Algorithmic game theory

Laurea Magistrale in Computer Science

2024/25

Lecture 10

Stackelberg duopoly (1934)

2 firms producing the same homogeneous commodity - competition over quantity

$$S_1 = S_2 = [0, +\infty) \qquad u_i(x_1, x_2) = x_i \max\{T - (x_1 + x_2), 0\} - cx_i \qquad (T > c)$$

inverse demand function prod. cost

$\Rightarrow$ firm 1 chooses x_1 first, firm 2 notices the choice and responds [optimally]

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$$u_1(x_1, R_2(x_1)) = \begin{cases} x_1(T - c - x_1)/2 & \text{if } x_1 \leq T - c \\ < 0 & \text{if } x_1 > T - c \end{cases}$$

$$\arg\max\{u_1(x_1, R_2(x_1)) \ : \ x_1 \in S_1\} = \{(T - c)/2\}$$

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$$R_2((T - c)/2) = \{(T - c)/4\}$$

Stackelberg equilibrium: $((T - c)/2, (T - c)/4)$

Monopoly, Cournot & Stackelberg duopolies: a comparison

	production	unitary price	utility per firm	system utility
monopoly	$(T-c)/2$	$(T+c)/2$	$(T-c)^2/4$	$(T-c)^2/4$
	$\wedge$	$\vee$	$\vee$	$\vee$
Cournot	$2(T-c)/3$	$(T+2c)/3$	$(T-c)^2/9$	$2(T-c)^2/9$
	$\wedge$	$\vee$	$\wedge$ $\vee$	$\vee$
Stackelberg	$3(T-c)/4$	$(T+3c)/4$	1 $(T-c)^2/8$ 2 $(T-c)^2/16$	$3(T-c)^2/16$

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Stackelberg	$3(T-c)/4$	$(T+3c)/4$	[1] $(T-c)^2/8$ [2] $(T-c)^2/16$	$3(T-c)^2/16$

Stackelberg duopoly

- dynamic/sequential game (game with successive moves)
- complete information (knowledge of other players' utilities)
- perfect information (knowledge of all the previous moves)

Stackelberg game

- $N = \{1, 2\} \longrightarrow 2$ players

  player 1 - the leader chooses first

  player 2 - the follower reacts to the leader's choice

- S_1, S_2 sets of (available) strategies
- $u_1, u_2 : S_1 \times S_2 \to \mathbb{R}$ utility functions

Working assumption:

$R_2(x_1) = \arg\max\{u_2(x_1, x_2) : x_2 \in S_2\}$ is a singleton for any $x_1 \in S_1$

Stackelberg equilibrium

$x_1^* \in S_1$ *is a Stackelberg solution if*

$$x_1^* \in \arg\max\{u_1(x_1, R_2(x_1)) \;:\; x_1 \in S_1\}$$

$(x_1^*, R_2(x_1^*))$ *is a Stackelberg equilibrium if* x_1^* *is a Stackelberg solution.*

Stackelberg and Nash equilibria may be different

leader/follower	ℓ_2	r_2
ℓ_1	(2,2)	(4,1)
r_1	(1,0)	$(3,\gamma)$

$(\gamma > 0)$

(ℓ_1, ℓ_2) unique Nash equilibrium

(r_1, r_2) unique Stackelberg equilibrium

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Proposition

If $(\bar{x}_1, \bar{x}_2) \in S_1 \times S_2$ is a Nash equilibrium and $x_1^* \in S_1$ is a Stackelberg solution, then

$$u_1(x_1^*, R_2(x_1^*)) \geq u_1(\bar{x}_1, \bar{x}_2).$$

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$$u_1(r_1, r_2) = 3 \geq 2 = u_1(\ell_1, \ell_2)$$

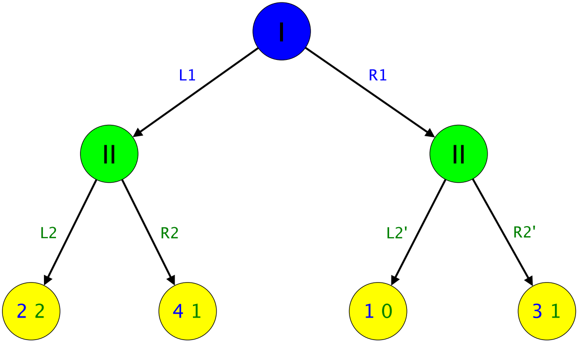
$$u_2(r_1, r_2) = \gamma \gtrless 2 = u_2(\ell_1, \ell_2)$$

(leadership gives some advantage)

