# Algorithmic game theory

Laurea Magistrale in Computer Science

2024/25

Lectures 0 & 1

# Game theory is not [mainly] about games

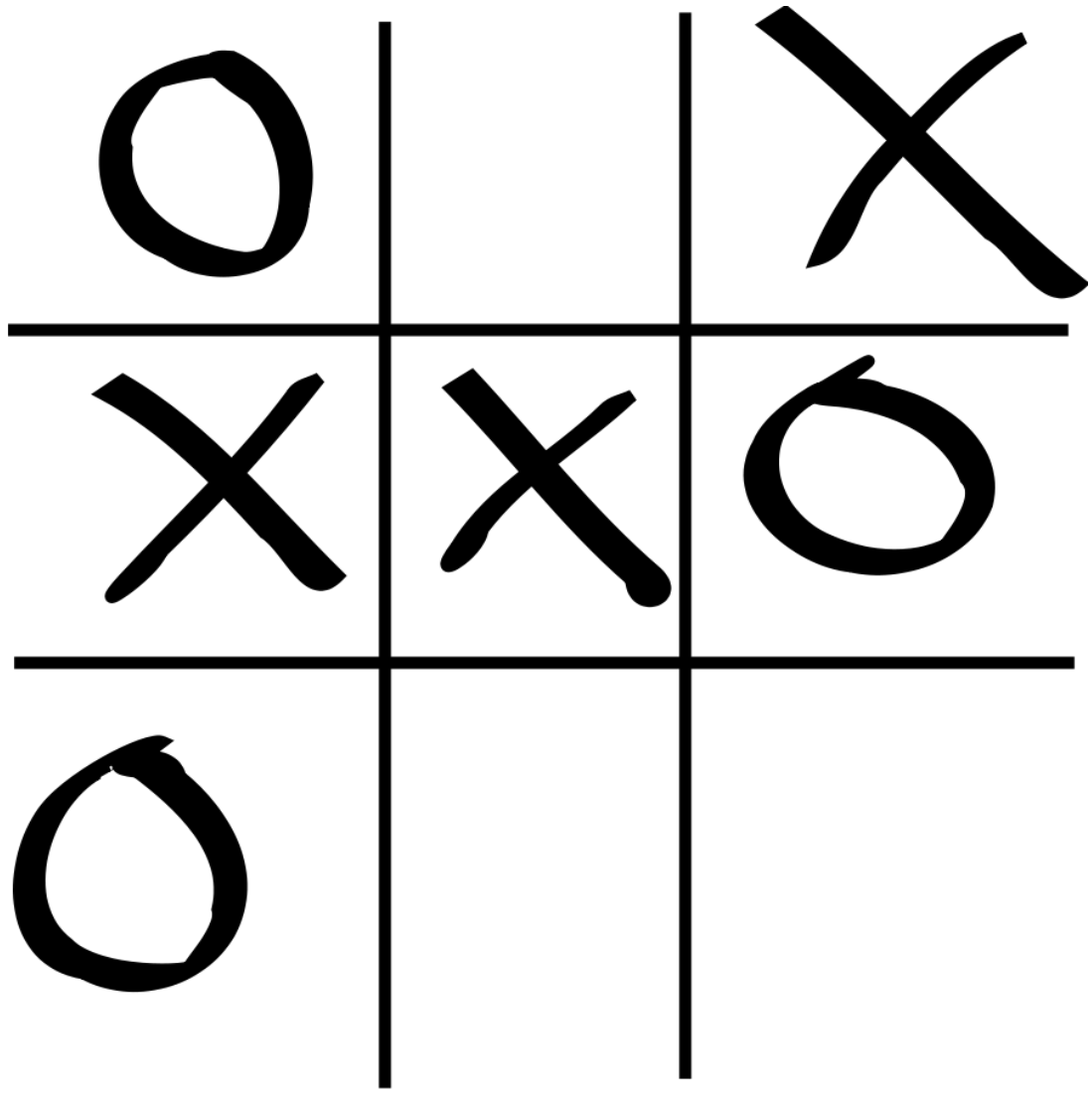
# though dealing with games as well

**(Zermelo's theorem, 