Algorithmic game theory

Laurea Magistrale in Computer Science

2024/25

Lecture 6

Security strategies: formal definition

Worst result for player i while playing strategy $x_i \in S_i$

$$w_i(x_i) = \min\{u_i(x_i, x_{-i}) \ : \ x_{-i} \in S_{-i}\}$$

Playing any

$$\bar{x}_i \in \arg\max\{w_i(x_i) \ : \ x_i \in S_i\},$$

player i gets at least $v_i = \max\{w_i(x_i) \ : \ x_i \in S_i\}$

Definition

Any such $\bar{x}_i$ is called a security strategy for player i

$v_i = \max\{w_i(x_i) \ : \ x_i \in S_i\}$ is called the security level of player i

Security strategies: failure in rock-paper-scissors

Worst result for player i while playing strategy $x_i \in S_i$

$$w_i(x_i) = \min\{u_i(x_i, x_{-i}) \ : \ x_{-i} \in S_{-i}\}$$

Playing any

$$\bar{x}_i \in \arg\max\{w_i(x_i) \ : \ x_i \in S_i\},$$

player i gets at least $v_i = \max\{w_i(x_i) \ : \ x_i \in S_i\}$

Definition

Any such $\bar{x}_i$ is called a security strategy for player i

$v_i = \max\{w_i(x_i) \ : \ x_i \in S_i\}$ is called the security level of player i

I/II	paper	scissors	rock	min
paper	(0,0)	(-1,1)	(1,-1)	-1
scissors	(1,-1)	(0,0)	(-1,1)	-1
rock	(-1,1)	(1,-1)	(0,0)	-1
min	-1	-1	-1	

every strategy is a security strategy

Security strategies: nice outcome for hawk-dove

Worst result for player i while playing strategy $x_i \in S_i$

$$w_i(x_i) = \min\{u_i(x_i, x_{-i}) \,:\, x_{-i} \in S_{-i}\}$$

Playing any

$$\bar{x}_i \in \arg\max\{w_i(x_i) \,:\, x_i \in S_i\},$$

player i gets at least $v_i = \max\{w_i(x_i) \,:\, x_i \in S_i\}$

Definition

Any such $\bar{x}_i$ is called a security strategy for player i

$v_i = \max\{w_i(x_i) \,:\, x_i \in S_i\}$ is called the security level of player i

I/II	hawk	dove	min
hawk	(-2,-2)	(2,0)	-2
dove	(0,2)	(1,1)	0
min	-2	0	

dove and dove are the security strategies

Strictly competitive games with two players

Definition

A two player game is strictly competitive if

$$u_1(x_1, x_2) > u_1(x_1', x_2') \iff u_2(x_1, x_2) < u_2(x_1', x_2')$$
$$u_1(x_1, x_2) < u_1(x_1', x_2') \iff u_2(x_1, x_2) > u_2(x_1', x_2')$$

hold for all pairs of strategy profiles $(x_1, x_2), (x_1', x_2') \in S_1 \times S_2$.

rock-paper-scissors is strictly competitive, hawk-dove is not

Strictly competitive games with two players

Definition

A two player game is strictly competitive if

$$u_1(x_1, x_2) > u_1(x_1', x_2') \iff u_2(x_1, x_2) < u_2(x_1', x_2')$$
$$u_1(x_1, x_2) < u_1(x_1', x_2') \iff u_2(x_1, x_2) > u_2(x_1', x_2')$$

hold for all pairs of strategy profiles $(x_1, x_2), (x_1', x_2') \in S_1 \times S_2$.

rock-paper-scissors is strictly competitive, hawk-dove is not

Definition

A two player game is a zero-sum game if

$$u_1(x_1, x_2) + u_2(x_1, x_2) = 0$$

holds for all strategy profiles $(x_1, x_2) \in S_1 \times S_2$.

Any strictly competitive game $\equiv$ zero-sum game by replacing u_2 by $-u_1$

Two player zero-sum games

Since $u_1 + u_2 \equiv 0$, a unique utility function $u = u_1$ can be considered: player 1 aims at maximizing u, player 2 at maximizing $-u$ (*minimizing* u)

Security level of player 1: $\underline{v} = \max\{\min\{u(x_1, x_2) : x_2 \in S_2\} : x_1 \in S_1\}$

Security level of player 2: $\overline{v} = \min\{\max\{u(x_1, x_2) : x_1 \in S_1\} : x_2 \in S_2\}$

Two player zero-sum games

Since $u_1 + u_2 \equiv 0$, a unique utility function $u = u_1$ can be considered: player 1 aims at maximizing u, player 2 at maximizing $-u$ (*minimizing* u)

