Algorithmic game theory

Laurea Magistrale in Computer Science

2024/25

Lecture 4

Strategic form of a game

- $N = \{1, \ldots, n\}$ finite set of players
- S_i set of strategies for player $i \in N$

  $S = S_1 \times \cdots \times S_n$ set of all the strategy profiles
- $u_i : S \to \mathbb{R}$ utility (or payoff) function for player $i \in N$

  each strategy profile $x \in S$ determines a unique outcome, which player i measures through $u_i(x)$: a larger value means a higher preference

$x_{-i} = (x_j)_{j \neq i}$ strategy profile for all players except i

$S_{-i} = \prod_{j \neq i} S_j$ set of all the strategy profiles for all players except i

Finite game: all the sets S_i are finite

Nash equilibria

Let $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a strategic game.

Definition

A Nash equilibrium is a strategy profile $x^* \in S$ such that the strategy x_i^* is a best response/reply to the strategy profile x_{-i}^* for all $i \in N$, i.e.,

$$x_i^* \in \arg\max\{u_i(x_i, x_{-i}^*) \ : \ x_i \in S_i\}$$

or equivalently

$$u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*) \qquad \text{for all } x_i \in S_i$$

holds for all players $i \in N$.

An equilibrium is a strategy profile with the property that no player can improve its utility changing strategy while all the other players do not

Players have no incentive to deviate from an equilibrium state unilateraly

Nash equilibria in the prisoner's dilemma

2 prisoners are accused of having committed a felony together

Years in jail are decided upon the prisoners' admissions of guilt

(negative values required in the framework of utility maximization)

I/II	not confess	confess
not confess	(-2,-2)	(-7,0)
confess	(0,-7)	(-5,-5)

(confess,confess) is the unique Nash equilibrium

Not Pareto optimal: both players could get a shorter conviction (by both not confessing)

Not socially optimal: it does not provide the joint best result (joint $\equiv$ sum of the utilities)

Nash equilibria in the battle of sexes

A couple's evening out:

she would prefer the movies, he would prefer the football game

both wish to go to the same place together rather than going alone

he/she	football	dancing
football	(2,1)	(0,0)
dancing	(0,0)	(1,2)

Nash equilibria in the battle of sexes

A couple's evening out:

she would prefer the movies, he would prefer the football game

both wish to go to the same place together rather than going alone

he/she	football	dancing
football	(2,1)	(0,0)
dancing	(0,0)	(1,2)

(football,football) and (dancing,dancing) are both Nash equilibria

Not equivalent: different equilibria may lead to different payoffs (not true in optimization)

Nash equilibria in the hawk-dove game (Maynard Smith-Price 1973)

Two animals to contest food:

hawk = aggressive behaviour (physically attack the other)

dove = cooperative behaviour (pacific attitude to share the food)

I/II	hawk	dove
hawk	(-2,-2)	(2,0)
dove	(0,2)	(1,1)

Nash equilibria in the hawk-dove game (Maynard Smith-Price 1973)

Two animals to contest food:

hawk = aggressive behaviour (physically attack the other)

dove = cooperative behaviour (pacific attitude to share the food)

I/II	hawk	dove
hawk	(-2,-2)	(2,0)
dove	(0,2)	(1,1)

(dove,hawk) and (hawk,dove) are both Nash equilibria

anti-coordination games: hawk-dove, chicken $\longrightarrow$ evolutionary stable equilibria

(brinkmanship in nuclear warfare)

No equilibria in rock-paper-scissors(-lizard-Spock)

I/II	paper	scissors	rock
paper	(0,0)	(-1,1)	(1,-1)
scissors	(1,-1)	(0,0)	(-1,1)
rock	(-1,1)	(1,-1)	(0,0)

No equilibria in rock-paper-scissors(-lizard-Spock)

I/II	paper	scissors	rock
paper	(0,0)	(-1,1)	(1,-1)
scissors	(1,-1)	(0,0)	(-1,1)
rock	(-1,1)	(1,-1)	(0,0)

