Algorithmic game theory

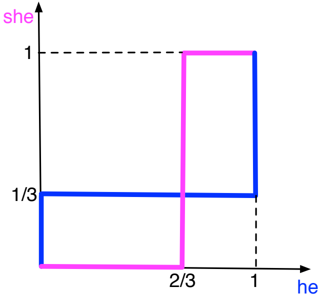
Laurea Magistrale in Computer Science
2024/25

Lecture 7

Mixed equilibria in the battle of sexes

		q	$1-q$
	he/she	football	dancing
p	football	(2,1)	(0,0)
$1-p$	dancing	(0,0)	(1,2)

$$R_{she}(p) = \begin{cases} 1 & \text{if } p > 2/3 \\ [0,1] & \text{if } p = 2/3 \\ 0 & \text{if } p < 2/3 \end{cases}$$



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Nash equilibria are mutual best responses

$$(p^*, q^*) = (1,1), (0,0), (2/3, 1/3)$$

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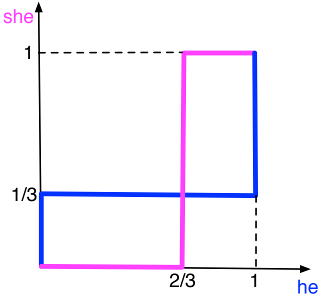

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Nash equilibria are mutual best responses

$$(p^*, q^*) = (1,1), (0,0), (2/3, 1/3)$$

- the best response to a pure strategy is the same pure strategy
- both pure strategies are best responses to the equilibrium mixed strategy

Mixed strategy equilibria as a combinatorial problem

Theorem

In a finite game $(N, (S_i)_{i\in N}, (u_i)_{i\in N})$ *a (mixed) strategy profile* $\sigma^* \in \Delta_S$ *is a Nash equilibrium in mixed strategies if and only if every pure strategy* $x_i \in S_i$ *such that* $\sigma_i^*(x_i) > 0$ *is a best response to* σ_{-i}^* *for each player* $i \in N$.

Corollary

Let $\sigma^* \in \Delta_S$ *be a Nash equilibrium in mixed strategies of a finite game. Every pure strategy* $x_i \in S_i$ *such that* $\sigma_i^*(x_i) > 0$ *yields player* i *the same payoff (provided the other players choose* σ_{-i}^**).*

$\{x_i \in S_i \ : \ \sigma_i(x_i) > 0\}$ support of $\sigma_i \in \Delta(S_i)$

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Corollary

Let $\sigma^* \in \Delta_S$ be a Nash equilibrium in mixed strategies of a finite game. Every pure strategy $x_i \in S_i$ such that $\sigma_i^*(x_i) > 0$ yields player i the same payoff (provided the other players choose σ_{-i}^*).

$$\{x_i \in S_i \ : \ \sigma_i(x_i) > 0\} \text{ support of } \sigma_i \in \Delta(S_i)$$

finding equilibria $\equiv$ finding a suitable support for each player

$$\begin{array}{l} \text{- choose `supports' (subsets of strategies)} \\ \left.\begin{array}{l} \text{- assign probabilities inside the supports} \\ \text{- check pure strategies entail the same payoff} \end{array}\right\} \longrightarrow \text{ system of equations} \end{array}$$

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Let $\sigma^* \in \Delta_S$ *be a Nash equilibrium in mixed strategies of a finite game. Every pure strategy* $x_i \in S_i$ *such that* $\sigma^*_i(x_i) > 0$ *yields player* i *the same payoff (provided the other players choose* σ^*_{-i}*).*

$$\{x_i \in S_i \ : \ \sigma_i(x_i) > 0\} \text{ support of } \sigma_i \in \Delta(S_i)$$

finding equilibria $\equiv$ finding a suitable support for each player

- choose 'supports' (subsets of strategies)

$$\left.\begin{array}{l}\text{- assign probabilities inside the supports}\\ \text{- check pure strategies entail the same payoff}\end{array}\right\} \longrightarrow \text{system of equations}$$