1913)**

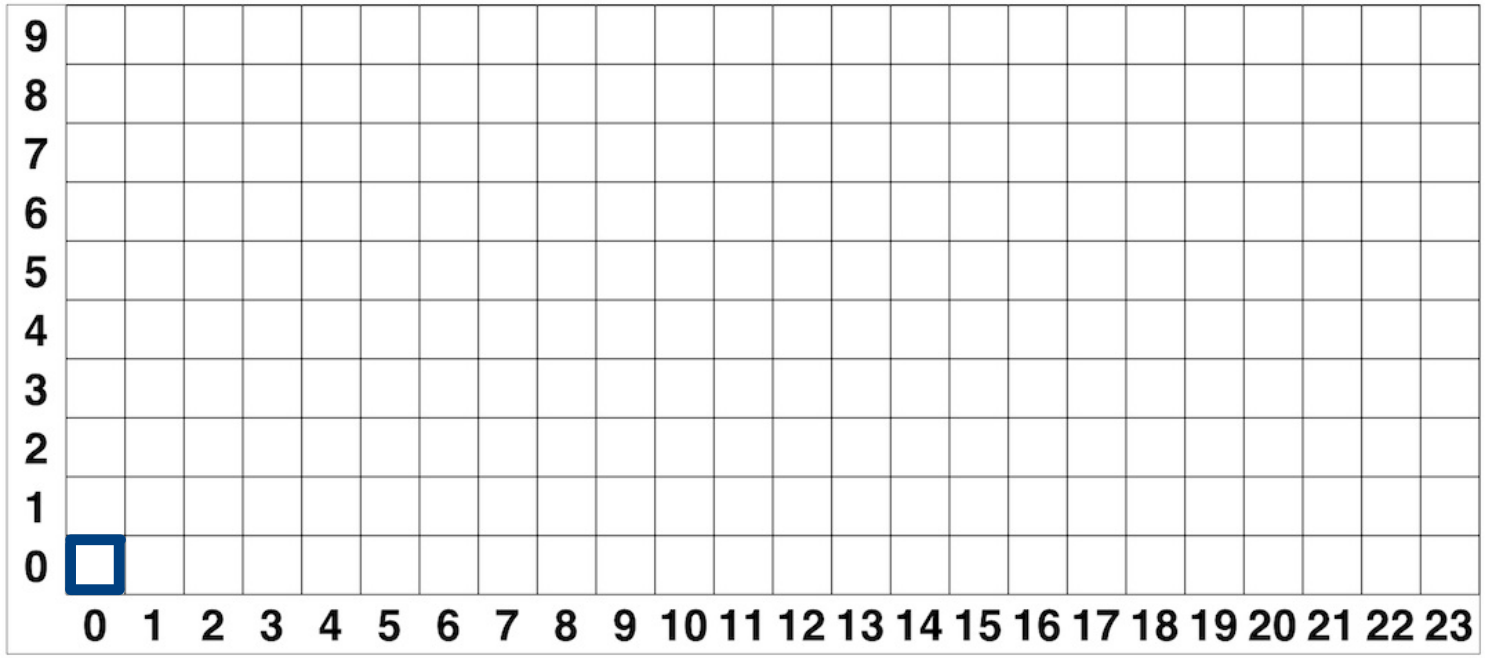
*In the game of chess one and only one of the following holds:*

- *the white player can enforce a win;*
- *the black player can enforce a win;*
- *both players can enforce a draw.*