No Nash equilibria exist

No equilibria in rock-paper-scissors(-lizard-Spock)

I/II	paper	scissors	rock
paper	(0,0)	(-1,1)	(1,-1)
scissors	(1,-1)	(0,0)	(-1,1)
rock	(-1,1)	(1,-1)	(0,0)

No Nash equilibria exist

Since $u_I + u_{II} \equiv 0$, a unique utility function $u = u_I$ can be considered: player I aims at maximizing u, player II at maximizing $-u$ (minimizing u)

I/II	① paper	② scissors	③ rock
① paper	0	-1	1
② scissors	1	0	-1
③ rock	-1	1	0

No equilibria in rock-paper-scissors(-lizard-Spock)

I/II	paper	scissors	rock
paper	(0,0)	(-1,1)	(1,-1)
scissors	(1,-1)	(0,0)	(-1,1)
rock	(-1,1)	(1,-1)	(0,0)

No Nash equilibria exist

Since $u_I + u_{II} \equiv 0$, a unique utility function $u = u_I$ can be considered: player I aims at maximizing u, player II at maximizing $-u$ (minimizing u)

I/II	② paper	② scissors	③ rock
① paper	0	-1	1
② scissors	1	0	-1
③ rock	-1	1	0

Proposition

The strategy profile $(\bar{k}, \bar{\ell})$ is a Nash equilibrium if and only if any k, ℓ satisfy

$$a_{k\bar{\ell}} \leq a_{\bar{k}\bar{\ell}} \leq a_{\bar{k}\ell}$$

.

($a_{k\ell} = u(k, \ell)$ payoff when player I plays k and player II plays ℓ)

Two player zero-sum finite symmetric games

Definition

A two player zero-sum finite game is symmetric if

- both players have the same number of strategies (m strategies each)
- $a_{ij} = -a_{ji}$ for all $i, j = 1, ..., m$ ($\implies a_{ii} = 0$)

(the matrix of the game A is antisymmetric, i.e., $A = -A^T$)

Two player zero-sum finite symmetric games

Definition

A two player zero-sum finite game is symmetric if

- both players have the same number of strategies (m strategies each)
- $a_{ij} = -a_{ji}$ for all $i, j = 1, \ldots, m$ ($\implies a_{ii} = 0$)

(the matrix of the game A is antisymmetric, i.e., $A = -A^T$)

Proposition

A two player zero-sum finite symmetric game has a Nash equilibrium if and only if there exists $\bar{k} \in \{1, \ldots, m\}$ such that $a_{\bar{k}\ell} \geq 0$ for all $\ell = 1, \ldots, m$. In that case, the strategy profile $(\bar{k}, \bar{k})$ is a Nash equilibrium.

Two player zero-sum finite symmetric games

Definition

A two player zero-sum finite game is symmetric if

- both players have the same number of strategies (m strategies each)
- $a_{ij} = -a_{ji}$ for all $i, j = 1, \dots, m$ $\quad (\implies a_{ii} = 0)$

(the matrix of the game A is antisymmetric, i.e., $A = -A^T$)

Proposition

A two player zero-sum finite symmetric game has a Nash equilibrium if and only if there exists $\bar{k} \in \{1, \dots, m\}$ such that $a_{\bar{k}\ell} \geq 0$ for all $\ell = 1, \dots, m$. In that case, the strategy profile $(\bar{k}, \bar{k})$ is a Nash equilibrium.

I/II	1 fire	2 water	3 wind
1 fire	0	-1	1
2 water	1	0	0
3 wind	-1	0	0

(water,water) is a Nash equilibrium

Symmetric games

Definition

A finite game is symmetric if

- all players have the same number of strategies
- any shuffling of the players (plus payoffs) does not change the game

examples: the prisoner's dilemma, hawk-dove

Symmetric games with two strategies per player

Definition

A finite game is symmetric if

- all players have the same number of strategies
- any shuffling of the players (plus payoffs) does not change the game

examples: the prisoner's dilemma, hawk-dove

Proposition

Every finite symmetric game with 2 strategies has a Nash equilibrium.

