Algorithmic game theory

Laurea Magistrale in Computer Science

2024/25

Lecture 11

Yet another Stackelberg finite game

1 = leader   2 = follower

I/II	①	②	③	④	⑤
①	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)
②	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)
③	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)
④	(12,0)	(9,3)	(3,4)	(3,3)	(0,0)
⑤	(12,0)	(8,4)	(4,4)	(0,0)	(1,1)

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③	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)
④	(12,0)	(9,3)	(3,4)	(3,3)	(0,0)
⑤	(12,0)	(8,4)	(4,4)	(0,0)	(1,1)

the leader anticipates the follower's responses

Yet another Stackelberg finite game

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I/II	①	②	③	④	⑤
①				(0,12)	(0,12)
②				(3,9)	
③			(6,6)	(4,6)	
④			(3,4)		
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optimistic attitude

Yet another Stackelberg finite game

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①				(0,12)	(0,12)
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the leader anticipates the follower's responses

optimistic attitude $\longrightarrow$ (⑤,②)

Yet another Stackelberg finite game

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the leader anticipates the follower's responses

optimistic attitude $\longrightarrow$ (⑤,②)

pessimistic attitude

Yet another Stackelberg finite game

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I/II	①	②	③	④	⑤
①				(0,12)	(0,12)
②				(3,9)	
③			(6,6)	(4,6)	
④			(3,4)		
⑤		(8,4)	(4,4)		

the leader anticipates the follower's responses

optimistic attitude $\longrightarrow$ (⑤,②)

pessimistic attitude $\longrightarrow$ (③,④) or (⑤,③)   which of the two strategies is preferable?

Sequential finite games with perfect information

Sequential finite game ≡ enumeration tree

Enumeration tree = directed rooted out-tree (oriented away from the root)

node = state of the game

leaf = outcome of the game

turn/ply = nodes with the same depth (distance from the root)

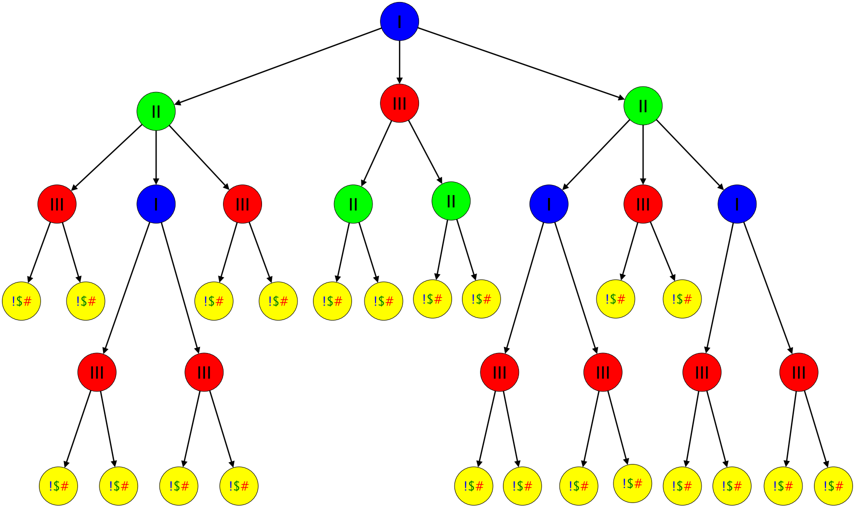
each node (but leafs) belongs to one player

arc = action taken by the "tail node" player

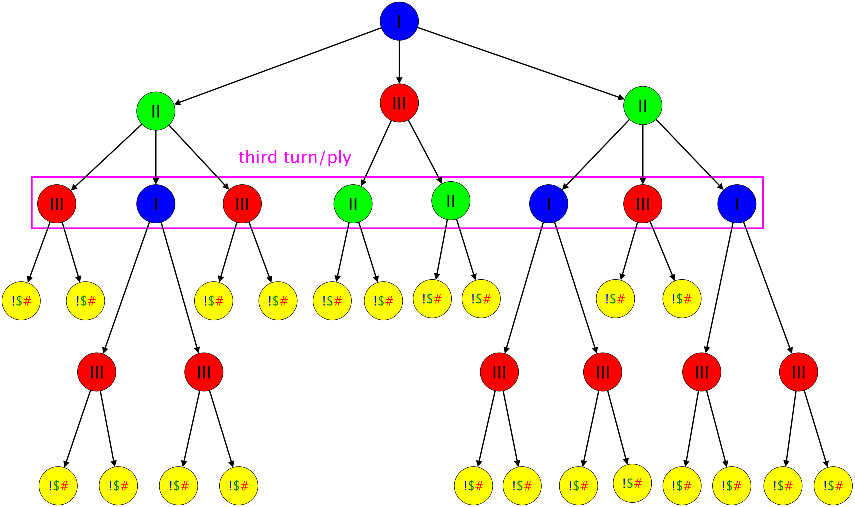
labels on a leaf = utilities/payoffs of the players

