Algorithmic game theory

Laurea Magistrale in Computer Science

2024/25

Lecture 2

What is a game?

A description of the strategic interactions between players

– (a finite number of) players

– strategies: the actions a player can take

– outcome: it depends on the strategies selected by all players

– preferences: a player's binary relation between outcomes
(complete, reflexive and transitive [total pre-order])

preferences are often given through an *utility function* [payoff]

The prisoner's dilemma

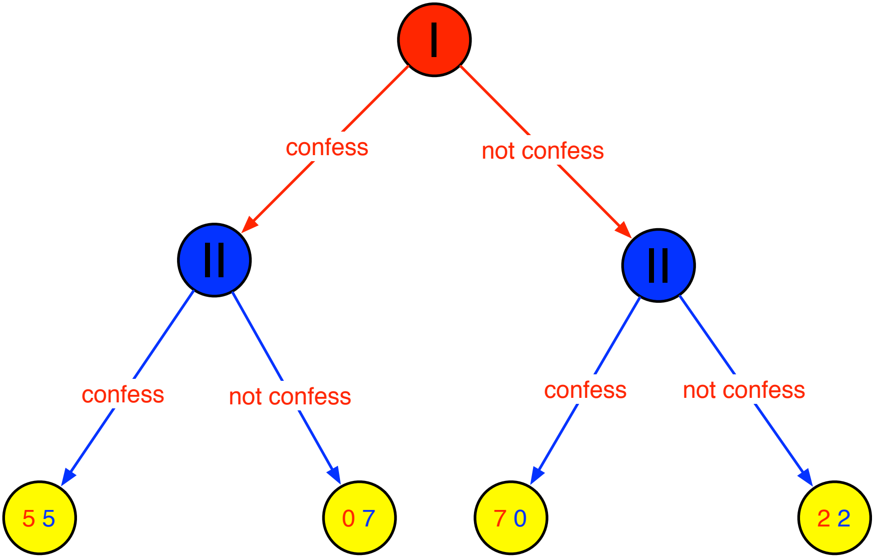
2 prisoners are accused of having committed a felony together

Years in jail are decided upon the prisoners' admissions of guilt

I/II	not confess	confess
not confess	(2,2)	(7,0)
confess	(0,7)	(5,5)

A.W.Tucker, A Two-Person Dilemma, memo at Stanford University, 1950

Another description of the prisoner's dilemma



A coordination game: the battle of sexes

A couple's evening out:

she would prefer go dancing, he would prefer the football game


Tonight we could go dancing!
Or... we could go to the football match!
I think Bob will go to the football match. Therefore, I'll go to the match.
I think Amy will expect me to be at the football match. Therefore she will go to the football match. So I shall go to the match.
I think Bob will expect me to be at the dance class. Hence he will be at dance class. So I will go dancing.
I think Amy will be at the dance class. Therefore I will go dancing.
I expected Bob to be at the football match. But he's not here! He must have thought I would go dancing.
I thought Amy would go dancing. But she isn't here! She must have expected me to go to the match.

A coordination game: the battle of sexes

A couple's evening out:

she would prefer go dancing, he would prefer the football game

both wish to go to the same place together rather than going alone

he/she	football	dancing
football	(2,1)	(0,0)
dancing	(0,0)	(1,2)

based on the stag hunt situation by Jean-Jacques Rousseau

Discours sur l'origine et les fondements de l'inégalité parmi les hommes, 1755

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he/she	football	dancing
football	(A,a)	(C,c)
dancing	(B,b)	(D,d)

$$A > B, \;\; D > C, \;\; a > c, \;\; d > b$$

An anti-coordination game: hawk-dove

(Maynard Smith-Price 1973)

Two animals to contest food:

hawk = aggressive behaviour (physically attack the other)

dove = cooperative behaviour (pacific attitude to share the food)

I/II	hawk	dove
hawk	(-2,-2)	(2,0)
dove	(0,2)	(1,1)

anti-coordination games: hawk-dove, chicken, graph colouring

(brinkmanship in nuclear warfare)

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I/II	hawk	dove
hawk	(A,A)	(B,b)
dove	(b,B)	(D,D)

anti-coordination games: $B > D > b > A$

(brinkmanship in nuclear warfare)

Rock-paper-scissors

paper covers rock - rock crushes scissors - scissors cuts paper

I/II	paper	scissors	rock
paper	(0,0)	(-1,1)	(1,-1)
scissors	(1,-1)	(0,0)	(-1,1)
rock	(-1,1)	(1,-1)	(0,0)