- check pure strategies are indeed best responses $\longrightarrow$ system of inequalities

Looking for suitable supports

I/II	①	②	③	④
①	(4,3)	(7,4)	(5,2)	(3,4)
②	(5,5)	(5,7)	(2,1)	(2,5)
③	(3,4)	(4,2)	(5,5)	(6,3)

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choose supports: {①, ③} {②, ③}

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choose supports: {①, ③} {②, ③}

assign probabilities: $p, 1-p$ $q, 1-q$ $(\sigma_1 = (p, 0, 1-p), \sigma_2 = (0, q, 1-q))$ $[p, q > 0]$)

Looking for suitable supports

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choose supports: $\{①, ③\}$ $\{②, ③\}$

assign probabilities: $p, 1-p$ $q, 1-q$ $(\sigma_1 = (p, 0, 1-p), \sigma_2 = (0, q, 1-q))$ $[p, q > 0]$)

same payoffs: $u_1(①, \sigma_2) = u_1(③, \sigma_2)$:

Looking for suitable supports

I/II	①	②	③	④
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assign probabilities: $p, 1-p$ $q, 1-q$ $(\sigma_1 = (p, 0, 1-p), \sigma_2 = (0, q, 1-q))$ $[p, q > 0])$

same payoffs: $u_1(①, \sigma_2) = u_1(③, \sigma_2)$: $7q + 5(1-q) = 4q + 5(1-q) \longrightarrow q = 0$ failure

Looking for suitable supports

I/II	①	②	③	④
①	(4,3)	(7,4)	(5,2)	(3,4)
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choose supports: {①, ③} {③}

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choose supports: $\{①, ③\}$ $\{③\}$

assign probabilities: $p, 1 - p$ 1 $(\sigma_1 = (p, 0, 1 - p), \sigma_2 = (0, 0, 1))$ $[p > 0]$)

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same payoffs: u_1(①, σ_2) = u_1(③, σ_2): 5=5

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same payoffs: u_1(①, σ_2) $=$ u_1(③, σ_2): 5=5

best responses:

Looking for suitable supports

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choose supports: {①, ③} {③}

assign probabilities: $p, 1-p$ 1 ($\sigma_1 = (p, 0, 1-p)$, $\sigma_2 = (0, 0, 1)$) [$p > 0$])

same payoffs: u_1(①, σ_2) $=$ u_1(③, σ_2): 5=5

best responses: ① and ③ are best responses to $\sigma_2 \equiv$ ③

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same payoffs: u_1(①, σ_2) = u_1(③, σ_2): 5=5

best responses: ① and ③ are best responses to $\sigma_2 \equiv$ ③

③ is a best response to σ_1

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same payoffs: $u_1(①, \sigma_2) = u_1(③, \sigma_2)$: 5=5

best responses: ① and ③ are best responses to $\sigma_2 \equiv$ ③

③ is a best response to σ_1: $2p + 5(1-p) \geq \begin{cases} 3p + 4(1-p) & ① \\ 4p + 2(1-p) & ② \\ 4p + 3(1-p) & ④ \end{cases} \longleftrightarrow 0 \leq p \leq 1/2$

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③ is a best response to σ_1:

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$\{((p, 0, 1-p), (0, 0, 1)) \,:\, 0 \leq p \leq 1/2\}$ are Nash equilibria in mixed strategies

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$\{((p, 0, 1-p), (0, 0, 1))\ :\ 0 \leq p \leq 1/2\}$ are Nash equilibria in mixed strategies

exercise: try supports {①, ③} {①, ③, ④}

Mathematical background: compactness

Definition

A set $S \subseteq \mathbb{R}^m$ is

(i) closed if the limit of any sequence of points $x^k \in S$ belongs to S, i.e.,

$$x^k \to x \implies x \in S$$

(ii) bounded if there exists $M > 0$ such that $S \subseteq \{x \in \mathbb{R}^m \ : \|x\| \leq M\}$