the formal mathematical definition is not very handy

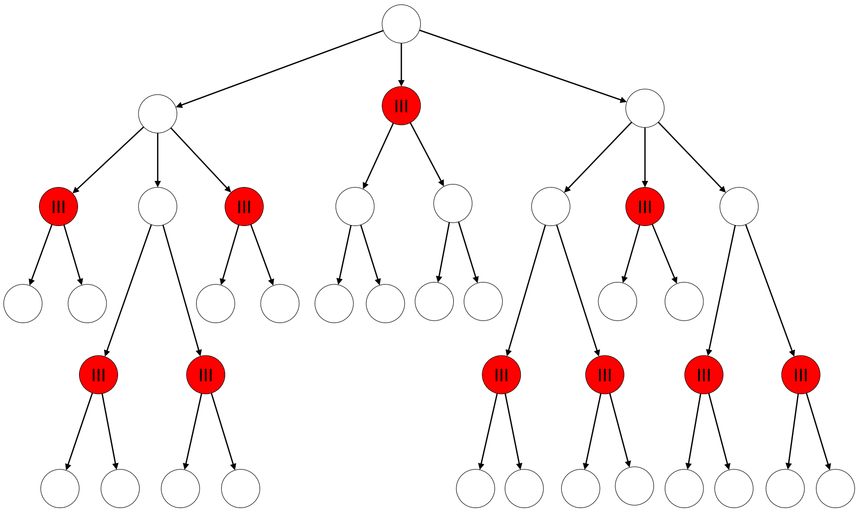
Enumeration tree



Enumeration tree: turn/ply



Enumeration tree: strategies



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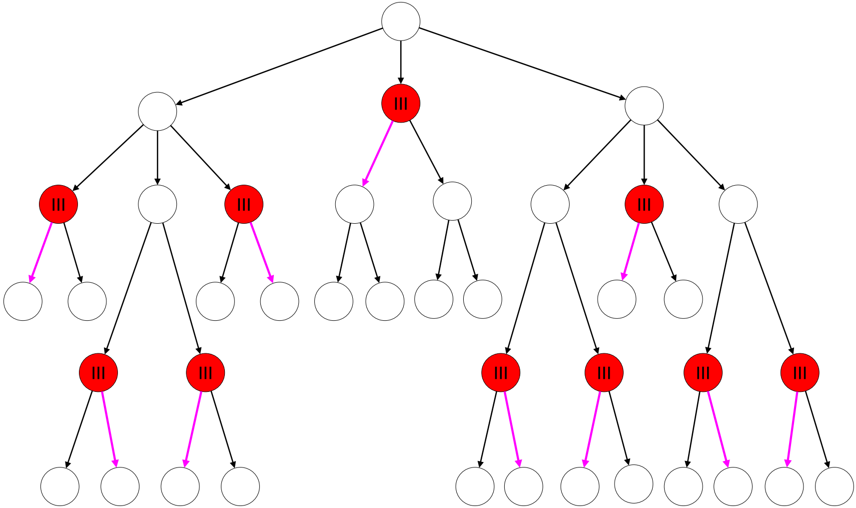
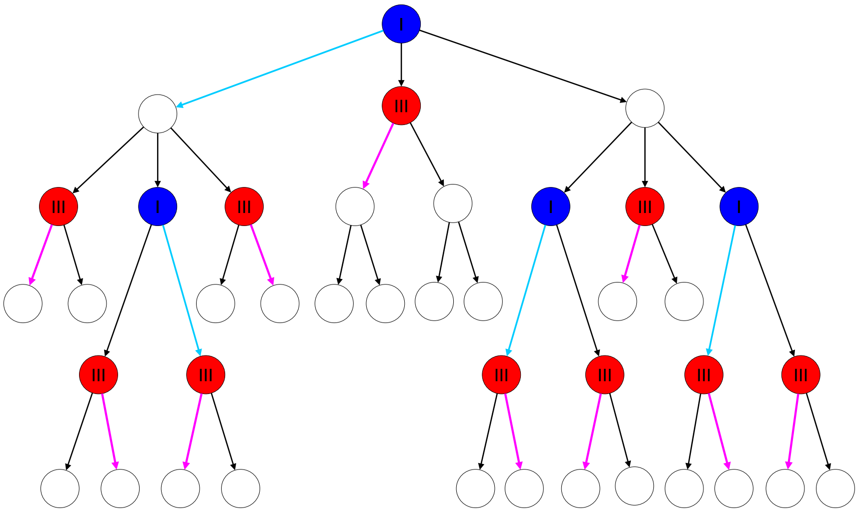
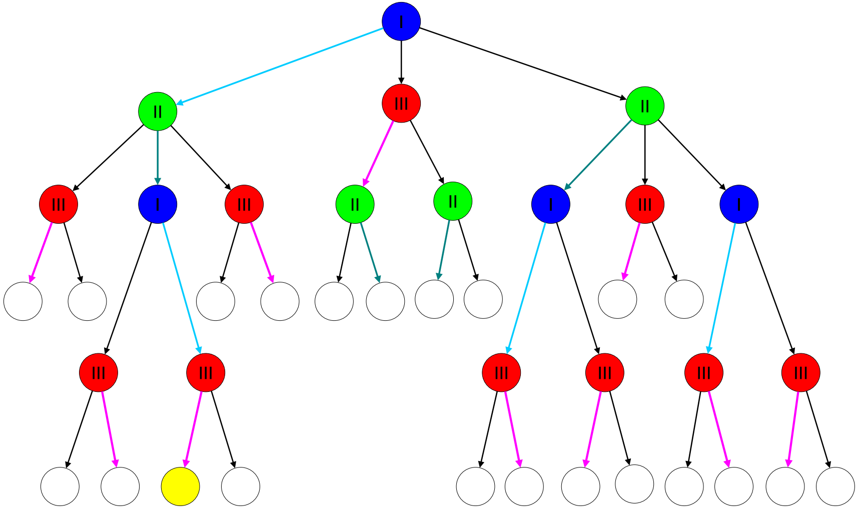


