Algorithmic game theory

Laurea Magistrale in Computer Science

2024/25

Lecture 8

Learning in games: best response dynamics

Algorithmic rephrasing of Cournot's basic idea

Synchronous distributed algorithm (Jacobi type algorithm)

1. $x^0 = (x_1^0, \dots, x_n^0) \in S_1 \times \cdots S_n,\ k = 0$
2. x_i^{k+1} *is a best response to* x_{-i}^k $(x_i^{k+1} \in R_i(x_{-i}^k))$ $\Big\} \Longrightarrow x^{k+1} \in R(x^k)$
   - *if* $x_i^k \in R_i(x_{-i}^k)$*, select* $x_i^{k+1} = x_i^k$
3. *if* $x^{k+1} = x^k$ *then STOP*
4. $k = k + 1$ *and go back to 2*

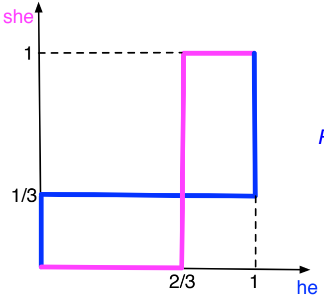
All players know the current state (x^k) and reply [simultaneously]

Knowledge of other players' utility functions is not required

Synchronous algorithm in finite games (mixed strategies)

		q	$1-q$
	he/she	football	dancing
p	football	(2,1)	(0,0)
$1-p$	dancing	(0,0)	(1,2)

$$R_{she}(p) = \begin{cases} 1 & \text{if } p > 2/3 \\ [0,1] & \text{if } p = 2/3 \\ 0 & \text{if } p < 2/3 \end{cases}$$



$$R_{he}(q) = \begin{cases} 1 & \text{if } q > 1/3 \\ [0,1] & \text{if } q = 1/3 \\ 0 & \text{if } q < 1/3 \end{cases}$$

$$x^0 = (1/2, 1/2) \longrightarrow x^1 = (1,0) \longrightarrow x^2 = (0,1) \longrightarrow x^3 = (1,0) \longrightarrow \textit{looping}$$

$$\left.\begin{array}{l} x^0 = (1/4, 1/4) \longrightarrow x^1 = (0,0) \\ x^0 = (3/4, 3/4) \longrightarrow x^1 = (1,1) \end{array}\right\} \longrightarrow \textit{converging}$$

Another view on successive best responses

Asynchronicity: updated responses of previous players are known too

Asynchronous distributed algorithm (Gauss-Seidel type algorithm)

1. $x^0 = (x_1^0, \ldots, x_n^0) \in S_1 \times \cdots S_n,\ k = 0$
2. *for* $i = 1, \ldots n$

   $x_i^{k+1} \in R_i(\bar{x}_{-i}^k)$ *with* $\bar{x}_{-i}^k = (x_1^{k+1}, \ldots, x_{i-1}^{k+1}, x_{i+1}^k, \ldots, x_n^k)$ $(\bar{x}_{-1}^k = x_{-1}^k)$

   - *if* $x_i^k \in R_i(\bar{x}_{-i}^k)$, *select* $x_i^{k+1} = x_i^k$
3. *if* $x^{k+1} = x^k$ *then STOP*
4. $k = k + 1$ *and go back to 2*

the way players are ordered may be relevant

Asynchronous algorithm in finite games

he/she	football	dancing
football	(2,1)	(0,0)
dancing	(0,0)	(1,2)

$$x^0 = (f, d) \xrightarrow{he} \bar{x}^0_{-2} = (d, d) \xrightarrow{she} x^1 = (d, d)$$

$$x^0 = (d, f) \xrightarrow{he} \bar{x}^0_{-2} = (f, f) \xrightarrow{she} x^1 = (f, f)$$

Asynchronous algorithm in finite games

he/she	football	dancing
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switching the order of players

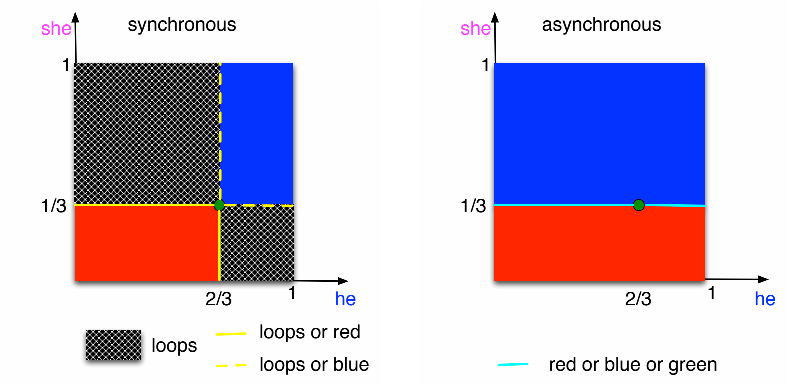
$$x^0 = (f, d) \xrightarrow{she} \bar{x}^0_{-2} = (f, f) \xrightarrow{he} x^1 = (f, f)$$

$$x^0 = (d, f) \xrightarrow{she} \bar{x}^0_{-2} = (d, d) \xrightarrow{he} x^1 = (d, d)$$