$(\|x\|_2 = \sqrt{x_1^2 + \cdots + x_m^2}$ is the Euclidean norm)

(iii) compact if it is closed and bounded

Extreme value theorem (Bolzano, Weierstrass): if $S \subseteq \mathbb{R}^m$ is compact, any continuous function $f : S \to \mathbb{R}$ has at least one maximum (minimum) point over S

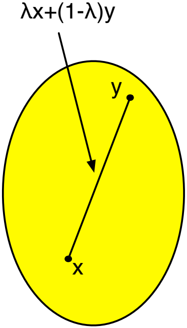
(any sequence in a compact set admits a convergent subsequence)

Mathematical background: convexity for sets

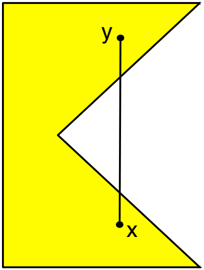
Definition

$S \subseteq \mathbb{R}^m$ is a convex set if

$$x, y \in S,\ \ \lambda \in [0,1] \implies \lambda x + (1-\lambda) y \in S$$



convex



nonconvex

Existence of Nash equilibria

Theorem (Nikaido-Isoda 1955)

Let $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic game. If any $i \in N$ satisfies

(i) $S_i \subseteq \mathbb{R}^{m_i}$ is convex and compact

(ii) u_i is continuous

(iii) the set of best responses $R_i(x_{-i})$ is convex for all $x_{-i} \in S_{-i}$

then the game has at least one Nash equilibrium.

the proof relies on Kakutani's fixed point theorem (1941):

$$x^* \text{ Nash equilibrium } \iff x^* \in R(x^*) = R_1(x^*_{-1}) \times \cdots \times R_n(x^*_{-n})$$

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the proof relies on Kakutani's fixed point theorem (1941):

x^* Nash equilibrium $\iff x^*\in R(x^*)=R_1(x^*_{-1})\times\cdots\times R_n(x^*_{-n})$

- $|R_i(x_{-i})|=1\Longrightarrow R_i(x_{-i})$ convex (uniqueness $\equiv$ Nash 1951)

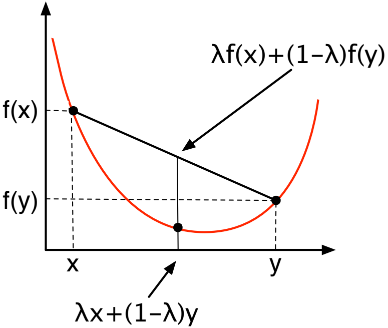
Mathematical background: convexity for functions

Definition

Let $S \subseteq \mathbb{R}^m$ be convex. $f : \mathbb{R}^m \to \mathbb{R}$ is a convex function on S if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x, y \in S$, $\lambda \in [0,1]$.



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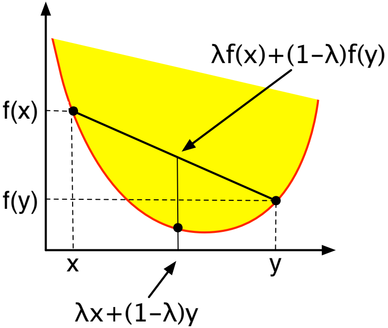

Mathematical background: convexity for functions

Definition

Let $S \subseteq \mathbb{R}^m$ be convex. $f : \mathbb{R}^m \to \mathbb{R}$ is a convex function on S if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

holds for all $x, y \in S,\ \lambda \in [0, 1]$.

Proposition

Let $S \subseteq \mathbb{R}^m$ be convex. $f : \mathbb{R}^m \to \mathbb{R}$ is a convex function on S if and only if (the restriction of) its epigraph (to S), namely,

$$epi_S(f) = \{(x, t) \in S \times \mathbb{R}\ :\ t \geq f(x)\}$$

is a convex set in $\mathbb{R}^{m+1}$.