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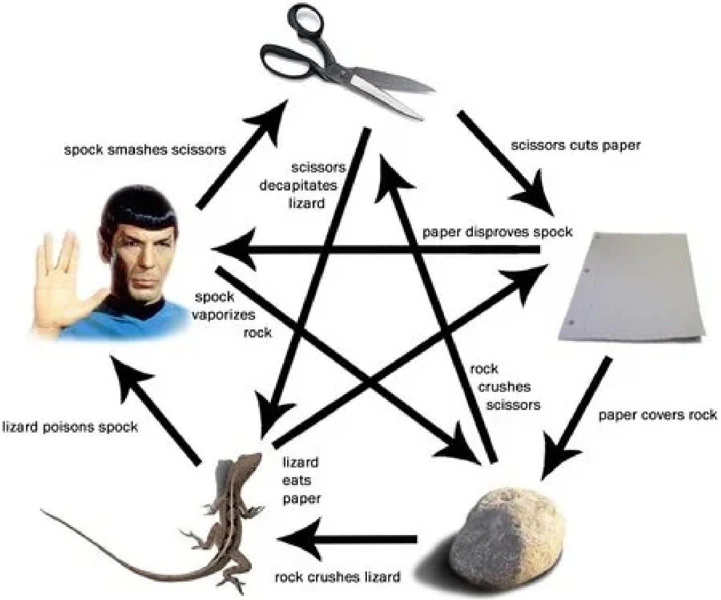
The Lizard-Spock expansion



Rock-paper-scissors-lizard-Spock

scissors cuts paper - paper covers rock - rock crushes lizard
lizard poisons Spock -Spock smashes scissors - scissors decapitates lizard
lizard eats paper- paper disproves Spock - Spock vaporizes rock
rock crushes scissors

I/II	paper	scissors	rock	lizard	Spock
paper	(0,0)	(-1,1)	(1,-1)	(-1,1)	(1,-1)
scissors	(1,-1)	(0,0)	(-1,1)	(1,-1)	(-1,1)
rock	(-1,1)	(1,-1)	(0,0)	(1,-1)	(-1,1)
lizard	(1,-1)	(-1,1)	(-1,1)	(0,0)	(1,-1)
Spock	(-1,1)	(1,-1)	(1,-1)	(-1,1)	(0,0)

Modeling noncooperative games

- Strategic (or normal) form

  best suited for "one shot" games

  separate description of the ingredients of the game

- Extensive form

  suitable for games with finitely many actions for each player

  enumerative description of the (sequential) structure of the actions taken by the players: "book of the game"

Games in an extensive form can be turned into a strategic form

Strategic form of a game

- $N = \{1, \ldots, n\}$ finite set of players
- S_i set of strategies for player $i \in N$

  $S = S_1 \times \cdots \times S_n$ set of all the strategy profiles

- $u_i : S \to \mathbb{R}$ utility (or payoff) function for player $i \in N$

  each strategy profile $x \in S$ determines a unique outcome, which player i measures through $u_i(x)$: a larger value means a higher preference

$x_{-i} = (x_j)_{j \neq i}$ strategy profile for all players except i

$S_{-i} = \prod_{j \neq i} S_j$ set of all the strategy profiles for all players except i

Finite game: all the sets S_i are finite

Colonel Blotto game(s)

2 players: colonel Blotto b, enemy e

limited amount of resources: R_b, R_e (> 0)

n battlefields, each with its own value: $w_1, \ldots, w_n$ (> 0)

battlefield winner: the player deploying most resources

how to allocate resources between the battlefields?

strategies:

$$x_{b/e} \in \mathbb{R}_+^n \text{ s.t. } (x_{b/e})_1 + \cdots + (x_{b/e})_n = R_{b/e}$$

utility functions:

$$u_b(x_b, x_e) = \sum_{i=1}^{n} w_i \, \mathrm{sign}((x_b)_i - (x_e)_i) = -u_e(x_b, x_e)$$

