Algorithmic game theory

Laurea Magistrale in Computer Science

2024/25

Lecture 3

Individual decision-making under risk

- a unique decision-maker
- n mutually exclusive events: $A_1, \ldots, A_n$ (exactly one will occur)
- A_i is preferred to A_{i+1}

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Lottery

$$L = [(A_1, p_1), ..., (A_i, p_i), ..., (A_n, p_n)] \quad \textit{with} \quad p_i \geq 0 \quad \textit{s.t.} \sum_{i=1}^{n} p_i = 1$$

p_i probability that A_i occurs

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Preferences over $\mathcal{L} = \{\text{lotteries}\}$ through a binary relation $\geq$ satisfying

- reflexivity: $L \geq L$
- transitivity: $L_1 \geq L_2,\ L_2 \geq L_3 \Longrightarrow L_1 \geq L_3$
- completeness: $L_1 \geq L_2$ or $L_2 \geq L_1$ holds

(antisymmetry not required: $L_1 \geq L_2,\ L_2 \geq L_1 \not\Longrightarrow L_1 = L_2$)

equivalence: $L_1 \sim L_2 \iff L_1 \geq L_2,\ L_2 \geq L_1$

Preferences versus utility

- Monotonicity

  $p, q \in [0,1]$: $\quad [(A_1, p), (A_n, 1-p)] \geq [(A_1, q), (A_n, 1-q)] \iff p \geq q$

- Continuity

  $\exists\ \mu_i \in [0,1]$ s.t. $[(A_i, 1)] \sim [(A_1, \mu_i), (A_n, 1-\mu_i)]$

- [De]composition

$$[(A_1, p_1), ..., (A_i, p_i), ..., (A_n, p_n)] \sim [(A_1, p_1 + p_i\mu_i), ..., (A_i, 0), ..., (A_n, p_n + p_i(1-\mu_i))]$$

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Expected utility theorem (von Neumann-Morgenstern 1944)

If the pair $(\mathcal{L}, \geq)$ satisfies the above monotonicity, continuity and [de]composition properties, then there exists $u : \mathcal{L} \to \mathbb{R}$ such that

$$L_1 \geq L_2 \iff u(L_1) \geq u(L_2).$$

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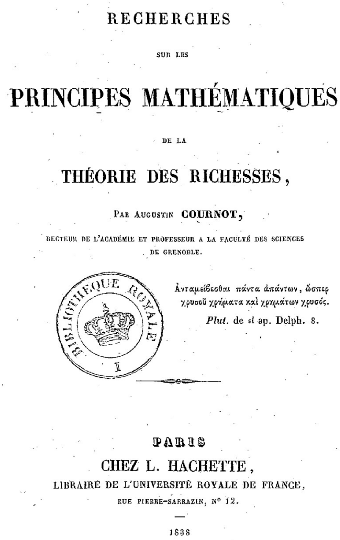
$$L_1 \geq L_2 \iff u(L_1) \geq u(L_2).$$

$$u(L) = \sum_{i=1}^{n} p_i \mu_i \quad \text{for } L = [(A_1, p_1), ..., (A_i, p_i), ..., (A_n, p_n)]$$

Antoine Augustin Cournot - introducing duopoly (1838)



RECHERCHES

SUR LES

PRINCIPES MATHÉMATIQUES

DE LA

THÉORIE DES RICHESSES,

PAR AUGUSTIN COURNOT,

RECTEUR DE L'ACADÉMIE ET PROFESSEUR A LA FACULTÉ DES SCIENCES DE GRENOBLE.

Ἀνταμείβεσθαι πάντα ἁπάντων, ὥσπερ χρυσοῦ χρήματα καὶ χρημάτων χρυσός.

Plut. de εἰ ap. Delph. 8.

PARIS

CHEZ L. HACHETTE,

LIBRAIRE DE L'UNIVERSITÉ ROYALE DE FRANCE,

RUE PIERRE-SARRAZIN, N° 12.

1838

Quantity competition à la Cournot

2 firms producing the same homogeneous commodity

competition over quantity: which level of production should a firm select?

$x_i \geq 0$ is the selected amount of production ($i = 1, 2$)

utility = total revenue minus production cost

$$u_i(x_1, x_2) = x_i p(x_1 + x_2) - c_i(x_i)$$

p inverse demand function: $p(t)$ unitary selling price for a total amount t
(highest price allowing to meet a total demand t)

c_i production cost function: $c_i(x_i)$ cost for a level of production x_i

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Data choice

$c_i(x_i) = c x_i$ with $c > 0$

$p(t) = \max\{T - \alpha t, 0\}$ with $T > c$ and wlog $\alpha = 1$ (otherwise rescale T, c)

$$u_i(x_1, x_2) = x_i \max\{T - (x_1 + x_2), 0\} - c x_i$$

Cournot duopoly: indivisible commodity

$x_i \in \mathbb{Z}_+$ units of the commodity to be produced $\longrightarrow$ at most $(T - c) - 1$

$$u_i(x_1, x_2) = x_i \max\{T - (x_1 + x_2), 0\} - c x_i$$

Example: $T = 10$, $c = 3$

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Cournot duopoly: divisible commodity

2 firms producing the same homogeneous commodity

competition over quantity: which level of production should a firm select?