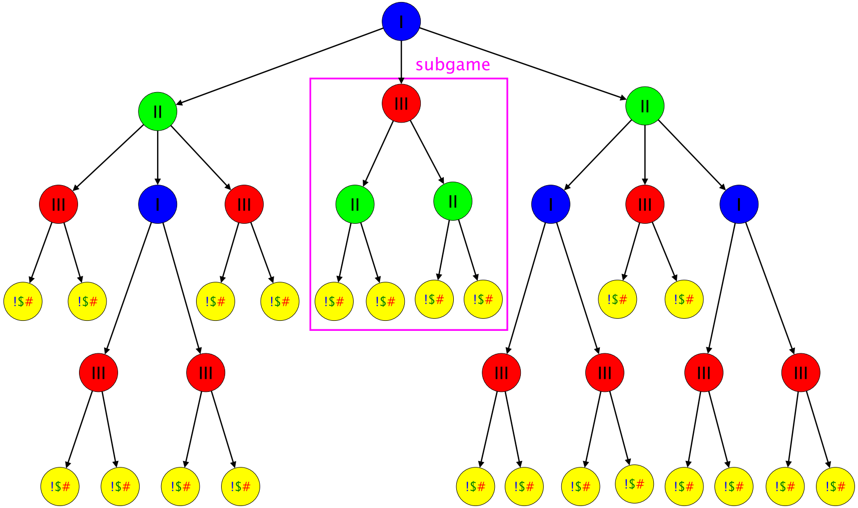
Enumeration tree: strategies



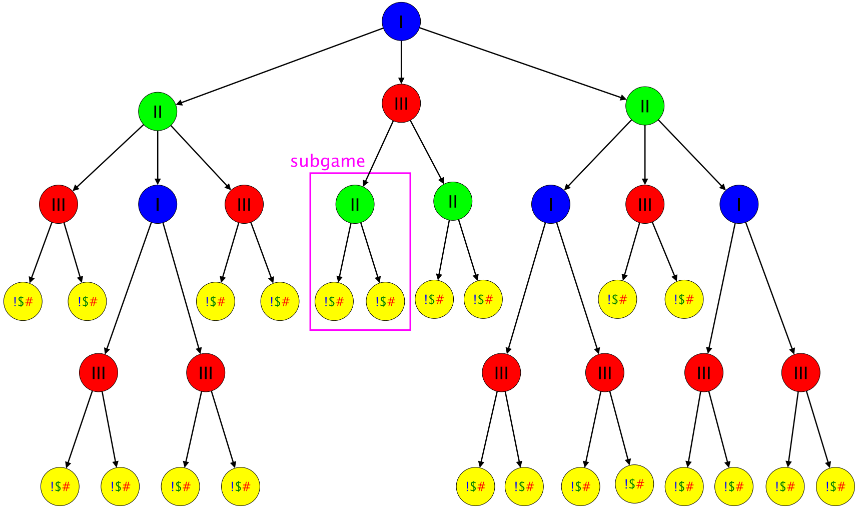
Enumeration tree: strategies and outcomes