Symmetric games with two strategies per player

Definition

A finite game is symmetric if

- all players have the same number of strategies
- any shuffling of the players (plus payoffs) does not change the game

examples: the prisoner's dilemma, hawk-dove

Proposition

Every finite symmetric game with 2 strategies has a Nash equilibrium.

the case of 2 players:

I/II	①	②
①	(a,a)	(c,d)
②	(d,c)	(b,b)

Symmetric games with two strategies per player

Definition

A finite game is symmetric if

- all players have the same number of strategies
- any shuffling of the players (plus payoffs) does not change the game

examples: the prisoner's dilemma, hawk-dove

Proposition

Every finite symmetric game with 2 strategies has a Nash equilibrium.

the case of 2 players:

I/II	①	②
①	(a,a)	(c,d)
②	(d,c)	(b,b)

- $a \geq d \implies$ (①, ①) is a Nash equilibrium
- $(a < d \; + \; c \geq b) \implies$ (①, ②) and (②, ①) are Nash equilibria
- $(a < d \; + \; c < b) \implies$ (②, ②) is a Nash equilibrium

Brute force computation of equilibria: 2 player finite games

Cournot duopoly with indivisible commodity ($T = 10$, $c = 3$)

I/II	0	1	2	3	4	5	6
0	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
1	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
2	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
3	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
4	(12,0)	(8,2)	(4,2)	(0,0)	(-4,-4)	(-8,-10)	(-12,-18)
5	(10,0)	(5,1)	(0,0)	(-5,-3)	(-10,-8)	(-15,-15)	(-15,-18)
6	(6,0)	(0,0)	(-6,-2)	(-12,-6)	(-18,-12)	(-18,-15)	(-18,-18)

Brute force computation of equilibria: 2 player finite games

Cournot duopoly with indivisible commodity ($T = 10$, $c = 3$)

I/II	0	1	2	3	4	5	6
0	(,0)	(,6)	(,10)	(,12)	(,12)	(,10)	(0,6)
1	(,0)	(,5)	(,8)	(,9)	(2,8)	(1,5)	(0,0)
2	(,0)	(,4)	(6,6)	(4,6)	(2,4)	(,0)	(,-6)
3	(12,0)	(9,3)	(6,4)	(,3)	(,0)	(,-5)	(,-12)
4	(12,0)	(,2)	(,2)	(,0)	(,-4)	(,-10)	(,-18)
5	(,0)	(,1)	(,0)	(,-3)	(,-8)	(,-15)	(,-18)
6	(,0)	(,0)	(,-2)	(,-6)	(,-12)	(,-15)	(,-18)

Brute force computation of equilibria: 2 player finite games

Cournot duopoly with indivisible commodity ($T = 10$, $c = 3$)

I/II	0	1	2	3	4	5	6
0	(,)	(,)	(,)	(,12)	(,12)	(,)	(0,)
1	(,)	(,)	(,)	(,9)	(2,)	(1,)	(0,)
2	(,)	(,)	(6,6)	(4,6)	(2,)	(,)	(,)
3	(12,)	(9,)	(6,4)	(,)	(,)	(,)	(,)
4	(12,)	(,2)	(,2)	(,)	(,)	(,)	(,)
5	(,)	(,1)	(,)	(,)	(,)	(,)	(,)
6	(,0)	(,0)	(,)	(,)	(,)	(,)	(,)

Brute force computation of equilibria: 2 player finite games

Cournot duopoly with indivisible commodity ($T = 10$, $c = 3$)

I/II	0	1	2	3	4	5	6
0	(,)	(,)	(,)	(,)	(,)	(,)	(,)
1	(,)	(,)	(,)	(,)	(,)	(,)	(,)
2	(,)	(,)	**(6,6)**	**(4,6)**	(,)	(,)	(,)
3	(,)	(,)	**(6,4)**	(,)	(,)	(,)	(,)
4	(,)	(,)	(,)	(,)	(,)	(,)	(,)
5	(,)	(,)	(,)	(,)	(,)	(,)	(,)
6	(,)	(,)	(,)	(,)	(,)	(,)	(,)