$x_i \in S_i = [0, +\infty)$ is the selected amount of production ($i = 1, 2$)

utility $=$ total revenue minus production cost

$$u_i(x_1, x_2) = x_i p(x_1 + x_2) - c_i(x_i)$$

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Cournot duopoly: binding agreement

Monopoly

unique firm (or equivalently $x_2 \equiv 0$):

$$u(x) = u_1(x,0) = x \max\{T - x, 0\} - cx = \begin{cases} (T-c)x - x^2 & \text{if } x \le T \\ -cx & \text{if } x \ge T \end{cases}$$

$$x^* = (T-c)/2 \text{ maximizes the utility } u(x) \text{ over } S_1$$

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Binding agreement

Jointly maximize $u_1(x_1,x_2) + u_2(x_1,x_2)$ and share the utility fairly

$$u_1(x_1,x_2) + u_2(x_1,x_2) = (x_1 + x_2) \max\{T - (x_1+x_2), 0\} - c(x_1+x_2)$$

$x = x_1 + x_2 \longrightarrow$ back to monopoly!

$(x_1^* = x_2^* = (T-c)/4)$

Cournot duopoly: best responses (no agreement allowed)

Best responses (replies): provided the other firm chooses x^*_{-i}, select any

$$x_i^* \in S_i \text{ such that } u_i(x_i^*, x^*_{-i}) \geq u_i(x_i, x^*_{-i}) \text{ for all } x_i \in S_i$$

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$$R_i(x^*_{-i}) = \arg\max\{u_i(x_i, x^*_{-i}) : x_i \in S_i\} = \begin{cases} (T - c - x^*_{-i})/2 & \text{if } x^*_{-i} \leq T - c \\ 0 & \text{if } x^*_{-i} \geq T - c \end{cases}$$

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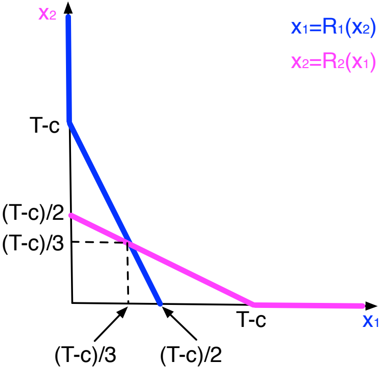

Cournot duopoly: equilibrium state

An equilibrium state is achieved at $(x_1^*, x_2^*) \in S_1 \times S_2$ such that

$$x_1^* = R_1(x_2^*) \text{ and } x_2^* = R_2(x_1^*)$$

equilibrium $\equiv$ best responses crosses

$(x_1^* = x_2^* = (T - c)/3)$

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A comparison between monopoly and Cournot duopoly

	production	unitary price	utility per firm	system utility
monopoly	$(T-c)/2$	$(T+c)/2$	$(T-c)^2/4$	$(T-c)^2/4$
	$\wedge$	$\vee$	$\vee$	$\vee$
duopoly	$2(T-c)/3$	$(T+2c)/3$	$(T-c)^2/9$	$2(T-c)^2/9$

binding agreement allowed: utility per firm $(T - c)^2/8$ (better than without)

Finite Cournot duopoly: best responses

$x_i \in \mathbb{Z}_+$ units of the commodity to be produced $\longrightarrow$ at most $(T-c)-1$

$$u_i(x_1, x_2) = x_i \max\{T-(x_1+x_2), 0\} - c\,x_i$$

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⓪	(,)	(,)	(,)	(,12)	(,12)	(,)	(0,)
①	(,)	(,)	(,)	(,9)	(2,)	(1,)	(0,)
②	(,)	(,)	(6,6)	(4,6)	(2,)	(,)	(,)
③	(12,)	(9,)	(6,4)	(,)	(,)	(,)	(,)
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⑥	(,0)	(,0)	(,)	(,)	(,)	(,)	(,)

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multiple equilibrium states exist

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binding agreement leads to $x_1 + x_2 = 3$ or $x_1 + x_2 = 4$

Strategic form of a game

- $N = \{1, \ldots, n\}$ finite set of players
- S_i set of strategies for player $i \in N$

  $S = S_1 \times \cdots \times S_n$ set of all the strategy profiles
- $u_i : S \to \mathbb{R}$ utility (or payoff) function for player $i \in N$

  each strategy profile $x \in S$ determines a unique outcome, which player i measures through $u_i(x)$: a larger value means a higher preference

$x_{-i} = (x_j)_{j \neq i}$ strategy profile for all players except i

$S_{-i} = \prod_{j \neq i} S_j$ set of all the strategy profiles for all players except i

Finite game: all the sets S_i are finite

Nash equilibria

Let $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a strategic game.

Definition

A Nash equilibrium is a strategy profile $x^* \in S$ such that the strategy x_i^* is a best response/reply to the strategy profile x_{-i}^* for all $i \in N$, i.e.,

$$x_i^* \in \arg\max\{u_i(x_i, x_{-i}^*) \ : \ x_i \in S_i\}$$

or equivalently

$$u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*) \qquad \text{for all } x_i \in S_i$$

holds for all players $i \in N$.

An equilibrium is a strategy profile with the property that no player can improve its utility changing strategy while all the other players do not

Players have no incentive to deviate from an equilibrium state unilateraly

Nash equilibria in the prisoner's dilemma

2 prisoners are accused of having committed a felony together

Years in jail are decided upon the prisoners' admissions of guilt

(negative values required in the framework of utility maximization)

I/II	not confess	confess
not confess	(-2,-2)	(-7,0)
confess	(0,-7)	(-5,-5)

(confess,confess) is the unique Nash equilibrium

Not Pareto optimal: both players could get a shorter conviction (by both not confessing)

Not socially optimal: it does not provide the joint best result (joint $\equiv$ sum of the utilities)