Enumeration tree: subgames



Enumeration tree: subgames



Subgame perfectness

Subgame

A subgame of a sequential finite game is a subgraph such that

- *it is a directed rooted out-tree*
- *its root is not a leaf*

Subgame perfectness

Subgame

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- *it is a directed rooted out-tree*
- *its root is not a leaf*

Subgame perfect equilibrium (Selten 1965)

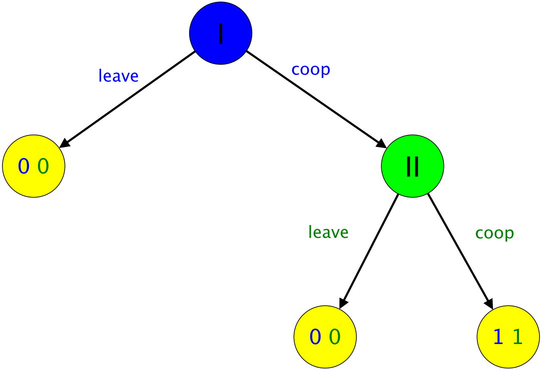
A profile of strategies is a subgame perfect equilibrium if its restriction to every subgame is a Nash equilibrium of the subgame.

- A subgame perfect equilibrium is a Nash equilibrium

  (a graph is a subgraph of itself)

- Not all Nash equilibria are subgame perfect

Pure coordination



I/II	leave	cooperate
leave	(0,0)	(0,0)
cooperate	(0,0)	(1,1)

(leave,leave) and (cooperate,cooperate) are both Nash equilibria

(leave,leave) is not subgame perfect

Backward induction

Backward induction procedure

- *solve the subgames rooted at nodes with the highest depth (last turn)*
- *delete non-equilibrium strategies (replace the subgames with equilibrium labels)*
- *proceed backwards to solve subgames with lower depth [till the root]*

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- solve ≡ compute a Nash equilibrium
- each restricted subgame amounts to a finite optimization problem
- multipla maxima could lead to difficulties (perform a choice)

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Backward induction procedure

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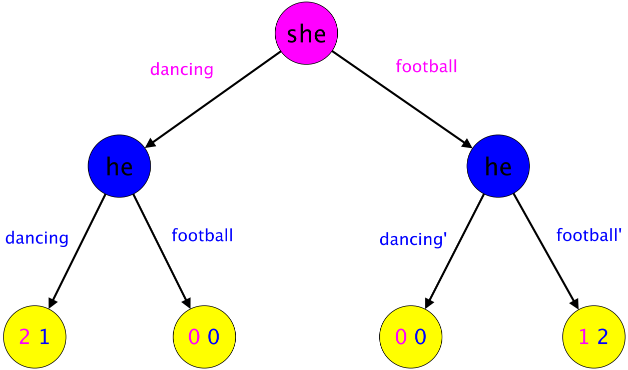
- solve $\equiv$ compute a Nash equilibrium
- each restricted subgame amounts to a finite optimization problem
- multipla maxima could lead to difficulties (perform a choice)

Theorem

Every sequential finite game has at least one subgame perfect equilibrium.

(backward induction provides subgame perfect equilibria)

Sequential battle of sexes

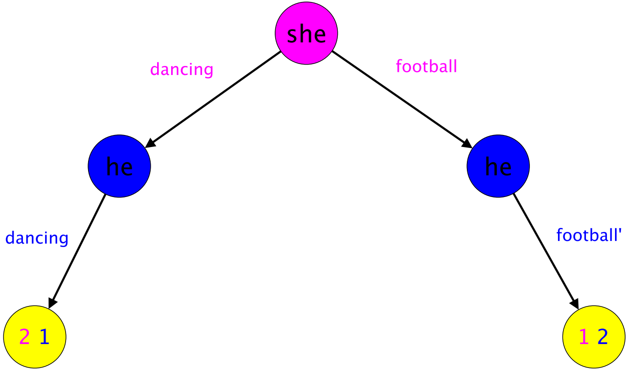


she/he	dd'	df'	fd'	ff'
dancing	(2,1)	(2,1)	(0,0)	(0,0)
football	(0,0)	(1,2)	(0,0)	(1,2)

(d,dd'), (d,df') and (f,ff') are Nash equilibria

what about subgame perfectness?