Security level of player 1: $\underline{v} = \max\{\min\{u(x_1, x_2) : x_2 \in S_2\} : x_1 \in S_1\}$

Security level of player 2: $\overline{v} = \min\{\max\{u(x_1, x_2) : x_1 \in S_1\} : x_2 \in S_2\}$

Proposition

$$\underline{v} \leq \overline{v}$$

Two player zero-sum games

Since $u_1 + u_2 \equiv 0$, a unique utility function $u = u_1$ can be considered: player 1 aims at maximizing u, player 2 at maximizing $-u$ (*minimizing* u)

Security level of player 1: $\underline{v} = \max\{\min\{u(x_1, x_2) : x_2 \in S_2\} : x_1 \in S_1\}$

Security level of player 2: $\overline{v} = \min\{\max\{u(x_1, x_2) : x_1 \in S_1\} : x_2 \in S_2\}$

Proposition

$$\underline{v} \leq \overline{v}$$

Definition

A two player zero-sum game has a value if $\underline{v} = \overline{v}$

Security strategies are referred to as minimax strategies

rock-paper-scissors does not have a value: $\underline{v} = -1 < 1 = \overline{v}$

Minimax strategies and Nash equilibria

Alternative formulation

(x_1^*, x_2^*) *is a Nash equilibrium if and only if it is a saddle point of* u, *i.e.,*

$$u(x_1, x_2^*) \leq u(x_1^*, x_2^*) \leq u(x_1^*, x_2) \quad \textit{for all } x_1 \in S_1, x_2 \in S_2.$$

Minimax strategies and Nash equilibria

Alternative formulation

(x_1^*, x_2^*) is a Nash equilibrium if and only if it is a saddle point of u, i.e.,

$$u(x_1, x_2^*) \leq u(x_1^*, x_2^*) \leq u(x_1^*, x_2) \quad \textit{for all } x_1 \in S_1, x_2 \in S_2.$$

Nash $\equiv$ minimax

(i) If (x_1^*, x_2^*) is a Nash equilibrium, then the game has a value $u(x_1^*, x_2^*)$ and x_1^* and x_2^* are minimax strategies.

(ii) If the game has a value, then any pair of minimax strategies (x_1^*, x_2^*) is a Nash equilibrium.

existence of Nash equilibria $\equiv$ existence of minimax value

Minimax strategies and Nash equilibria

Alternative formulation

(x_1^*, x_2^*) is a Nash equilibrium if and only if it is a saddle point of u, i.e.,

$$u(x_1, x_2^*) \le u(x_1^*, x_2^*) \le u(x_1^*, x_2) \quad \textit{for all } x_1 \in S_1, x_2 \in S_2.$$

Nash $\equiv$ minimax

(i) If (x_1^*, x_2^*) is a Nash equilibrium, then the game has a value $u(x_1^*, x_2^*)$ and x_1^* and x_2^* are minimax strategies.

(ii) If the game has a value, then any pair of minimax strategies (x_1^*, x_2^*) is a Nash equilibrium.

existence of Nash equilibria $\equiv$ existence of minimax value

Shuffling equilibria

Let (x_1^*, x_2^*) and (x_1°, x_2°) be two Nash equilibria.

(i) $u(x_1^*, x_2^*) = u(x_1^\circ, x_2^\circ)$

(ii) (x_1^*, x_2°) and (x_1°, x_2^*) are also Nash equilibria.

Two player zero-sum games: the finite case

$$S_1 = \{1, ..., m_1\}, \quad S_2 = \{1, ..., m_2\}$$

$a_{k\ell} = u(k, \ell)$ payoff when player 1 plays k and player 2 plays ℓ

Alternative formulation

The strategy profile $(\bar{k}, \bar{\ell})$ is a Nash equilibrium if and only if any k,ℓ satisfy

$$a_{k\bar{\ell}} \leq a_{\bar{k}\bar{\ell}} \leq a_{\bar{k}\ell}$$

I/II	①	②	③	④	
①	4	7	2	3	
②	1	5	2	4	
③	3	4	3	6	

Two player zero-sum games: the finite case

$$S_1 = \{1, ..., m_1\}, \quad S_2 = \{1, ..., m_2\}$$

$a_{k\ell} = u(k, \ell)$ payoff when player 1 plays k and player 2 plays ℓ

Alternative formulation

The strategy profile $(\bar{k}, \bar{\ell})$ is a Nash equilibrium if and only if any k,ℓ satisfy