Distributed algorithms and mixed strategies

		q	$1-q$
	he/she	football	dancing
p	football	(2,1)	(0,0)
$1-p$	dancing	(0,0)	(1,2)

Nash equilibria in mixed strategies: $(1,1)$, $(0,0)$, $(2/3,1/3)$



starting points are coloured according to the reached equilibrium

Distributed algorithms and mixed strategies

		q_1	q_2	q_3
	I/II	paper	scissors	rock
p_1	paper	0	-1	1
p_2	scissors	1	0	-1
p_3	rock	-1	1	0

Unique Nash equilibrium: $\big((1/3, 1/3, 1/3), (1/3, 1/3, 1/3)\big)$

$$h(p,q) = p^T A q = p_1(q_3 - q_2) + p_2(q_1 - q_3) + p_3(q_2 - q_1)$$

$$\min\{p_2 - p_3, p_3 - p_1, p_1 - p_2\} \leq h(p,q) \leq \max\{q_3 - q_2, q_1 - q_3, q_2 - q_1\}$$

Distributed algorithms and mixed strategies

	I/II	paper (q_1)	scissors (q_2)	rock (q_3)
p_1	paper	0	-1	1
p_2	scissors	1	0	-1
p_3	rock	-1	1	0

Unique Nash equilibrium: $\big((1/3, 1/3, 1/3), (1/3, 1/3, 1/3)\big)$

$$h(p,q) = p^T A q = p_1(q_3 - q_2) + p_2(q_1 - q_3) + p_3(q_2 - q_1)$$

$$\min\{p_2 - p_3, p_3 - p_1, p_1 - p_2\} \leq h(p,q) \leq \max\{q_3 - q_2, q_1 - q_3, q_2 - q_1\}$$

Best responses* always include at most 2 pure strategies

Both the sync and the async algorithm loop or never converge

*to strategies which are not part of the equilibrium

Convergence of distributed algorithms

Basic convergence

Let $\{x^k\}$ be the sequence generated by the [a]synchronous algorithm.

(i) If the sequence is finite, then the last point is a Nash equilibrium.

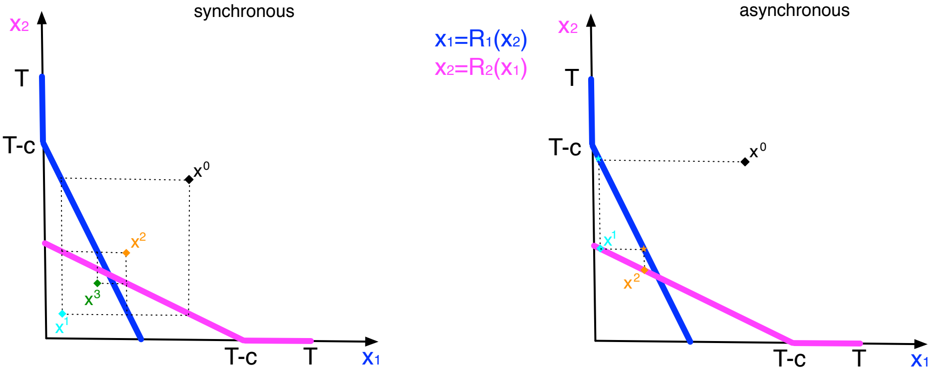
(ii) In any finite game the sequence is finite or the algorithm loops.

(iii) Suppose that $S \subseteq \mathbb{R}^m$ is closed and u_i is continuous on S for all $i \in N$. If the sequence is infinite and $x^k \to x^*$ for some $x^* \in S$, then x^* is a Nash equilibrium.

- finite games in mixed strategies fall within *(iii)*
- no guarantee that the infinite sequence in *(iii)* converges

  (rock-paper-scissors in mixed strategies)

Convergence of distributed algorithms: Cournot duopoly



Potential games: exactness

Definition (Monderer-Shapley 1996)

G is an exact potential game if there exists $P : S \to \mathbb{R}$ such that whatever $i \in N$ is considered the equality

$$u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) = P(x_i, x_{-i}) - P(x_i', x_{-i})$$

holds for all $x_i, x_i' \in S_i$ and all $x_{-i} \in S_{-i}$.