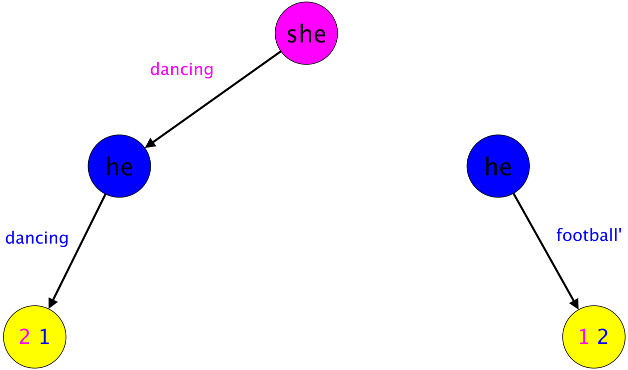
Sequential battle of sexes



she/he	dd’	df’	fd’	ff’
dancing	(2,1)	(2,1)	(0,0)	(0,0)
football	(0,0)	(1,2)	(0,0)	(1,2)

(d,dd’), (d,df’) and (f,ff’) are Nash equilibria

Sequential battle of sexes



she/he	dd'	df'	fd'	ff'
dancing	(2,1)	(2,1)	(0,0)	(0,0)
football	(0,0)	(1,2)	(0,0)	(1,2)

(d,dd'), (d,df') and (f,ff') are Nash equilibria

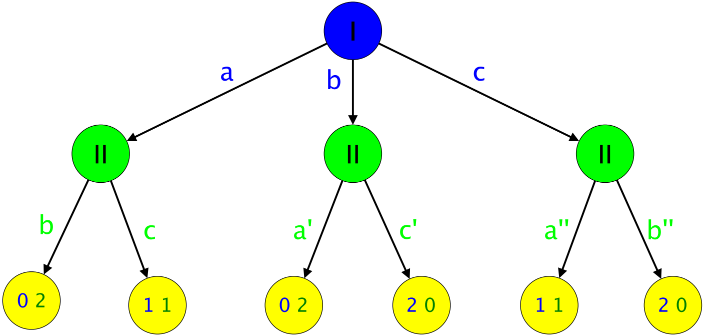
(d,df') is the unique subgame perfect equilibrium

Veto driven choice

2 players to agree a common alternative between a, b and c

preferences: $a \succ b \succ c$ for player 1, $c \succ b \succ a$ for player 2

At its turn each player vetoes an alternative



I/II	ba'a"	ba'b"	bc'a"	bc'b"	ca'a"	ca'b"	cc'a"	cc'b"
a	(0,2)	(0,2)	(0,2)	(0,2)	(1,1)	(1,1)	(1,1)	(1,1)
b	(0,2)	(0,2)	(2,0)	(2,0)	(0,2)	(0,2)	(2,0)	(2,0)
c	(1,1)	(2,0)	(1,1)	(2,0)	(1,1)	(2,0)	(1,1)	(2,0)

Veto driven choice

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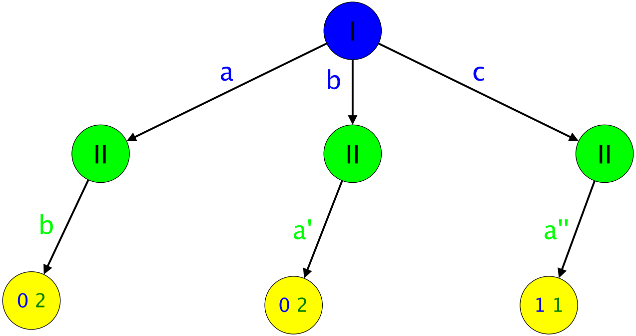


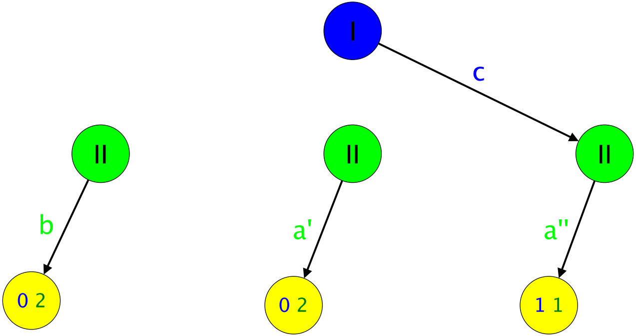
I/II	ba'a"	ba'b"	bc'a"	bc'b"	ca'a"	ca'b"	cc'a"	cc'b"
a	(0,2)	(0,2)	(0,2)	(0,2)	(1,1)	(1,1)	(1,1)	(1,1)
b	(0,2)	(0,2)	(2,0)	(2,0)	(0,2)	(0,2)	(2,0)	(2,0)
c	(1,1)	(2,0)	(1,1)	(2,0)	(1,1)	(2,0)	(1,1)	(2,0)