$$a_{k\bar{\ell}} \leq a_{\bar{k}\bar{\ell}} \leq a_{\bar{k}\ell}$$

I/II	①	②	③	④	min
①	4	7	2	3	
②	1	5	2	4	
③	3	4	3	6	

Two player zero-sum games: the finite case

$$S_1 = \{1, ..., m_1\}, \quad S_2 = \{1, ..., m_2\}$$

$a_{k\ell} = u(k, \ell)$ payoff when player 1 plays k and player 2 plays ℓ

Alternative formulation

The strategy profile $(\bar{k}, \bar{\ell})$ is a Nash equilibrium if and only if any k,ℓ satisfy

$$a_{k\bar{\ell}} \leq a_{\bar{k}\bar{\ell}} \leq a_{\bar{k}\ell}$$

I/II	①	②	③	④	min
①	4	7	2	3	2
②	1	5	2	4	1
③	3	4	3	6	3

Two player zero-sum games: the finite case

$$S_1 = \{1, ..., m_1\}, \quad S_2 = \{1, ..., m_2\}$$

$a_{k\ell} = u(k, \ell)$ payoff when player 1 plays k and player 2 plays ℓ

Alternative formulation

The strategy profile $(\bar{k}, \bar{\ell})$ is a Nash equilibrium if and only if any k,ℓ satisfy

$$a_{k\bar{\ell}} \leq a_{\bar{k}\bar{\ell}} \leq a_{\bar{k}\ell}$$

I/II	①	②	③	④	min
①	4	7	2	3	2
②	1	5	2	4	1
③	3	4	3	6	3

maxmin: $\underline{v} = 3$

Two player zero-sum games: the finite case

$$S_1 = \{1, ..., m_1\}, \quad S_2 = \{1, ..., m_2\}$$

$a_{k\ell} = u(k, \ell)$ payoff when player 1 plays k and player 2 plays ℓ

Alternative formulation

The strategy profile $(\bar{k}, \bar{\ell})$ is a Nash equilibrium if and only if any k,ℓ satisfy

$$a_{k\bar{\ell}} \leq a_{\bar{k}\bar{\ell}} \leq a_{\bar{k}\ell}$$

I/II	①	②	③	④	min
①	4	7	2	3	2
②	1	5	2	4	1
③	3	4	3	6	3
max	4	7	3	6	

maxmin: $\underline{v} = 3$

Two player zero-sum games: the finite case

$$S_1 = \{1, ..., m_1\}, \quad S_2 = \{1, ..., m_2\}$$

$a_{k\ell} = u(k, \ell)$ payoff when player 1 plays k and player 2 plays ℓ

Alternative formulation

The strategy profile $(\bar{k}, \bar{\ell})$ is a Nash equilibrium if and only if any k,ℓ satisfy

$$a_{k\bar{\ell}} \leq a_{\bar{k}\bar{\ell}} \leq a_{\bar{k}\ell}$$

I/II	①	②	③	④	min
①	4	7	2	3	2
②	1	5	2	4	1
③	3	4	3	6	3
max	4	7	3	6	

maxmin: $\underline{v} = 3$

minimax $\overline{v} = 3$

Two player zero-sum games: the finite case

$$S_1 = \{1, \dots, m_1\}, \quad S_2 = \{1, \dots, m_2\}$$

$a_{k\ell} = u(k,\ell)$ payoff when player 1 plays k and player 2 plays ℓ

Alternative formulation

The strategy profile $(\bar{k}, \bar{\ell})$ is a Nash equilibrium if and only if any k,ℓ satisfy

$$a_{k\bar{\ell}} \leq a_{\bar{k}\bar{\ell}} \leq a_{\bar{k}\ell}$$

I/II	①	②	③	④	min
①	4	7	2	3	2
②	1	5	2	4	1
③	3	4	3	6	3
max	4	7	3	6	

maxmin: $\underline{v} = 3$

minimax $\overline{v} = 3$

the game has value 3 and (③,③) is a Nash equilibrium

Two player zero-sum games: the finite case

$$S_1 = \{1, ..., m_1\}, \quad S_2 = \{1, ..., m_2\}$$

$a_{k\ell} = u(k, \ell)$ payoff when player 1 plays k and player 2 plays ℓ

Alternative formulation

The strategy profile $(\bar{k}, \bar{\ell})$ is a Nash equilibrium if and only if any k,ℓ satisfy