Mathematical background: convexity for functions

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Let $S \subseteq \mathbb{R}^m$ be convex. $f : \mathbb{R}^m \to \mathbb{R}$ is a convex function on S if

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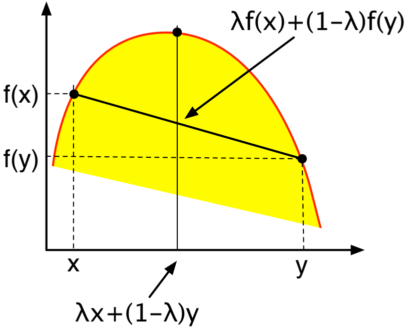
holds for all $x, y \in S,\ \lambda \in [0,1]$.

$f : \mathbb{R}^m \to \mathbb{R}$ is a concave function on S if $-f$ is a convex function on S



Existence of Nash equilibria: finite games

Theorem (Nikaido-Isoda 1955)

Let $(N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a strategic game. If any $i \in N$ satisfies

(i) $S_i \subseteq \mathbb{R}^{m_i}$ *is convex and compact*

(ii) u_i *is continuous*

(iii) the set of best responses $R_i(x_{-i})$ *is convex for all* $x_{-i} \in S_{-i}$

then the game has at least one Nash equilibrium.

- $u_i(\cdot, x_{-i})\colon x_i \longmapsto u_i(x_i, x_{-i})$ concave + S_i convex $\Longrightarrow R_i(x_{-i})$ convex

finite game in mixed strategies: $\begin{cases} u_i(\cdot, x_{-i}) \text{ linear} \\ S_i = \Delta_{m_i} \text{ convex and compact} \end{cases}$

Corollary

Every finite game has at least one Nash equilibrium in mixed strategies.

Existence of Nash equilibria: two player zero-sum games

Minimax theorem (von Neumann 1928)

Let $(\{1,2\},\{S_1,S_2\}),u)$ *be a two player zero-sum game. If*

(i) $S_i \subseteq \mathbb{R}^{m_i}$ *is convex and compact* $(i=1,2)$

(ii) u *is continuous*

(iii) $u(\cdot,x_2) : x_1 \longmapsto u(x_1,x_2)$ *is concave for all* $x_2 \in S_2$

(iv) $u(x_1,\cdot) : x_2 \longmapsto u(x_1,x_2)$ *is convex for all* $x_1 \in S_1$

then

$$\max_{x_1\in S_1}\ \min_{x_2\in S_2} u(x_1,x_2) = \min_{x_2\in S_2}\ \max_{x_1\in S_1} u(x_1,x_2).$$

Hence, the game has at least one Nash equilibrium.

minimax equality ⟶ security/minimax strategies ⟷ Nash equilibrium

convexity/concavity can be replaced by quasiconvexity/concavity

Learning a game

Can equilibria be learnt?

Learning a game

Can equilibria be learnt?

players/agents:

- choose their strategies
- observe the state of the game/system
- update strategies if profitable
- observe the new state of the game/system

....

$$\downarrow$$

dynamics

(another view of the basic idea in Cournot's approach to duopoly)

Learning a Cournot duopoly

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
④	(12,0)	(8,2)	(4,2)	(0,0)	(-4,-4)	(-8,-10)	(-12,-18)
⑤	(10,0)	(5,1)	(0,0)	(-5,-3)	(-10,-8)	(-15,-15)	(-15,-18)
⑥	(6,0)	(0,0)	(-6,-2)	(-12,-6)	(-18,-12)	(-18,-15)	(-18,-18)

Learning a Cournot duopoly

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
④	(12,0)	(8,2)	(4,2)	(0,0)	(-4,-4)	(-8,-10)	(-12,-18)
⑤	(10,0)	(5,1)	(0,0)	(-5,-3)	(-10,-8)	(-15,-15)	(-15,-18)
⑥	(6,0)	(0,0)	(-6,-2)	(-12,-6)	(-18,-12)	(-18,-15)	(-18,-18)

initial state: (②,⑤)