Veto driven choice

2 players to agree a common alternative between a, b and c

preferences: $a \succ b \succ c$ for player 1, $c \succ b \succ a$ for player 2

At its turn each player vetoes an alternative



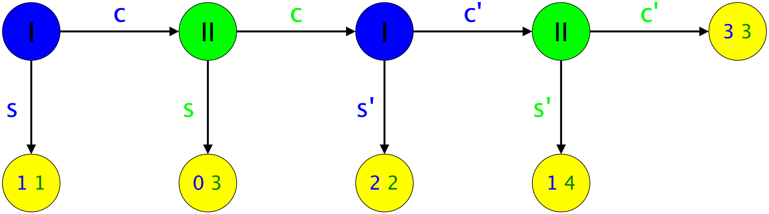
I/II	ba'a"	ba'b"	bc'a"	bc'b"	ca'a"	ca'b"	cc'a"	cc'b"
a	(0,2)	(0,2)	(0,2)	(0,2)	(1,1)	(1,1)	(1,1)	(1,1)
b	(0,2)	(0,2)	(2,0)	(2,0)	(0,2)	(0,2)	(2,0)	(2,0)
c	(1,1)	(2,0)	(1,1)	(2,0)	(1,1)	(2,0)	(1,1)	(2,0)

Centipede game (Rosenthal 1981)

At its turn the player decides to continue or stop the game

continue: 1€ moved to the other player's wallet, that gets 1€ in addition

stop: the game ends with the current wallets



I/II	ss'	sc'	cs'	cc'
ss'	(1,1)	(1,1)	(1,1)	(1,1)
sc'	(1,1)	(1,1)	(1,1)	(1,1)
cs'	(0,3)	(0,3)	(2,2)	(2,2)
cc'	(0,3)	(0,3)	(1,4)	(3,3)

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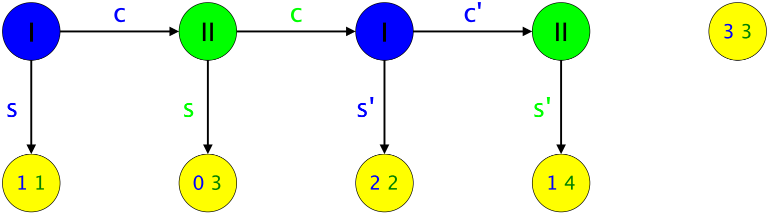

I/II	ss'	sc'	cs'	cc'
ss'	(1,1)	(1,1)	(1,1)	(1,1)
sc'	(1,1)	(1,1)	(1,1)	(1,1)
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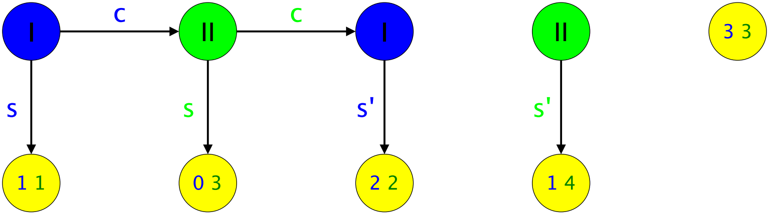

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ss'	(1,1)	(1,1)	(1,1)	(1,1)
sc'	(1,1)	(1,1)	(1,1)	(1,1)
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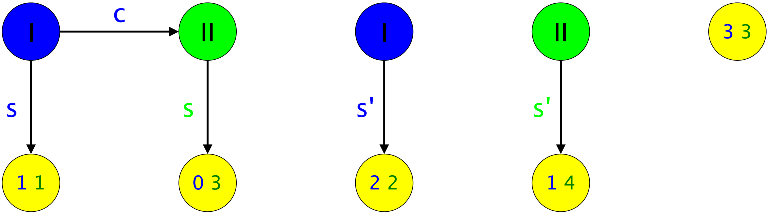

I/II	ss'	sc'	cs'	cc'
ss'	(1,1)	(1,1)	(1,1)	(1,1)
sc'	(1,1)	(1,1)	(1,1)	(1,1)
cs'	(0,3)	(0,3)	(2,2)	(2,2)
cc'	(0,3)	(0,3)	(1,4)	(3,3)