$$a_{k\bar{\ell}} \leq a_{\bar{k}\bar{\ell}} \leq a_{\bar{k}\ell}$$

I/II	①	②	③	④	min
①	4	7	2	3	2
②	1	5	2	4	1
③	3	4	3	6	3
max	4	7	3	6	

maxmin: $\underline{v} = 3$

minimax $\overline{v} = 3$

the game has value 3 and (③,③) is a Nash equilibrium

computational complexity:
compute maxima/minima in $(m_1 + m_2 + 2)$ arrays

No equilibria in rock-paper-scissors

I/II	paper	scissors	rock
paper	0	-1	1
scissors	1	0	-1
rock	-1	1	0

No Nash equilibria exist

No equilibria in rock-paper-scissors: enter randomness

I/II	paper	scissors	rock
paper	0	-1	1
scissors	1	0	-1
rock	-1	1	0

No Nash equilibria exist

what about repeating the game over and over?

rock and scissors half of times each (probabilities 1/2) never play paper (probability 0)

[do people really randomise?]

No equilibria in rock-paper-scissors: enter randomness

I/II	paper	scissors	rock
paper	0	-1	1
scissors	1	0	-1
rock	-1	1	0

No Nash equilibria exist

what about repeating the game over and over?

rock and scissors half of times each (probabilities 1/2) never play paper (probability 0)

[do people really randomise?]

what will the evolution/dynamics of the repeated game be?

Mixed strategies

A mixed strategy is a probability measure over the set of (pure) strategies

Mixed strategies: rock-paper-scissors

A mixed strategy is a probability measure over the set of (pure) strategies

mixed strategies: $p, q \in \mathbb{R}^3_+$ such that $p_1 + p_2 + p_3 = 1$ and $q_1 + q_2 + q_3 = 1$

		q_1	q_2	q_3
	I/II	paper	scissors	rock
p_1	paper	0	-1	1
p_2	scissors	1	0	-1
p_3	rock	-1	1	0

Mixed strategies

A mixed strategy is a probability measure over the set of (pure) strategies

mixed strategies: $p, q \in \mathbb{R}^3_+$ such that $p_1 + p_2 + p_3 = 1$ and $q_1 + q_2 + q_3 = 1$

		q_1	q_2	q_3
	I/II	paper	scissors	rock
p_1	paper	0	-1	1
p_2	scissors	$p_2 q_1$	0	-1
p_3	rock	-1	1	0

Mixed strategies

A mixed strategy is a probability measure over the set of (pure) strategies

mixed strategies: $p, q \in \mathbb{R}^3_+$ such that $p_1 + p_2 + p_3 = 1$ and $q_1 + q_2 + q_3 = 1$

		q_1	q_2	q_3
	I/II	paper	scissors	rock
p_1	paper	0	-$p_1 q_2$	$p_1 q_3$
p_2	scissors	$p_2 q_1$	0	-$p_2 q_3$
p_3	rock	-$p_3 q_1$	$p_3 q_2$	0

Mixed strategies

A mixed strategy is a probability measure over the set of (pure) strategies

mixed strategies: $p, q \in \mathbb{R}^3_+$ such that $p_1 + p_2 + p_3 = 1$ and $q_1 + q_2 + q_3 = 1$

		q_1	q_2	q_3
	I/II	paper	scissors	rock
p_1	paper	0	-$p_1 q_2$	$p_1 q_3$
p_2	scissors	$p_2 q_1$	0	-$p_2 q_3$
p_3	rock	-$p_3 q_1$	$p_3 q_2$	0

expected utility: $h(p, q) = -p_1 q_2 + p_1 q_3 + p_2 q_1 - p_2 q_3 - p_3 q_1 + p_3 q_2$

Mixed strategies

A mixed strategy is a probability measure over the set of (pure) strategies

mixed strategies: $p, q \in \mathbb{R}^3_+$ such that $p_1 + p_2 + p_3 = 1$ and $q_1 + q_2 + q_3 = 1$

	I/II	paper (q_1)	scissors (q_2)	rock (q_3)
p_1	paper	0	$-p_1 q_2$	$p_1 q_3$
p_2	scissors	$p_2 q_1$	0	$-p_2 q_3$
p_3	rock	$-p_3 q_1$	$p_3 q_2$	0

expected utility: $h(p, q) = -p_1 q_2 + p_1 q_3 + p_2 q_1 - p_2 q_3 - p_3 q_1 + p_3 q_2$

extension of the game:

- players: I, II
- strategy sets: Δ_3, Δ_3 (unitary simplices: $\Delta_m = \{v \in \mathbb{R}^m_+ : v_1 + \cdots + v_m = 1\}$)
- utility functions: h, $-h$

Mixed strategies: the finite case

$G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ finite (strategic) game: all the sets S_i are finite