Stackelberg versus Nash equilibria in extensive form

Nash equilibria somehow neglect sequential moves

Extensive form suits sequential (finite) games much better



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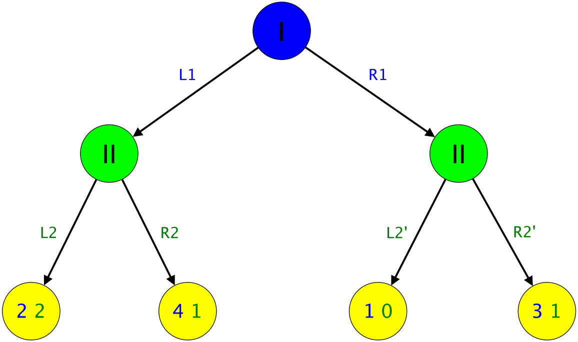

leader/follower	$\ell_2 \ell_2'$	$\ell_2 r_2'$	$r_2 \ell_2'$	$r_2 r_2'$
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r_1	(1,0)	(3,1)	(1,0)	(3,1)

$(\ell_1, \ell_2 \ell_2')$ and $(r_1, \ell_2 r_2')$ are both Nash equilibria

Lack of uniqueness in best replies

leader/follower	ℓ_2	c_2	r_2
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$\longrightarrow$ $u_1(x_1, R_2(x_1))$ is not well-defined if $R_2(x_1)$ is not a singleton

what possible meanings for $\arg\max\{u_1(x_1, R_2(x_1)) \ : \ x_1 \in S_1\}$?

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Optimistic Stackelberg problem •

$$\max\{u_1(x_1, x_2) \; : \; x_1 \in S_1, x_2 \in R_2(x_1)\} \qquad (OS)$$

(hierarchical/bilevel optimization)

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(hierarchical/bilevel optimization)

Pessimistic Stackelberg problem (Leitmann 1978) •

$$\max\{\min\{u_1(x_1, x_2) \ : x_2 \in R_2(x_1)\} \ : \ x_1 \in S_1\} \qquad (PS)$$

(security strategy for the leader)

Existence of Stackelberg equilibria

optimistic Stackelberg equilibria $\equiv$ maximum points (x_1^*, x_2^*) of (OS)

Theorem (Simaan-Cruz 1973)

Let $(\{1,2\}, \{S_1, S_2\}, \{u_1, u_2\})$ *be a Stackelberg game.*

If each player $i \in \{1,2\}$ *satisfies*

(i) $S_i \subseteq \mathbb{R}^{m_i}$ *is compact*

(ii) u_i *is continuous on* $S_1 \times S_2$

then the game has at least one optimistic Stackelberg equilibrium.

((OS) satisfies the assumptions of Weierstrass extreme value theorem)

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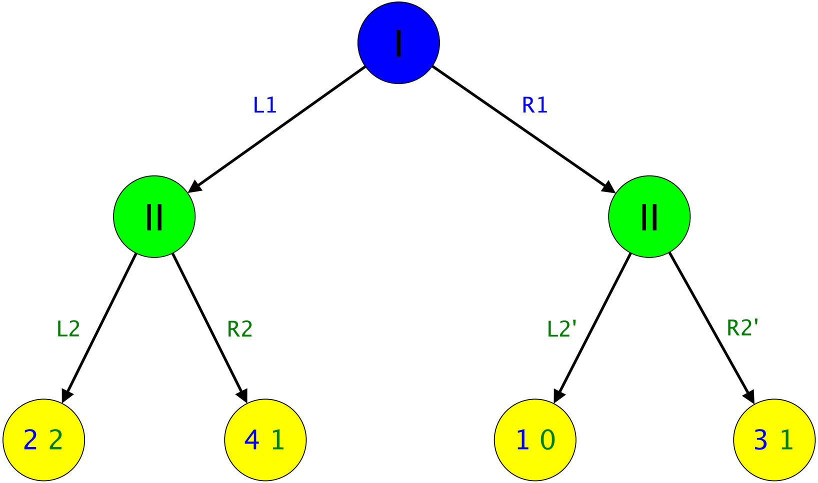
pessimistic Stackelberg equilibria: continuity + compactness $\not\Longrightarrow$ existence

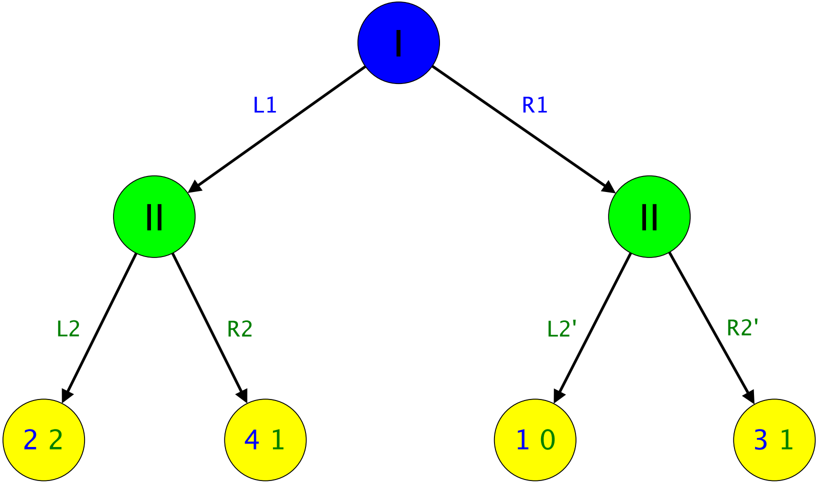
$(S_1 = S_2 = [-1,1], \quad u_1(x_1,x_2) = x_1 - x_2, \quad u_2(x_1,x_2) = x_1 x_2)$