Learning a Cournot duopoly

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
④	(12,0)	(8,2)	(4,2)	(0,0)	(-4,-4)	(-8,-10)	(-12,-18)
⑤	(10,0)	(5,1)	(0,0)	(-5,-3)	(-10,-8)	(-15,-15)	(-15,-18)
⑥	(6,0)	(0,0)	(-6,-2)	(-12,-6)	(-18,-12)	(-18,-15)	(-18,-18)

initial state: (②,⑤)

profitable updates:

Learning a Cournot duopoly

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
④	(12,0)	(8,2)	(4,2)	(0,0)	(-4,-4)	(-8,-10)	(-12,-18)
⑤	(10,0)	(5,1)	(0,0)	(-5,-3)	(-10,-8)	(-15,-15)	(-15,-18)
⑥	(6,0)	(0,0)	(-6,-2)	(-12,-6)	(-18,-12)	(-18,-15)	(-18,-18)

initial state: (②,⑤)

profitable updates: ① is a best response to ⑤, ② to ②

Learning a Cournot duopoly

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
④	(12,0)	(8,2)	(4,2)	(0,0)	(-4,-4)	(-8,-10)	(-12,-18)
⑤	(10,0)	(5,1)	(0,0)	(-5,-3)	(-10,-8)	(-15,-15)	(-15,-18)
⑥	(6,0)	(0,0)	(-6,-2)	(-12,-6)	(-18,-12)	(-18,-15)	(-18,-18)

initial state: (②,⑤)

profitable updates: ① is a best response to ⑤, ② to ②

new state: (①,②)

Learning a Cournot duopoly

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
④	(12,0)	(8,2)	(4,2)	(0,0)	(-4,-4)	(-8,-10)	(-12,-18)
⑤	(10,0)	(5,1)	(0,0)	(-5,-3)	(-10,-8)	(-15,-15)	(-15,-18)
⑥	(6,0)	(0,0)	(-6,-2)	(-12,-6)	(-18,-12)	(-18,-15)	(-18,-18)

initial state: (②,⑤)

profitable updates: ① is a best response to ⑤, ② to ②

new state: (①,②)

profitable updates: ③ is a best response to ②, ③ to ①

new state: (③,③)

Learning a Cournot duopoly

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
④	(12,0)	(8,2)	(4,2)	(0,0)	(-4,-4)	(-8,-10)	(-12,-18)
⑤	(10,0)	(5,1)	(0,0)	(-5,-3)	(-10,-8)	(-15,-15)	(-15,-18)
⑥	(6,0)	(0,0)	(-6,-2)	(-12,-6)	(-18,-12)	(-18,-15)	(-18,-18)

initial state: (②,⑤)

profitable updates: ① is a best response to ⑤, ② to ②

new state: (①,②)

profitable updates: ③ is a best response to ②, ③ to ①

new state: (③,③)

profitable updates: ② is a best response to ③, ② to ③

new state: (②,②)

Learning a Cournot duopoly

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
④	(12,0)	(8,2)	(4,2)	(0,0)	(-4,-4)	(-8,-10)	(-12,-18)
⑤	(10,0)	(5,1)	(0,0)	(-5,-3)	(-10,-8)	(-15,-15)	(-15,-18)
⑥	(6,0)	(0,0)	(-6,-2)	(-12,-6)	(-18,-12)	(-18,-15)	(-18,-18)

initial state: (②,⑤)

profitable updates: ① is a best response to ⑤, ② to ②

new state: (①,②)

profitable updates: ③ is a best response to ②, ③ to ①

new state: (③,③)