A mixed strategy σ_i assigns a probability $\sigma_i(x_i)$ to each strategy $x_i \in S_i$

$$\Delta(S_i) = \{\sigma_i : S_i \to [0,1] \; : \; \sum_{x_i \in S_i} \sigma_i(x_i) = 1 \; \}$$

(mixed strategy $\equiv$ vector of $m_i = |S_i|$ nonnegative components whose sum is 1)

pure strategy $x_i \in S_i \; \equiv \; \sigma_{x_i} \in \Delta(S_i)$ such that $\sigma_{x_i}(x_i) = 1$

$$h_i(\sigma) = \sum_{x \in S} u_i(x_1, \ldots, x_n)\, \sigma_1(x_1) \ldots \sigma_n(x_n)$$

Mixed strategies: the finite case

$G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ finite (strategic) game: all the sets S_i are finite

A mixed strategy σ_i assigns a probability $\sigma_i(x_i)$ to each strategy $x_i \in S_i$

$$\Delta(S_i) = \{\sigma_i : S_i \to [0,1] \ : \ \sum_{x_i \in S_i} \sigma_i(x_i) = 1 \ \}$$

(mixed strategy $\equiv$ vector of $m_i = |S_i|$ nonnegative components whose sum is 1)

pure strategy $x_i \in S_i \ \equiv \ \sigma_{x_i} \in \Delta(S_i)$ such that $\sigma_{x_i}(x_i) = 1$

$$h_i(\sigma) = \sum_{x \in S} u_i(x_1, \ldots, x_n)\, \sigma_i(x_i) \sigma_{-i}(x_{-i})$$

$$\text{with } \sigma_{-i}(x_{-i}) = \prod_{j \neq i} \sigma_j(x_j)$$

Nash equilibria in mixed strategies

Mixed extension

Let $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ *be a finite (strategic) game. Then, the game*

$$G_{me} = (N, \{\Delta(S_i)_i\}_{i\in N}, \{h_i\}_{i\in N})$$

is called the extension of G *to mixed strategies.*

(all strategy profiles for G are included in G_{me})

Nash equilibria in mixed strategies = Nash equilibria of the mixed extension

Nash equilibria in mixed strategies

Mixed extension

Let $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ *be a finite (strategic) game. Then, the game*

$$G_{me} = (N, \{\Delta(S_i)_i\}_{i\in N}, \{h_i\}_{i\in N})$$

is called the extension of G *to mixed strategies.*

(all strategy profiles for G are included in G_{me})

Nash equilibria in mixed strategies = Nash equilibria of the mixed extension

How are Nash equilibria for G and G_{me} related?

Proposition

If a strategy profile $x^* = (x_1^*, ..., x_n^*) \in S$ *is a Nash equilibrium for* G*, then* $\sigma_{x^*} = (\sigma_{x_1^*}, ..., \sigma_{x_n^*}) \in \Delta_S$ *is a Nash equilibrium for* G_{me}*.*

$(\Delta_S = \Delta(S_1) \times \cdots \times \Delta(S_n))$

Mixed equilibria in rock-paper-scissors

		q_1	q_2	q_3
	I/II	paper	scissors	rock
p_1	paper	0	-1	1
p_2	scissors	1	0	-1
p_3	rock	-1	1	0

$$h(p, q) = p^T A q = q_1(p_2 - p_3) + q_2(p_3 - p_1) + q_3(p_1 - p_2)$$

Mixed equilibria in rock-paper-scissors

		q_1	q_2	q_3
	I/II	paper	scissors	rock
p_1	paper	0	-1	1
p_2	scissors	1	0	-1
p_3	rock	-1	1	0

$$h(p,q) = p^T A q = q_1(p_2 - p_3) + q_2(p_3 - p_1) + q_3(p_1 - p_2)$$

security/minimax strategy for player I

$$w_1(p) = \min\{h(p,q) \,:\, q \in \Delta_3\} = \min\{(p_2 - p_3), (p_3 - p_1), (p_1 - p_2)\}$$

Mixed equilibria in rock-paper-scissors

		q_1	q_2	q_3
	I/II	paper	scissors	rock
p_1	paper	0	-1	1
p_2	scissors	1	0	-1
p_3	rock	-1	1	0

$$h(p,q) = p^T A q = q_1(p_2 - p_3) + q_2(p_3 - p_1) + q_3(p_1 - p_2)$$

security/minimax strategy for player I

$$w_1(p) = \min\{h(p,q) \,:\, q \in \Delta_3\} = \min\{(p_2 - p_3), (p_3 - p_1), (p_1 - p_2)\}$$

$$\big((p_2 - p_3) + (p_3 - p_1) + (p_1 - p_2) = 0 \implies w_1(p) \leq 0 \text{ for all } p\big)$$