Stackelberg games drive to backward induction



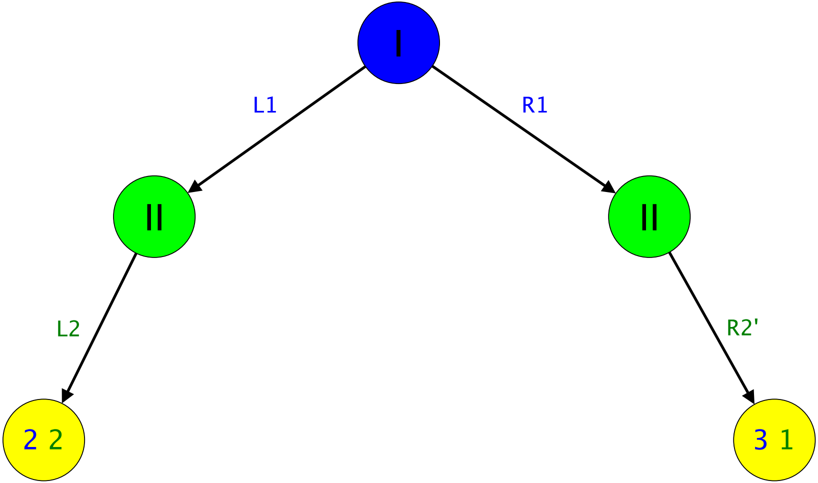
Stackelberg games drive to backward induction

the leader may anticipate the follower's responses



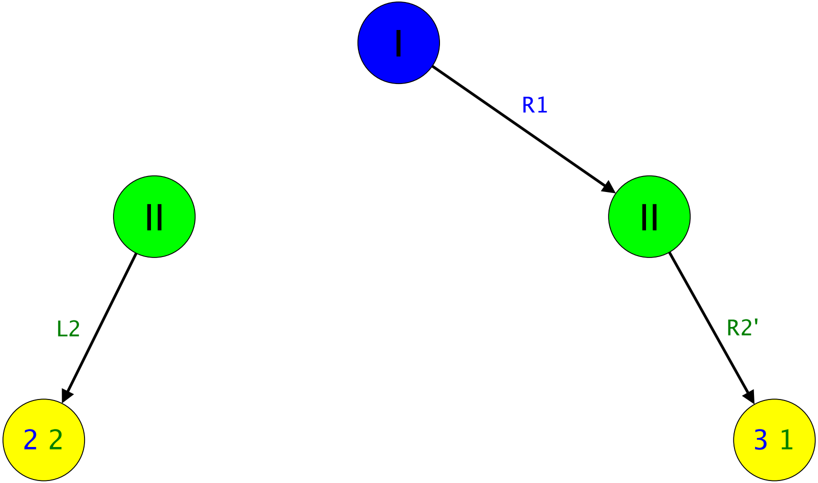
Stackelberg games drive to backward induction

non-optimal responses of the follower are deleted



Stackelberg games drive to backward induction

the leader performs the optimal choice in the restricted game



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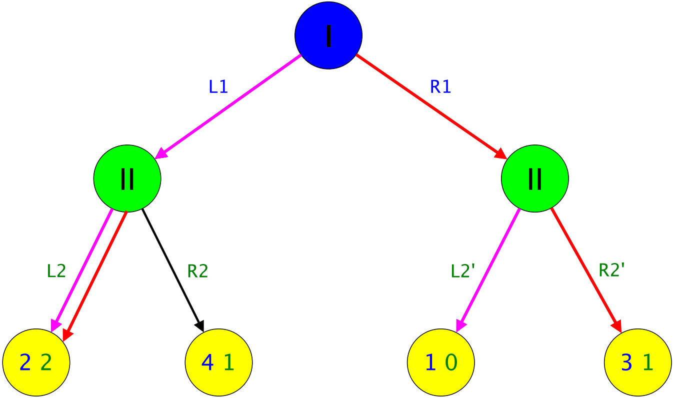