# tic tac toe: enforcing a draw



# another board game: chomp



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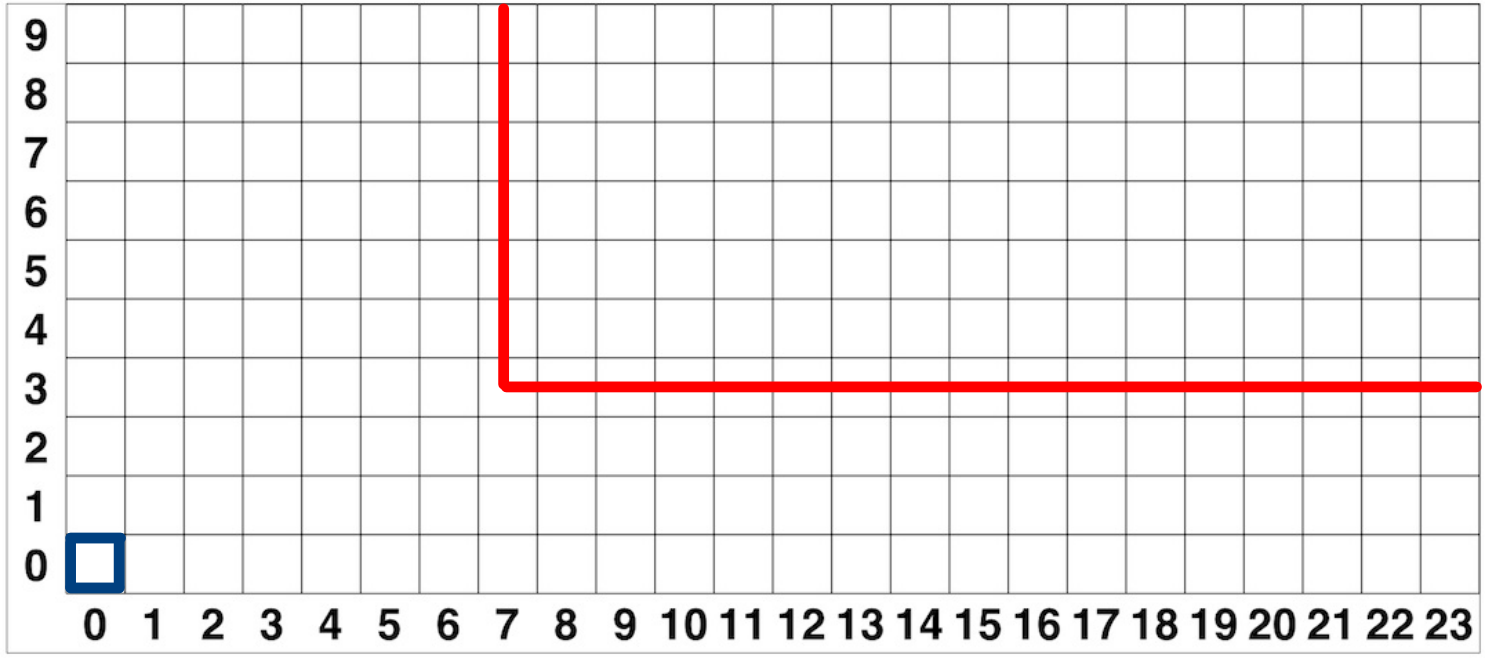