Mixed equilibria in rock-paper-scissors

	I/II	paper (q_1)	scissors (q_2)	rock (q_3)
p_1	paper	0	-1	1
p_2	scissors	1	0	-1
p_3	rock	-1	1	0

$$h(p,q) = p^T A q = q_1(p_2 - p_3) + q_2(p_3 - p_1) + q_3(p_1 - p_2)$$

security/minimax strategy for player I

$$w_1(p) = \min\{h(p,q) \,:\, q \in \Delta_3\} = \min\{(p_2 - p_3), (p_3 - p_1), (p_1 - p_2)\}$$

$$\arg\max\{w_1(p) \,:\, p \in \Delta_3\} = \{(1/3, 1/3, 1/3)\} \longrightarrow \underline{v} = 0$$

Mixed equilibria in rock-paper-scissors

		q_1	q_2	q_3
	I/II	paper	scissors	rock
p_1	paper	0	-1	1
p_2	scissors	1	0	-1
p_3	rock	-1	1	0

$$h(p,q) = p^T A q = q_1(p_2 - p_3) + q_2(p_3 - p_1) + q_3(p_1 - p_2)$$

security/minimax strategy for player I

$$w_1(p) = \min\{h(p,q) \,:\, q \in \Delta_3\} = \min\{(p_2 - p_3), (p_3 - p_1), (p_1 - p_2)\}$$

$$\arg\max\{w_1(p) \,:\, p \in \Delta_3\} = \{(1/3, 1/3, 1/3)\} \;\longrightarrow\; \underline{v} = 0$$

Same for player II

Mixed strategy equilibrium: each pure strategies is played with probability 1/3

Security/minimax strategies and linear programming

$w_1(p) = \min\{(p_2 - p_3), (p_3 - p_1), (p_1 - p_2)\}$ $\qquad$ $w_2(q) = \max\{(q_3 - q_2), (q_1 - q_3), (q_2 - q_1)\}$

$\max\{w_1(p) \; : \; p \in \Delta_3\}$ reads

$$\begin{aligned} \max \quad & u \\ & \text{subject to} \\ & u \leq (p_2 - p_3) \\ & u \leq (p_3 - p_1) \\ & u \leq (p_1 - p_2) \\ & p_1 + p_2 + p_3 = 1 \\ & p_1 \geq 0,\; p_2 \geq 0,\; p_3 \geq 0 \end{aligned}$$

$\min\{w_2(q) \; : \; q \in \Delta_3\}$ reads

$$\begin{aligned} \min \quad & v \\ & \text{subject to} \\ & v \geq (q_3 - q_2) \\ & v \geq (q_1 - q_3) \\ & v \geq (q_2 - q_1) \\ & q_1 + q_2 + q_3 = 1 \\ & q_1 \geq 0,\; q_2 \geq 0,\; q_3 \geq 0 \end{aligned}$$

dual linear programming problems

Security/minimax strategies and linear programming

$w_1(p) = \min\{(p_2 - p_3), (p_3 - p_1), (p_1 - p_2)\}$

$w_2(q) = \max\{(q_3 - q_2), (q_1 - q_3), (q_2 - q_1)\}$

$\max\{w_1(p) \ : \ p \in \Delta_3\}$ reads

$$\begin{array}{ll} \max & u \\ & \text{subject to} \\ & u \leq (p_2 - p_3) \\ & u \leq (p_3 - p_1) \\ & u \leq (p_1 - p_2) \\ & p_1 + p_2 + p_3 = 1 \\ & p_1 \geq 0, \ p_2 \geq 0, \ p_3 \geq 0 \end{array}$$

$\min\{w_2(q) \ : \ q \in \Delta_3\}$ reads

$$\begin{array}{ll} \min & v \\ & \text{subject to} \\ & v \geq (q_3 - q_2) \\ & v \geq (q_1 - q_3) \\ & v \geq (q_2 - q_1) \\ & q_1 + q_2 + q_3 = 1 \\ & q_1 \geq 0, \ q_2 \geq 0, \ q_3 \geq 0 \end{array}$$

dual linear programming problems

strong duality theorem in linear programming

$\Downarrow$

existence of a Nash equilibrium in mixed strategies

Finite two player zero-sum games and linear programming

finite two player zero-sum game $\longrightarrow$ $A \in \mathbb{R}^{m_1 \times m_2}$