The function P is called an exact potential for G.

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holds for all $x_i, x_i' \in S_i$ and all $x_{-i} \in S_{-i}$.

The function P is called an exact potential for G.

Proposition

(i) If P is an exact potential function for G, then the function $P + a$ is an exact potential for G for any $a \in \mathbb{R}$.

(ii) If P and Q are exact potential functions for G, then $P - Q$ is constant.

(iii) Suppose S_i is convex, P and $u_i(\cdot, x_{-i})$ are continuously differentiable on S_i for any $x_{-i} \in S_{-i}$ and any $i \in N$. Then, P is an exact potential for G if and only if each $i \in N$ satisfies $\nabla_i P(x) = \nabla_i u_i(x)$ for all $x \in S$.

Potential games: exactness

exact potential for Cournot duopoly (inverse demand function $p(t) = T - t$)

$$P(x_1, x_2) = T(x_1 + x_2) - x_1^2 - x_2^2 - x_1 x_2 - c(x_1 + x_2)$$

Potential games: exactness

exact potential for Cournot duopoly (inverse demand function $p(t) = T - t$)

$$P(x_1, x_2) = T(x_1 + x_2) - x_1^2 - x_2^2 - x_1 x_2 - c(x_1 + x_2)$$

duopoly with indivisible commodity

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
④	(12,0)	(8,2)	(4,2)	(0,0)	(-4,-4)	(-8,-10)	(-12,-18)
⑤	(10,0)	(5,1)	(0,0)	(-5,-3)	(-10,-8)	(-15,-15)	(-15,-18)
⑥	(6,0)	(0,0)	(-6,-2)	(-12,-6)	(-18,-12)	(-18,-15)	(-18,-18)

Potential games: exactness

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$$P(x_1, x_2) = T(x_1 + x_2) - x_1^2 - x_2^2 - x_1 x_2 - c(x_1 + x_2)$$

duopoly with indivisible commodity

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
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exact potential

I/II	⓪	①	②	③	④	⑤	⑥
⓪	0	6	10	10	10	8	5
①	6	11	14	13	12	9	5
②	10	14	16	14	12	8	3
③	12	15	16	13	10	5	-1
④	12	14	14	10	6	0	-7
⑤	10	11	10	5	0	-7	-10
⑥	6	6	4	-2	-8	-10	-13

Looking for an exact potential

the battle of sexes

he/she	football	dancing
football	(2,1)	(0,0)
dancing	(0,0)	(1,2)

he/she	football	dancing
football		
dancing		

Looking for an exact potential

the battle of sexes

he/she	football	dancing
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dancing	(0,0)	(1,2)

he/she	football	dancing
football		
dancing	0	

Looking for an exact potential

the battle of sexes

he/she	football	dancing
football	(2,1)	(0,0)
dancing	(0,0)	(1,2)

he/she	football	dancing
football	2	
dancing	0	

Looking for an exact potential

the battle of sexes

he/she	football	dancing
football	(2,1)	(0,0)
dancing	(0,0)	(1,2)

he/she	football	dancing
football	2	1
dancing	0	

Looking for an exact potential

the battle of sexes

he/she	football	dancing
football	(2,1)	(0,0)
dancing	(0,0)	(1,2)

its exact potential

he/she	football	dancing
football	2	1
dancing	0	2

Looking for an exact potential

the battle of sexes

he/she	football	dancing
football	(2,1)	(0,0)
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its exact potential

he/she	football	dancing
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dancing	0	2

Looking for an exact potential

the battle of sexes

he/she	football	dancing
football	(2,1)	(0,0)
dancing	(0,0)	(1,2)

its exact potential

he/she	football	dancing
football	2	1
dancing	0	2

Characterization of exact potential games

A game has an exact potential if and only if

$$\sum_{k=1}^{\ell} \left(u_{i_k}(x^k) - u_{i_k}(x^{k-1}) \right) = 0$$

holds for any cycle $(x^0, \ldots, x^\ell)$, *i.e., any finite sequence of strategy profiles such that* $x^0 = x^\ell$ *and for any* $k = 1, \ldots, \ell$ *there exists* i_k *s.t.* $x^k_{-i_k} = x^{k-1}_{-i_k}$.

($\ell = 4$ is enough: cycles involving 2 players only suffice)

Potential games: ordinals

Definition (Monderer-Shapley 1996)

G is an ordinal potential game if there exists $P : S \to \mathbb{R}$ such that whatever $i \in N$ is considered the equivalence

$$u_i(x_i, x_{-i}) > u_i(x_i', x_{-i}) \iff P(x_i, x_{-i}) > P(x_i', x_{-i})$$

holds for all $x_i, x_i' \in S_i$ and all $x_{-i} \in S_{-i}$.