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Extensive form of a game

It can be represented by an enumeration tree

node = state of the game

leaf = outcome of the game

each node (but leafs) belongs to one player

arc = action taken by the "tail node" player

labels on a leaf = utilities/payoffs of the players

the formal mathematical definition is not very handy

A sequential allocation game

2 (identical) objects to be shared by 2 players

player 1 suggests the allocation, players 2 accepts or declines

in case of no agreement they both get nothing

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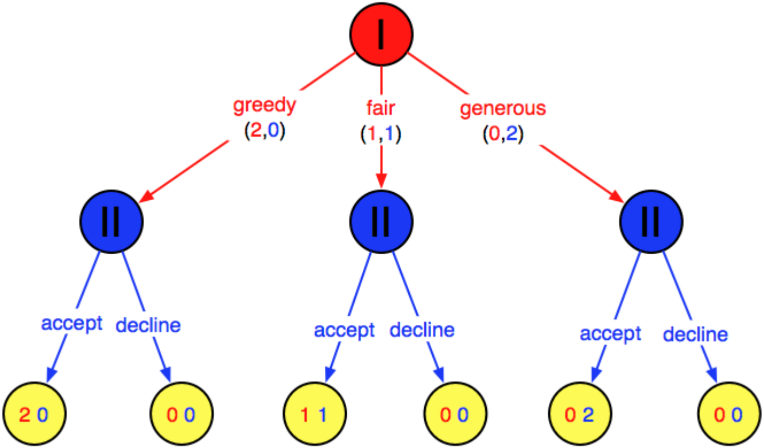
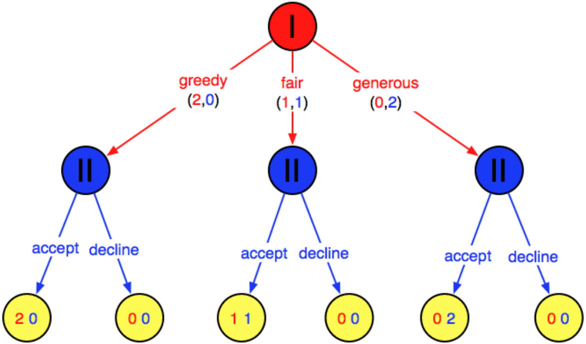


Turning extensive into strategic form



strategy $=$ choice of a forward arc at each node owned by the player

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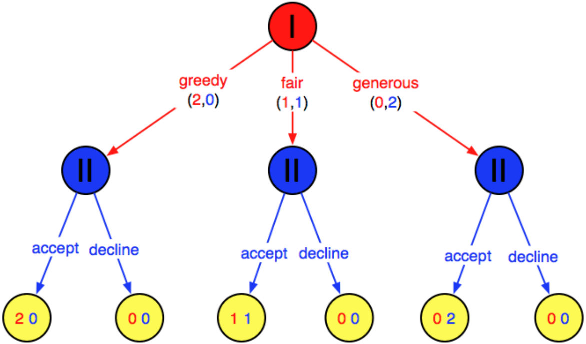

strategy = choice of a forward arc at each node owned by the player

I/II	a a a	a a d	a d a	d a a	d d a	d a d	a d d	d d d
greedy	(2,0)	(2,0)	(2,0)	(0,0)	(0,0)	(0,0)	(2,0)	(0,0)
fair	(1,1)	(1,1)	(0,0)	(1,1)	(0,0)	(1,1)	(0,0)	(0,0)
generous	(0,2)	(0,0)	(0,2)	(0,2)	(0,2)	(0,0)	(0,0)	(0,0)

$3 \times 8 = 24$ pairings of strategies to describe only 6 real situations

Individual decision-making under risk

- a unique decision-maker
- n mutually exclusive events: $A_1, \ldots, A_n$ (exactly one will occur)
- A_i is preferred to A_{i+1}

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Lottery

$$L = [(A_1, p_1), ..., (A_i, p_i), ..., (A_n, p_n)] \quad \textit{with} \quad p_i \geq 0 \quad \textit{s.t.} \sum_{i=1}^{n} p_i = 1$$

p_i probability that A_i occurs

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Preferences over $\mathcal{L} = \{\text{lotteries}\}$ through a binary relation $\geq$ satisfying

- reflexivity: $L \geq L$
- transitivity: $L_1 \geq L_2, \ L_2 \geq L_3 \Longrightarrow L_1 \geq L_3$
- completeness: $L_1 \geq L_2$ or $L_2 \geq L_1$ holds

(antisymmetry not required: $L_1 \geq L_2, \ L_2 \geq L_1 \not\Longrightarrow L_1 = L_2$)

equivalence: $L_1 \sim L_2 \Longleftrightarrow L_1 \geq L_2, \ L_2 \geq L_1$

Preferences versus utility

- Monotonicity

  $p, q \in [0,1]$: $\quad [(A_1, p), (A_n, 1-p)] \geq [(A_1, q), (A_n, 1-q)] \iff p \geq q$

- Continuity

  $\exists\ \mu_i \in [0,1]$ s.t. $[(A_i, 1)] \sim [(A_1, \mu_i), (A_n, 1-\mu_i)]$

- [De]composition

$$[(A_1, p_1), \ldots, (A_i, p_i), \ldots, (A_n, p_n)] \sim [(A_1, p_1 + p_i\mu_i), \ldots, (A_i, 0), \ldots, (A_n, p_n + p_i(1-\mu_i))]$$

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Expected utility theorem (von Neumann-Morgenstern 1944)

If the pair $(\mathcal{L}, \geq)$ satisfies the above monotonicity, continuity and [de]composition properties, then there exists $u : \mathcal{L} \to \mathbb{R}$ such that

$$L_1 \geq L_2 \iff u(L_1) \geq u(L_2).$$

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$$u(L) = \sum_{i=1}^{n} p_i \mu_i \quad \text{for } L = [(A_1, p_1), ..., (A_i, p_i), ..., (A_n, p_n)]$$