$h(p, q) = p^T A q$ utility/payoff function

Finite two player zero-sum games and linear programming

finite two player zero-sum game $\longrightarrow$ $A \in \mathbb{R}^{m_1 \times m_2}$

$h(p, q) = p^T A q$ utility/payoff function

$$w_1(p) = \min\{(A^T p)_i \ : \ i = 1, \ldots m_2\} \qquad w_2(q) = \max\{(Aq)_i \ : \ i = 1, \ldots m_1\}$$

Finite two player zero-sum games and linear programming

finite two player zero-sum game $\longrightarrow$ $A \in \mathbb{R}^{m_1 \times m_2}$

$h(p,q) = p^T A q$ utility/payoff function

$$w_1(p) = \min\{(A^T p)_i \ : \ i = 1, \dots m_2\} \qquad w_2(q) = \max\{(Aq)_i \ : \ i = 1, \dots m_1\}$$

$\max\{w_1(p) \ : \ p \in \Delta_3\}$ reads

$$\begin{array}{ll} \max & u \\ & \text{subject to} \\ & u \le (A^T p)_i \quad i = 1, \dots m_2 \\ & p_1 + \dots + p_{m_1} = 1 \\ & p_i \ge 0 \quad i = 1, \dots m_1 \end{array}$$

$\min\{w_2(q) \ : \ q \in \Delta_3\}$ reads

$$\begin{array}{ll} \min & v \\ & \text{subject to} \\ & v \ge (Aq)_i \quad i = 1, \dots m_1 \\ & q_1 + \dots + q_{m_2} = 1 \\ & q_i \ge 0 \quad i = 1, \dots m_2 \end{array}$$

dual linear programming problems

Finite two player zero-sum games and linear programming

finite two player zero-sum game $\longrightarrow$ $A \in \mathbb{R}^{m_1 \times m_2}$

$h(p, q) = p^T A q$ utility/payoff function

$$w_1(p) = \min\{(A^T p)_i \ : \ i = 1, \dots m_2\} \qquad w_2(q) = \max\{(Aq)_i \ : \ i = 1, \dots m_1\}$$

$\max\{w_1(p) \ : \ p \in \Delta_3\}$ reads

$$\begin{array}{ll} \max & u \\ & \text{subject to} \\ & u \le (A^T p)_i \quad i = 1, \dots m_2 \\ & p_1 + \cdots + p_{m_1} = 1 \\ & p_i \ge 0 \quad i = 1, \dots m_1 \end{array}$$

$\min\{w_2(q) \ : \ q \in \Delta_3\}$ reads

$$\begin{array}{ll} \min & v \\ & \text{subject to} \\ & v \ge (Aq)_i \quad i = 1, \dots m_1 \\ & q_1 + \cdots + q_{m_2} = 1 \\ & q_i \ge 0 \quad i = 1, \dots m_2 \end{array}$$

dual linear programming problems

existence of a Nash equilibrium in mixed strategies

$\equiv$

strong duality theorem in linear programming

algorithms for linear programming to compute Nash equilibria

Mixed equilibria in the battle of sexes

Any $p, q \in [0,1]$ identify a pair of mixed strategies

(football with probability p vs q, movies with probability $1-p$ vs $1-q$)

		q	$1-q$
	he/she	football	dancing
p	football	(2,1)	(0,0)
$1-p$	dancing	(0,0)	(1,2)

Mixed equilibria in the battle of sexes

Any $p, q \in [0,1]$ identify a pair of mixed strategies

(football with probability p vs q, movies with probability $1-p$ vs $1-q$)

		q	$1-q$
	he/she	football	dancing
p	football	(2,1)	(0,0)
$1-p$	dancing	(0,0)	(1,2)

$$h_{he}(p,q) = 2pq + (1-p)(1-q), \quad h_{she}(p,q) = pq + 2(1-p)(1-q)$$

Mixed equilibria in the battle of sexes

Any $p, q \in [0,1]$ identify a pair of mixed strategies

(football with probability p vs q, movies with probability $1-p$ vs $1-q$)

		q	$1-q$
	he/she	football	dancing
p	football	(2,1)	(0,0)
$1-p$	dancing	(0,0)	(1,2)

$$h_{he}(p,q) = 2pq + (1-p)(1-q), \quad h_{she}(p,q) = pq + 2(1-p)(1-q)$$

Best responses for her

$$h_{she}(p,q) = q(p) + (1-q)\big(2(1-p)\big) \leq \max\{p, 2(1-p)\} \quad (q \in [0,1])$$