Centipede game (Rosenthal 1981)

At its turn the player decides to continue or stop the game

continue: 1€ moved to the other player's wallet, that gets 1€ in addition

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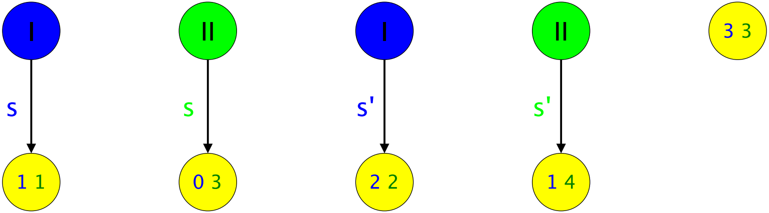


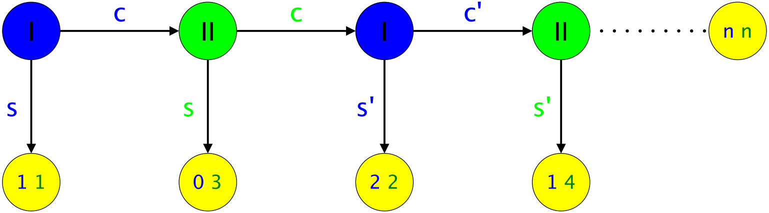
I/II	ss'	sc'	cs'	cc'
ss'	(1,1)	(1,1)	(1,1)	(1,1)
sc'	(1,1)	(1,1)	(1,1)	(1,1)
cs'	(0,3)	(0,3)	(2,2)	(2,2)
cc'	(0,3)	(0,3)	(1,4)	(3,3)

Centipede game: inefficiency

At its turn the player decides to continue or stop the game

continue: 1€ moved to the other player's wallet, that gets 1€ in addition

stop: the game ends with the current wallets



- According to the unique [subgame perfect] equilibrium the game stops immediately
- Cooperation could enforce a(n arbitrarily) larger gain for both players

The chain store paradox (Selten 1978)

A chain store with branches in n towns

n potential independent competitors, one in each town

competitors: *enter the market?* chain store: *cooperate or act aggressively?*

choices are taken one town after the other (with perfect information)

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competitors: *enter the market?* chain store: *cooperate or act aggressively?*

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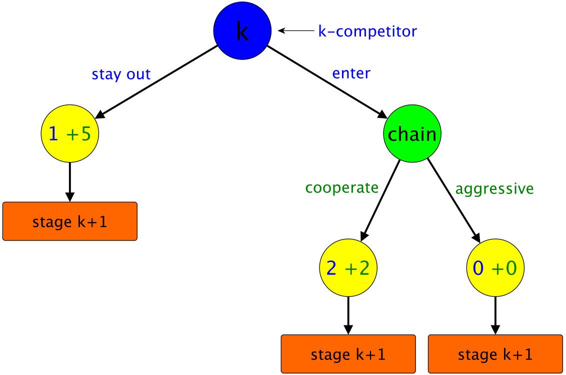

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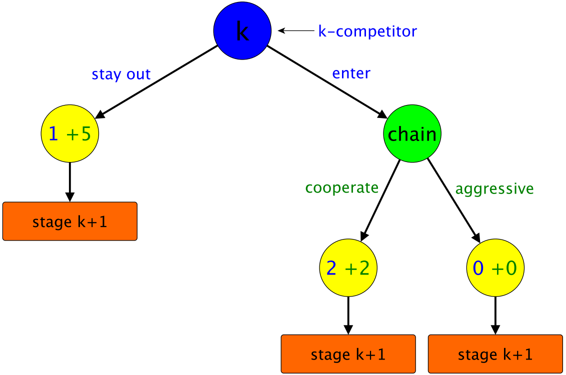
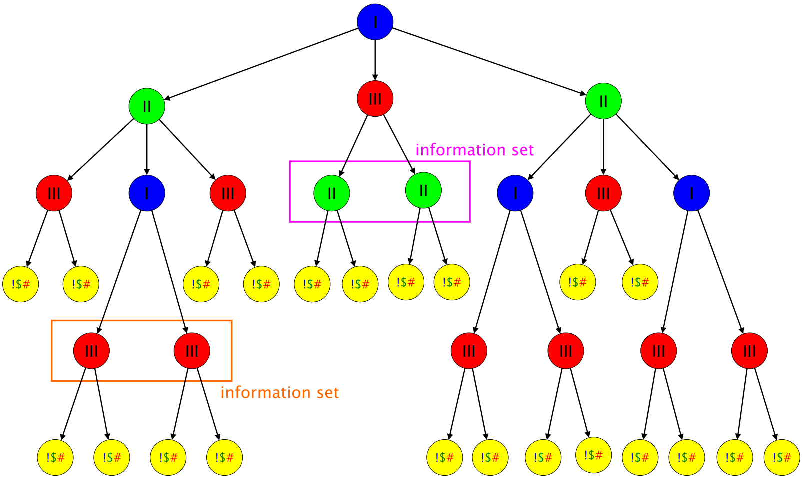


backward induction: cooperate at each stage – human plausible behaviour?