profitable updates: ② is a best response to ③, ② to ③

new state: (②,②) ⟶ equilibrium state reached

Learning a Cournot duopoly

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
④	(12,0)	(8,2)	(4,2)	(0,0)	(-4,-4)	(-8,-10)	(-12,-18)
⑤	(10,0)	(5,1)	(0,0)	(-5,-3)	(-10,-8)	(-15,-15)	(-15,-18)
⑥	(6,0)	(0,0)	(-6,-2)	(-12,-6)	(-18,-12)	(-18,-15)	(-18,-18)

initial state: (②,⑤) *utilities (0,0)*

profitable updates: ① is a best response to ⑤, ② to ②

new state: (①,②) *utilities (4,8)*

profitable updates: ③ is a best response to ②, ③ to ①

new state: (③,③) *utilities (3,3)*

profitable updates: ② is a best response to ③, ② to ③

new state: (②,②) ⟶ equilibrium state reached *utilities (6,6)*

Learning a Cournot duopoly

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
④	(12,0)	(8,2)	(4,2)	(0,0)	(-4,-4)	(-8,-10)	(-12,-18)
⑤	(10,0)	(5,1)	(0,0)	(-5,-3)	(-10,-8)	(-15,-15)	(-15,-18)
⑥	(6,0)	(0,0)	(-6,-2)	(-12,-6)	(-18,-12)	(-18,-15)	(-18,-18)

initial state: (②,⑤) *utilities (0,0)*

profitable updates: ① is a best response to ⑤, ② to ② *(second not unique)*

new state: (①,②) *utilities (4,8)*

profitable updates: ③ is a best response to ②, ③ to ① *(first not unique)*

new state: (③,③) *utilities (3,3)*

profitable updates: ② is a best response to ③, ② to ③

new state: (②,②) ⟶ equilibrium state reached *utilities (6,6)*

Best response dynamics

Algorithmic rephrasing of Cournot's basic idea

Synchronous distributed algorithm (Jacobi type algorithm)

1. $x^0 = (x_1^0, \ldots, x_n^0) \in S_1 \times \cdots S_n,\ k = 0$
2. x_i^{k+1} *is a best response to* x_{-i}^k $\left(x_i^{k+1} \in R_i(x_{-i}^k)\right)$ $\Bigg\} \Longrightarrow x^{k+1} \in R(x^k)$
   - *if* $x_i^k \in R_i(x_{-i}^k)$, *select* $x_i^{k+1} = x_i^k$
3. *if* $x^{k+1} = x^k$ *then STOP*
4. $k = k + 1$ *and go back to 2*

All players know the current state (x^k) and reply [simultaneously]

Knowledge of other players' utility functions is not required

Avoiding useless switching may prevent looping

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
④	(12,0)	(8,2)	(4,2)	(0,0)	(-4,-4)	(-8,-10)	(-12,-18)
⑤	(10,0)	(5,1)	(0,0)	(-5,-3)	(-10,-8)	(-15,-15)	(-15,-18)
⑥	(6,0)	(0,0)	(-6,-2)	(-12,-6)	(-18,-12)	(-18,-15)	(-18,-18)

current state: (②,②)

equilibrium state

Avoiding useless switching may prevent looping

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
④	(12,0)	(8,2)	(4,2)	(0,0)	(-4,-4)	(-8,-10)	(-12,-18)
⑤	(10,0)	(5,1)	(0,0)	(-5,-3)	(-10,-8)	(-15,-15)	(-15,-18)
⑥	(6,0)	(0,0)	(-6,-2)	(-12,-6)	(-18,-12)	(-18,-15)	(-18,-18)

current state: (②,②) equilibrium state

possible updates: ③ is a best response to ②, ③ to ②

Avoiding useless switching may prevent looping

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
④	(12,0)	(8,2)	(4,2)	(0,0)	(-4,-4)	(-8,-10)	(-12,-18)
⑤	(10,0)	(5,1)	(0,0)	(-5,-3)	(-10,-8)	(-15,-15)	(-15,-18)
⑥	(6,0)	(0,0)	(-6,-2)	(-12,-6)	(-18,-12)	(-18,-15)	(-18,-18)

current state: (②,②) equilibrium state

possible updates: ③ is a best response to ②, ③ to ②

new state: (③,③)

profitable updates: ② is a best response to ③, ② to ③

new state: (②,②)