Mixed equilibria in the battle of sexes

Any $p, q \in [0,1]$ identify a pair of mixed strategies

(football with probability p vs q, movies with probability $1-p$ vs $1-q$)

	he/she	football (q)	dancing ($1-q$)
p	football	(2,1)	(0,0)
$1-p$	dancing	(0,0)	(1,2)

$$h_{he}(p,q) = 2pq + (1-p)(1-q), \quad h_{she}(p,q) = pq + 2(1-p)(1-q)$$

Best responses for her

$$h_{she}(p,q) = q(p) + (1-q)\big(2(1-p)\big) \le \max\{p, 2(1-p)\} \quad (q \in [0,1])$$

If $p > 2(1-p)$ $[p > 2/3]$, then $q_{best} = 1$

If $p < 2(1-p)$ $[p < 2/3]$, then $q_{best} = 0$

If $p = 2(1-p)$ $[p = 2/3]$, then $q_{best} \in [0,1]$

Mixed equilibria in the battle of sexes

Any $p, q \in [0,1]$ identify a pair of mixed strategies
(football with probability p vs q, movies with probability $1-p$ vs $1-q$)

	he/she	football (q)	dancing ($1-q$)
p	football	(2,1)	(0,0)
$1-p$	dancing	(0,0)	(1,2)

$$h_{he}(p,q) = 2pq + (1-p)(1-q), \quad h_{she}(p,q) = pq + 2(1-p)(1-q)$$

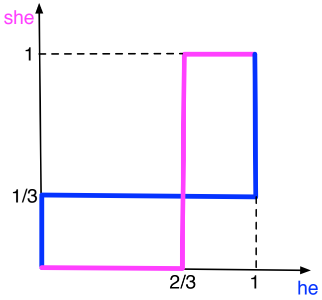
Best responses for her

$$h_{she}(p,q) = q(p) + (1-q)\big(2(1-p)\big) \leq \max\{p, 2(1-p)\} \quad (q \in [0,1])$$

$$R_{she}(p) = \begin{cases} 1 & \text{if } p > 2/3 \\ [0,1] & \text{if } p = 2/3 \\ 0 & \text{if } p < 2/3 \end{cases}$$

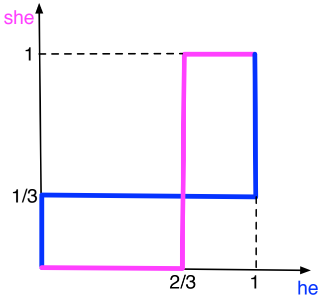
Mixed equilibria in the battle of sexes

$$R_{she}(p) = \begin{cases} 1 & \text{if } p > 2/3 \\ [0,1] & \text{if } p = 2/3 \\ 0 & \text{if } p < 2/3 \end{cases} \qquad R_{he}(q) = \begin{cases} 1 & \text{if } q > 1/3 \\ [0,1] & \text{if } q = 1/3 \\ 0 & \text{if } q < 1/3 \end{cases}$$



Mixed equilibria in the battle of sexes

$$R_{she}(p) = \begin{cases} 1 & \text{if } p > 2/3 \\ [0,1] & \text{if } p = 2/3 \\ 0 & \text{if } p < 2/3 \end{cases} \qquad R_{he}(q) = \begin{cases} 1 & \text{if } q > 1/3 \\ [0,1] & \text{if } q = 1/3 \\ 0 & \text{if } q < 1/3 \end{cases}$$

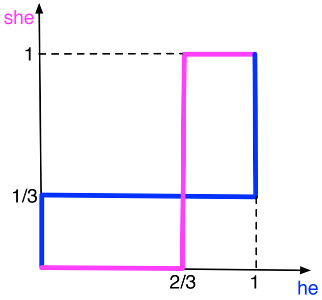


Nash equilibria are mutual best responses

$$(p^*, q^*) = (1,1), (0,0), (2/3, 1/3)$$

Mixed equilibria in the battle of sexes

$$R_{she}(p) = \begin{cases} 1 & \text{if } p > 2/3 \\ [0,1] & \text{if } p = 2/3 \\ 0 & \text{if } p < 2/3 \end{cases} \qquad R_{he}(q) = \begin{cases} 1 & \text{if } q > 1/3 \\ [0,1] & \text{if } q = 1/3 \\ 0 & \text{if } q < 1/3 \end{cases}$$



Nash equilibria are mutual best responses

$$(p^*, q^*) = (1,1), (0,0), (2/3, 1/3)$$

change the utility values 2 and 1 with 3 and 2:
pure equilibria don't change, the equilibrium in mixed strategies does