3 - 2 = 0$, starting from $x_0 = 2$.

(7) Use Gaussian elimination, without reordering equations, to solve the following linear systems:

(a)

$$\begin{cases} x_1 - x_2 + 2x_3 &= 1 \\ -x_1 + 3x_3 &= -4 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{cases},$$

(b)

$$\begin{cases} x_1 - x_2 + 2x_3 &= 1 \\ -x_1 - 4x_2 + 3x_3 &= 4 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{cases}.$$

(8) Show that the function $f(\boldsymbol{x}) = (\sqrt{|x_1|} + \sqrt{|x_2|})^2$ is not a norm for the vectors $\boldsymbol{x} \in \mathbb{R}^2$.

(9) Compute the eigenvalues and the eigenvectors of the following matrices, and say if they are diagonalizable:

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}.$$

(10) Compute the interpolating polynomial of degree 2 for the function $f(x) = \cos(x)$ in the points $x_0 = -\pi/2, x_1 = 0, x_2 = \pi/2$.