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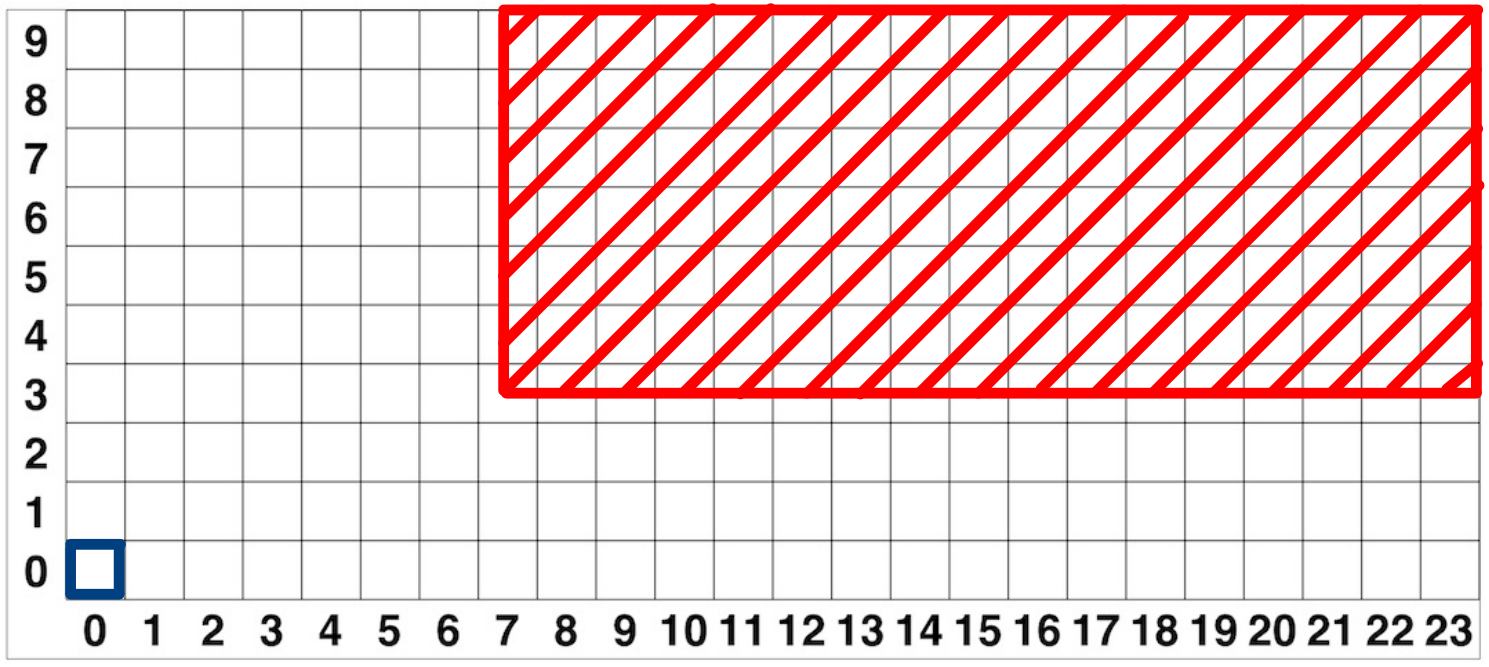

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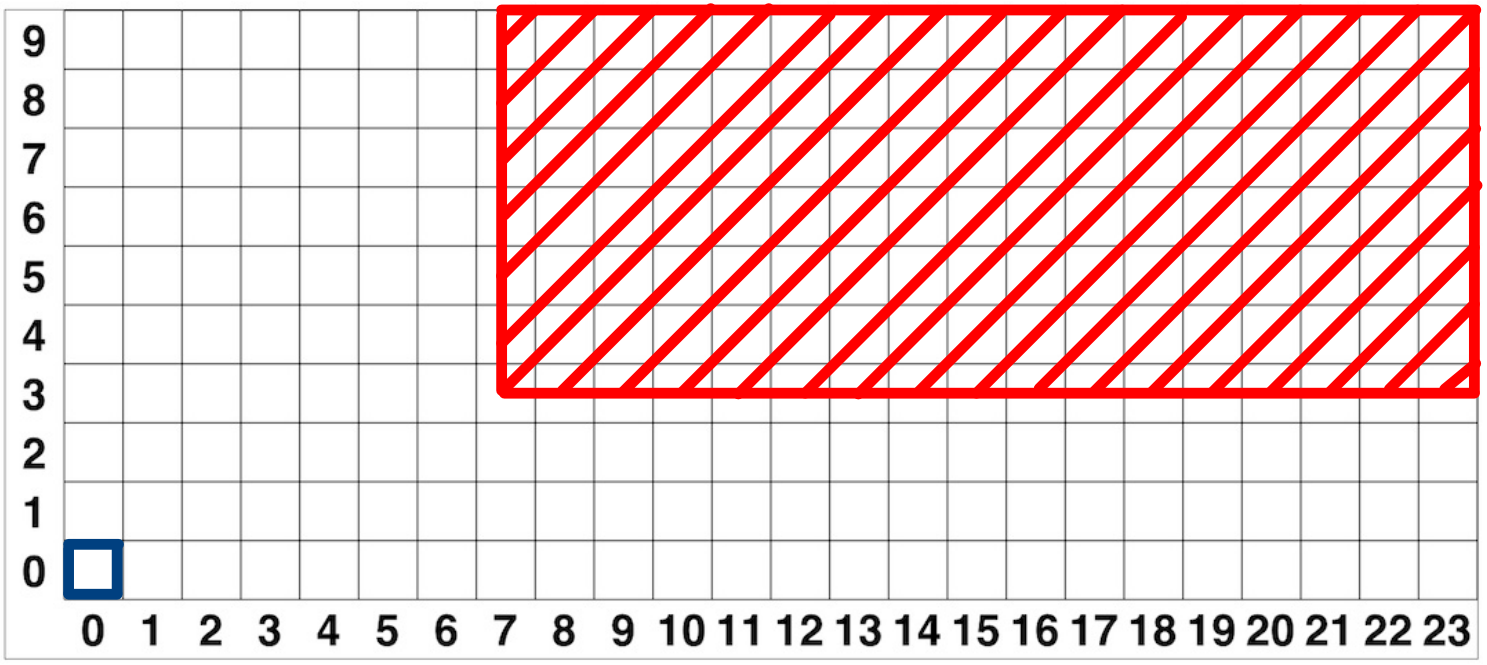