Brute force computation of equilibria: 2 player finite games

$$S_i = \{1, ..., m_i\} \quad (i = 1, 2)$$

player 1: compute all best responses to any strategy of player 2

$$L_1 = \{(x_1, x_2) \in S_1 \times S_2 \ : \ x_1 \in R_1(x_2)\}$$

player 2: compute all best responses to any strategy of player 1

$$L_2 = \{(x_1, x_2) \in S_1 \times S_2 \ : \ x_2 \in R_2(x_1)\}$$

$\longrightarrow$ compute maxima in $(m_1 + m_2)$ arrays

Brute force computation of equilibria: 2 player finite games

$$S_i = \{1, ..., m_i\} \quad (i = 1, 2)$$

player 1: compute all best responses to any strategy of player 2

$$L_1 = \{(x_1, x_2) \in S_1 \times S_2 \ : \ x_1 \in R_1(x_2)\}$$

player 2: compute all best responses to any strategy of player 1

$$L_2 = \{(x_1, x_2) \in S_1 \times S_2 \ : \ x_2 \in R_2(x_1)\}$$

$\longrightarrow$ compute maxima in $(m_1 + m_2)$ arrays

$$\{\text{equilibria}\} = L_1 \cap L_2$$

$\longrightarrow$ compute common element(s) of two arrays

Brute force computation of equilibria: 2 player finite games

$$S_i = \{1, ..., m_i\} \quad (i = 1, 2)$$

player 1: compute all best responses to any strategy of player 2

$$L_1 = \{(x_1, x_2) \in S_1 \times S_2 \ : \ x_1 \in R_1(x_2)\}$$

player 2: compute all best responses to any strategy of player 1

$$L_2 = \{(x_1, x_2) \in S_1 \times S_2 \ : \ x_2 \in R_2(x_1)\}$$

$\longrightarrow$ compute maxima in $(m_1 + m_2)$ arrays

$$\{\text{equilibria}\} = L_1 \cap L_2$$

$\longrightarrow$ compute common element(s) of two arrays

Polynomial computational complexity

Looking for a winning strategy

2 prisoners are accused of having committed a felony together

Years in jail are decided upon the prisoners' admissions of guilt

I/II	not confess	confess
not confess	(-2,-2)	(-7,0)
confess	(0,-7)	(-5,-5)

Looking for a winning strategy

2 prisoners are accused of having committed a felony together

Years in jail are decided upon the prisoners' admissions of guilt

I/II	not confess	confess
not confess	(-2,-2)	(-7,0)
confess	(0,-7)	(-5,-5)

confess is always a best response for prisoner I

$$u_1(c, nc) > u_1(nc, nc), \quad u_1(c, c) > u_1(nc, c)$$

Looking for a winning strategy

2 prisoners are accused of having committed a felony together

Years in jail are decided upon the prisoners' admissions of guilt

I/II	not confess	confess
not confess	(-2,-2)	(-7,0)
confess	(0,-7)	(-5,-5)

confess is always a best response for prisoner I

$$u_1(c, nc) > u_1(nc, nc), \quad u_1(c, c) > u_1(nc, c)$$

confess is always a best response for prisoner II

$$u_2(nc, c) > u_2(nc, nc), \quad u_2(c, c) > u_2(c, nc)$$

(actually they are both the unique best response)

Strategic dominance

Let $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic game.

Definition

(i) A strategy $x_i^* \in S_i$ is dominant (for player i) if it is a best response to every strategy profile $x_{-i} \in S_{-i}$, i.e., $x_i^* \in R_i(x_{-i})$ or equivalently

$$u_i(x_i^*, x_{-i}) \geq u_i(x_i, x_{-i}) \quad \text{for all } x_i \in S_i \text{ and all } x_{-i} \in S_{-i}.$$

Strategic dominance

Let $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic game.