Avoiding useless switching may prevent looping

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
④	(12,0)	(8,2)	(4,2)	(0,0)	(-4,-4)	(-8,-10)	(-12,-18)
⑤	(10,0)	(5,1)	(0,0)	(-5,-3)	(-10,-8)	(-15,-15)	(-15,-18)
⑥	(6,0)	(0,0)	(-6,-2)	(-12,-6)	(-18,-12)	(-18,-15)	(-18,-18)

current state: (②,②) equilibrium state

possible updates: ③ is a best response to ②, ③ to ② *useless switch*

new state: (③,③)

profitable updates: ② is a best response to ③, ② to ③

new state: (②,②)

a possibly endless loop between the two states might occur

avoid useless switches: if $x_i^k \in R_i(x_{-i}^k)$, select $x_i^{k+1} = x_i^k$

Synchronous algorithm in finite games

Prisoner's dilemma

I/II	not confess	confess
not confess	(-2,-2)	(-7,0)
confess	(0,-7)	(-5,-5)

$$x^0 = (nc, nc) \longrightarrow x^1 = (c, c), \;\; x^0 = (c, nc) \longrightarrow x^1 = (c, c)$$

Synchronous algorithm in finite games

Prisoner's dilemma

I/II	not confess	confess
not confess	(-2,-2)	(-7,0)
confess	(0,-7)	(-5,-5)

$$x^0 = (nc, nc) \longrightarrow x^1 = (c, c), \;\; x^0 = (c, nc) \longrightarrow x^1 = (c, c)$$

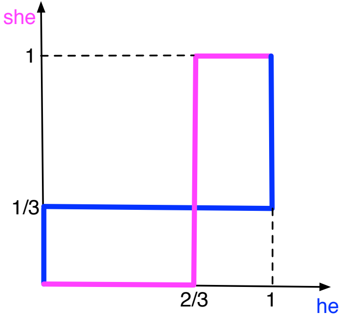
The battle of sexes

he/she	football	dancing
football	(2,1)	(0,0)
dancing	(0,0)	(1,2)

$$x^0 = (f, d) \longrightarrow x^1 = (d, f) \longrightarrow x^2 = (f, d) = x^0$$

the algorithm loops

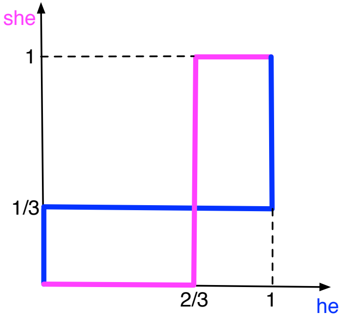
Synchronous algorithm with mixed strategies



$$x^0 = (1/2, 1/2) \longrightarrow x^1 = (1, 0) \longrightarrow x^2 = (0, 1) \longrightarrow x^3 = (1, 0)$$

the algorithm loops

Synchronous algorithm with mixed strategies



$$x^0 = (1/2, 1/2) \longrightarrow x^1 = (1, 0) \longrightarrow x^2 = (0, 1) \longrightarrow x^3 = (1, 0)$$

the algorithm loops

$$\hat{x}^0 = (2/3, 1/3) \longrightarrow \hat{x}^1 = (1/2, 1/2)$$

$$\left.\begin{array}{l} 1/2 \in R_{he}(1/3) \text{ but } 2/3 \in R_{he}(1/3) \\ 1/2 \in R_{she}(2/3) \text{ but } 1/3 \in R_{she}(2/3) \end{array}\right\} \Longrightarrow \hat{x}^0 \to \hat{x}^1 \text{ not allowed}$$