no draw is possible: which player can enforce a win?

# Goals of game theory

phenomena/systems with interactions between multiple decision-makers

(decision-makers may be individuals or groups, nature, abstract entities, etc.)

- analyse situations in which their goals may conflict

  (the outcome for each one depends also on the choices of the others)

- understand inner mechanisms of
  - competition and cooperation
  - threats and promises

- forecast the behaviour of decision-makers [players]

- design mechanisms to steer systems towards desired objectives

Basic assumptions: players are rational and reason strategically

game = [math] description of the strategic interactions between players

# Classification of games

– Cooperative games

agreements between players are allowed

which coalition(s) will be formed?

how will the outcome be split?

– Noncooperative games

agreements between players are not allowed

players aiming at their own best individually

- Strategic games

  "one shot": actions taken simultaneously at the beginning

  complete/incomplete information: whether or not all data are common knowledge

- Extensive games

  ordered events: actions taken sequentially (games with moves)

  perfect/imperfect information: whether or not all the past moves are disclosed

- Repeated games

  number of repetitions of some base (strategic or extensive) game

# Breakthrough work

E.Zermelo, Uber eine Anwendung der Mengenlehre auf die Theorie des Schachspiels, in E.W.Hobson and A.E.H.Love (eds.) *Proceedings of the Fifth International Congress of Mathematicians, Volume II*, Cambridge University Press, 1913, pp. 501-504

J.von Neumann, Zur Theorie der Gesellschaftsspiele, *Mathematische Annalen* 100 (1928) 295-320.

J.von Neumann, O.Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, 1944.

J.F.Nash, Equilibrium Points in N-Person Games, *Proceedings of the National Academy of Sciences of the United States of America* 36 (1950) 48-49.

J.F.Nash, Non-Cooperative Games, *Annals of Mathematics* 54 (1951) 286-295.

L.S.Shapley, A Value for n-Person Games, in H.W. Kuhn and A.W. Tucker (eds.) *Contributions to the Theory of Games, Volume II*, Princeton University Press, 1953, pp. 307-317

# Historical curiosity: Talmud Bavlì and game theory

Oldest known example of game theory (500 A.D.)

A man dies leaving debts larger than his estate.
How to divide the estate between the creditors?

The Talmud Bavlì provides rules for some cases with 3 creditors

R.J.Aumann, M.Maschler, Game Theoretic Analysis of a Bankruptcy Problem from the Talmud, *Journal of Economic Theory* 36 (1985) 195-213