Definition

(i) A strategy $x_i^* \in S_i$ is dominant (for player i) if it is a best response to every strategy profile $x_{-i} \in S_{-i}$, i.e., $x_i^* \in R_i(x_{-i})$ or equivalently

$$u_i(x_i^*, x_{-i}) \geq u_i(x_i, x_{-i}) \quad \text{for all } x_i \in S_i \text{ and all } x_{-i} \in S_{-i}.$$

(ii) A strategy $x_i^* \in S_i$ is strictly dominant (for player i) if it is the unique best response to every strategy profile $x_{-i} \in S_{-i}$, i.e., $R_i(x_{-i}) = \{x_i^*\}$ or equivalently

$$u_i(x_i^*, x_{-i}) > u_i(x_i, x_{-i}) \quad \text{for all } x_i \in S_i, x_i \neq x_i^*, \text{and all } x_{-i} \in S_{-i}.$$

Strategic dominance

Let $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic game.

Definition

(i) A strategy $x_i^* \in S_i$ is dominant (for player i) if it is a best response to every strategy profile $x_{-i} \in S_{-i}$, i.e., $x_i^* \in R_i(x_{-i})$ or equivalently

$$u_i(x_i^*, x_{-i}) \geq u_i(x_i, x_{-i}) \quad \text{for all } x_i \in S_i \text{ and all } x_{-i} \in S_{-i}.$$

(ii) A strategy $x_i^* \in S_i$ is strictly dominant (for player i) if it is the unique best response to every strategy profile $x_{-i} \in S_{-i}$, i.e., $R_i(x_{-i}) = \{x_i^*\}$ or equivalently

$$u_i(x_i^*, x_{-i}) > u_i(x_i, x_{-i}) \quad \text{for all } x_i \in S_i, x_i \neq x_i^*, \text{and all } x_{-i} \in S_{-i}.$$

Dominant strategies of a player are completely equivalent: same payoffs

If it exists, a strictly dominant strategy is unique

Strategic dominance

Let $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic game.

Definition

(i) A strategy $x_i^* \in S_i$ is dominant (for player i) if it is a best response to every strategy profile $x_{-i} \in S_{-i}$, i.e., $x_i^* \in R_i(x_{-i})$ or equivalently

$$u_i(x_i^*, x_{-i}) \geq u_i(x_i, x_{-i}) \quad \text{for all } x_i \in S_i \text{ and all } x_{-i} \in S_{-i}.$$

(ii) A strategy $x_i^* \in S_i$ is strictly dominant (for player i) if it is the unique best response to every strategy profile $x_{-i} \in S_{-i}$, i.e., $R_i(x_{-i}) = \{x_i^*\}$ or equivalently

$$u_i(x_i^*, x_{-i}) > u_i(x_i, x_{-i}) \quad \text{for all } x_i \in S_i, x_i \neq x_i^*, \text{and all } x_{-i} \in S_{-i}.$$

Dominant strategies of a player are completely equivalent: same payoffs

If it exists, a strictly dominant strategy is unique

A (rational) player with a strictly dominant strategy is totally predictable

Strategic dominance and equilibria

Definition

A strategy profile $x^* \in S$ is [strictly] dominant if for each player i the strategy $x_i^* \in S_i$ is [strictly] dominant.

Proposition

(i) A dominant strategy profile is a Nash equilibrium of the game.

(ii) A strictly dominant strategy profile is the unique Nash equilibrium of the game.

Dominant strategy profiles are unlikely to exist

(none in the battle of sexes and Cournot duopoly)

Both reverse relationships do not hold

An asymmetric prisoner's dilemma: lack of dominance

2 prisoners are accused of having committed a felony together

Years in jail are decided upon the prisoners' admissions of guilt

I/II	not confess	confess
not confess	(-2,-2)	(-7,0)
confess	(-3,-7)	(-5,-5)

(confess, confess) is the unique Nash equilibrium

(confess, confess) is not a dominant strategy profile

$$u_1(c, nc) < u_1(nc, nc)$$