Imperfect information

Not all the previous moves are known

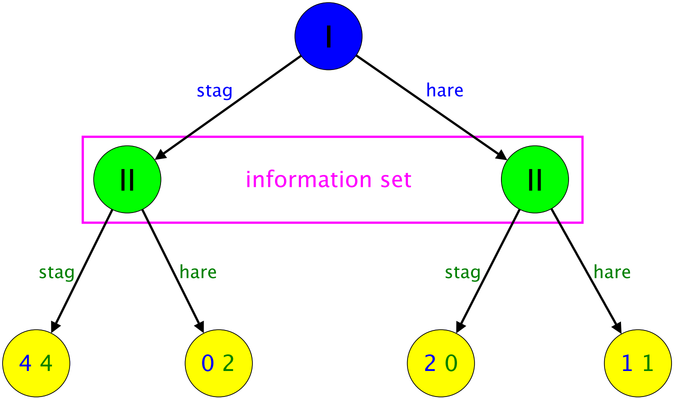


Information set

set of nodes of the same player with same parent and same possible actions

Turning strategic games into sequential games

I/II	stag	hare
stag	(4,4)	(0,2)
hare	(2,0)	(1,1)



Imperfect information and forward induction

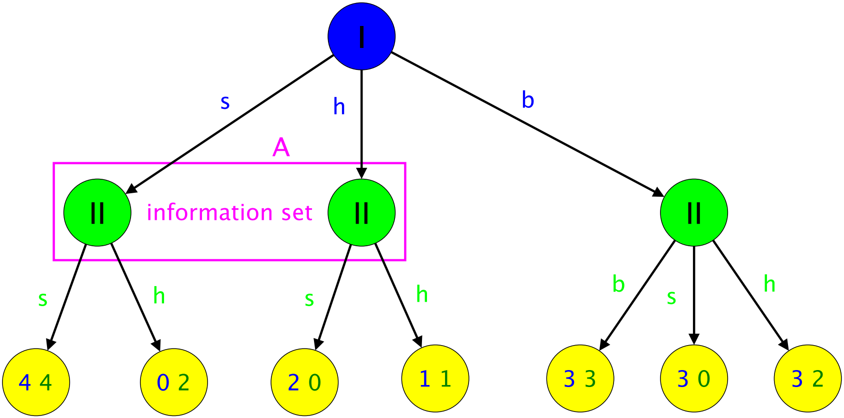
backward induction: future moves will be rational

forward induction: past moves have been rational

Imperfect information and forward induction

backward induction: future moves will be rational

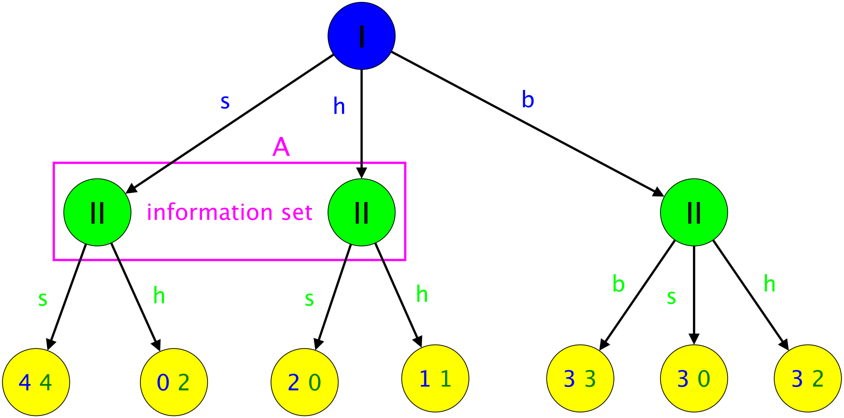
forward induction: past moves have been rational



Imperfect information and forward induction

backward induction: future moves will be rational

forward induction: past moves have been rational



If the information set A has been reached, player 1 has [likely] chosen s