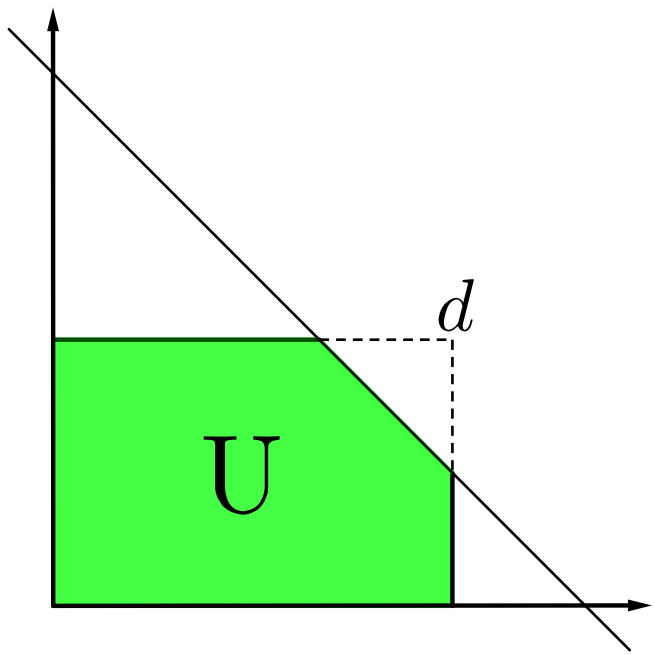
# Bankruptcy

- business failure for a company
- asset of the company 4 m€
- $n$ creditors - $d_i$ m€ the liability towards creditor $i$
- $d_1 + \cdots + d_n > 4$

how to divide the asset between the creditors?

$$U = \{u \in \mathbb{R}^n_+ \ : \ u_1 + \cdots + u_n \leq 4, \ u_i \leq d_i \ i = 1, \ldots n\}$$

(feasible outcomes for an agreement/arbitration)



# Some forerunners in economics

A.A.Cournot, *Recherches sur les Principes Mathematiques de la Theorie des Richesses*, Hachette, 1838

Competition between producers: duopoly (foreseeing Nash equilibria)

F.Y.Edgeworth, *Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences*, Kegan Paul, 1881.

Trading between people: allocations of 2 commodities between 2 people

# 13 Nobel laureates ..... and some more

The Sveriges Riksbank Prize in Economic Sciences
(in memory of Alfred Nobel)


1994 J.F.Nash, J.C.Harsanyi, R.Selten

for their pioneering analysis of equilibria
in the theory of non-cooperative games

2005 R.J.Aumann, T.C.Schelling

for having enhanced our understanding of conflict
and cooperation through game-theory analysis

2007 L.Hurwicz, E.S.Maskin, R.B.Myerson

for having laid the foundations of mechanism design theory

2012 A.E.Roth, L.S.Shapley

for the theory of stable allocations and the practice of market design

2014 J.Tirole

for his analysis of market power and regulation

2020 P.R.Milgrom, R.B.Wilson

for improvements to auction theory and inventions of new auction formats

# AGT contents

## Basic topics

- Noncooperative games
  Nash (normal) games, multilevel games, sequential games, algorithms
- Allocations and bargaining
  divisible and indivisible goods, fairness, Nash solution to bargaining, algorithms
- Cooperative games
  transferable utility, core and nucleoli, Shapley value, power indices

## Elective topics

minimax algorithms, network games, consensus theory, matching, auctions
reinforcement learning and no regret minimization in multi-agent systems
paradoxes, complexity of equilibria computation, computational social choice

## Applications to economics and computer science

- Cournot, Bertrand and Stackelberg oligopolies
- Exchange economies
- Networks: routing and security
- Mining blockchains
- .....

# Why should I take AGT?

- truly interdisciplinary

  unique blend of mathematics and computer science with economics, psychology and much more!

- expected and unexpected applications

  ranging from political sciences to engineering, from criminology to biology

- to open up your mind to strategic thinking

- first step towards winning a Nobel prize

- (nice?) instructor who keeps playing despite age

- just to have fun

# What is a game?

A description of the strategic interactions between players

– (a finite number of) players

– strategies: the actions a player can take

– outcome: it depends on the strategies selected by all players

– preferences: a player's binary relation between outcomes
(complete, reflexive and transitive [total pre-order])

preferences are often given through an *utility function* [payoff]

# The prisoner's dilemma

2 prisoners are accused of having committed a felony together

Years in jail are decided upon the prisoners' admissions of guilt

| I/II | not confess | confess |
|---|---|---|
| not confess | (2,2) | (7,0) |
| confess | (0,7) | (5,5) |

A.W.Tucker, A Two-Person Dilemma, memo at Stanford University, 1950