The function P is called an ordinal potential for G.

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The function P is called an ordinal potential for G.

Proposition

(i) If P is an ordinal potential function for G, then the function $P + a$ is an ordinal potential for G for any $a \in \mathbb{R}$.

(ii) If P is an exact potential function for G, then it is also an ordinal potential function for G.

(iii) Suppose that P is an ordinal potential for G. Then,

$$x_i^* \in R_i(x_{-i}^*) \iff x_i^* \in \arg\max\{P(x_i, x_{-i}^*) \ : \ x_i \in S_i\}$$

Potential games: ordinals

ordinal potential for Cournot duopoly (any inverse demand function p)

$$P(x_1, x_2) = x_1 x_2 (p(x_1 + x_2) - c)$$

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game

I/II	①	②
①	(1,1)	(2,3)
②	(2,1)	(3,2)

its ordinal potential

I/II	①	②
①	1	2
②	2	3

Potential games: ordinals

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its ordinal potential

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I/II	①	②
①		
②		

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its ordinal potential

I/II	①	②
①	1	2
②	2	3

I/II	①	②
①	0	
②		

Potential games: ordinals

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its ordinal potential

I/II	①	②
①	1	2
②	2	3

I/II	①	②
①	0	2
②		

Potential games: ordinals

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its ordinal potential

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②	2	3

I/II	①	②
①	0	2
②		3

Potential games: ordinals

ordinal potential for Cournot duopoly (any inverse demand function p)

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game

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its ordinal potential

I/II	①	②
①	1	2
②	2	3

I/II	①	②
①	0	2
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Potential games: ordinals

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game

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its ordinal potential

I/II	①	②
①	1	2
②	2	3

no exact potential exists

I/II	①	②
①	~~0~~ 1	2
②	2	3

Potential games: ordinals

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game

I/II	①	②
①	(1,1)	(2,3)
②	(2,1)	(3,2)

its ordinal potential

I/II	①	②
①	1	2
②	2	3

Proposition

If G is an ordinal potential game, then there are no weak improvement cycles, i.e., no cycle $(x^0, \ldots, x^\ell)$ satisfies

$$u_{i_k}(x^k) \geq u_{i_k}(x^{k-1})$$

for any $k = 1, \ldots, \ell$ with $u_{i_{\bar{k}}}(x^{\bar{k}}) > u_{i_{\bar{k}}}(x^{\bar{k}-1})$ for at least one $\bar{k}$.

The vice versa holds whenever G is finite. *(but $\ell = 4$ is not enough)*

Potential games: computation of equilibria

Maxima and equilibria

Suppose P is an ordinal potential for G. Then, any maximum point of P over S is a Nash equilibrium.

base of the proof

$$x^* \in \arg\max\{P(x) \ : \ x \in S\} \Longrightarrow x_i^* \in \arg\max\{P(x_i, x_{-i}^*) \ : \ x_i \in S_i\}$$

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not all Nash equilibria maximize exact/ordinal potentials

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not all Nash equilibria maximize exact/ordinal potentials

asymmetric battle of sexes

I/II	①	②
①	(3,2)	(0,0)
②	(0,0)	(1,2)

its exact potential

I/II	①	②
①	3	1
②	0	2

(②, ②) is a Nash equilibrium not maximizing the potential

Potential games: existence and computation of equilibria

existence of maximizers of P over S $\implies$ existence of equilibria

Potential games: existence and computation of equilibria

existence of maximizers of P over S $\implies$ existence of equilibria

existence for continuous games

Suppose $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is an exact potential game.
If u_i is continuous and $S_i \subseteq \mathbb{R}^{m_i}$ is compact for any $i \in N$, then G has at least one Nash equilibrium. *(false for ordinal potential games)*

Potential games: existence and computation of equilibria

existence of maximizers of P over S $\implies$ existence of equilibria

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Suppose $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is an exact potential game.
If u_i is continuous and $S_i \subseteq \mathbb{R}^{m_i}$ is compact for any $i \in N$, then G has at least one Nash equilibrium. *(false for ordinal potential games)*

existence for finite games

Any finite ordinal potential game has at least one Nash equilibrium.

Potential games: existence and computation of equilibria

existence of maximizers of P over S $\Longrightarrow$ existence of equilibria

existence for continuous games

Suppose $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is an exact potential game.
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existence for finite games

Any finite ordinal potential game has at least one Nash equilibrium.

Convergence of the asynchronous algorithm

Suppose that G is a finite ordinal potential game. Then, the asynchronous distributed algorithm stops at a Nash equilibrium (after a finite number of iterations).

(the value of the potential is improved at each iteration)