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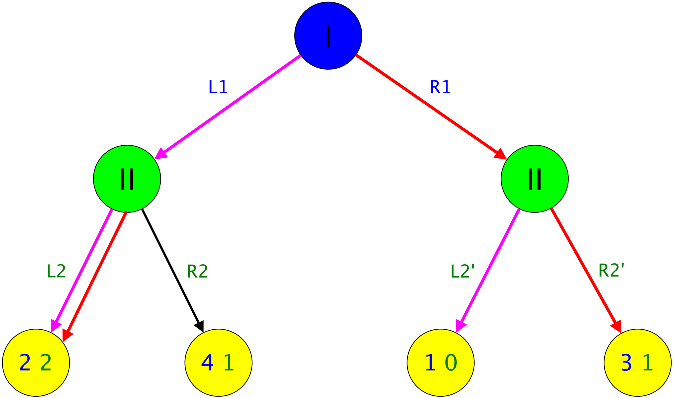

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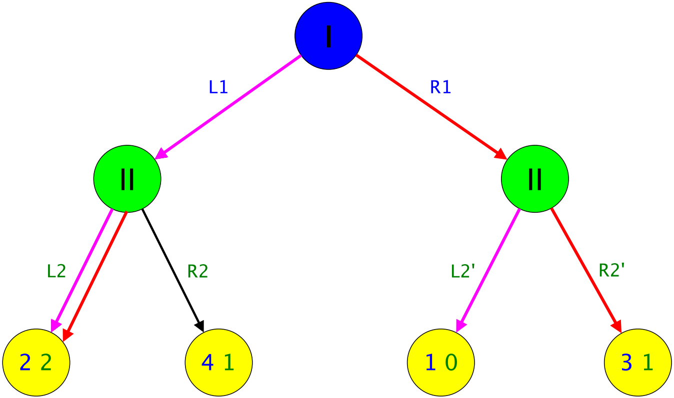

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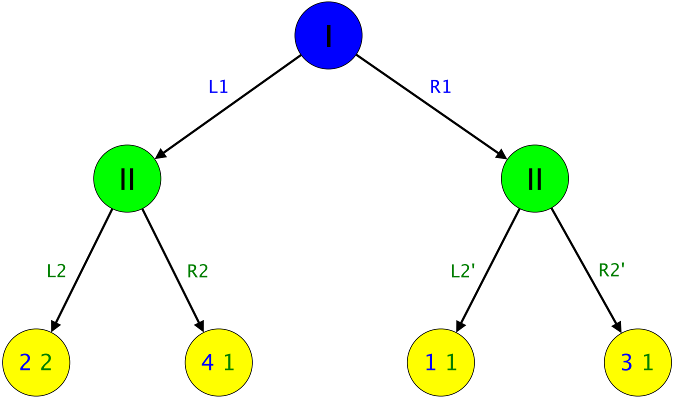
ℓ'_2 is not the best choice for the follower if its tail node is reached

$(\ell_1, \ell_2 \ell'_2)$ is not "subgame perfect"

Backward induction with nonunique best replies



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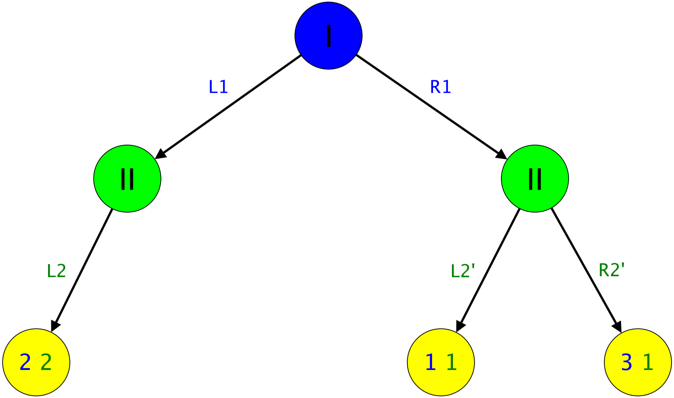
Backward induction with nonunique best replies: failure



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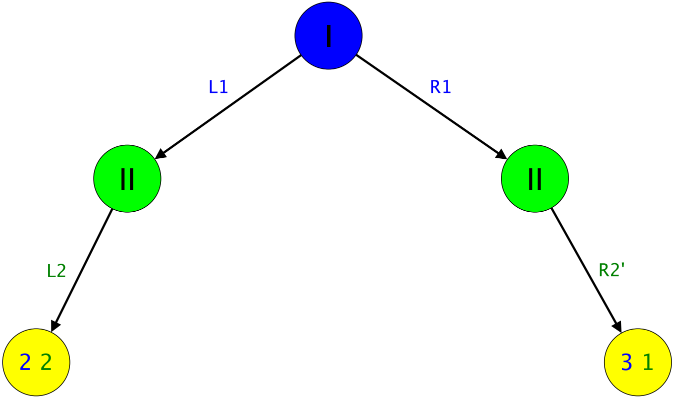
Backward induction with an optimistic attitude



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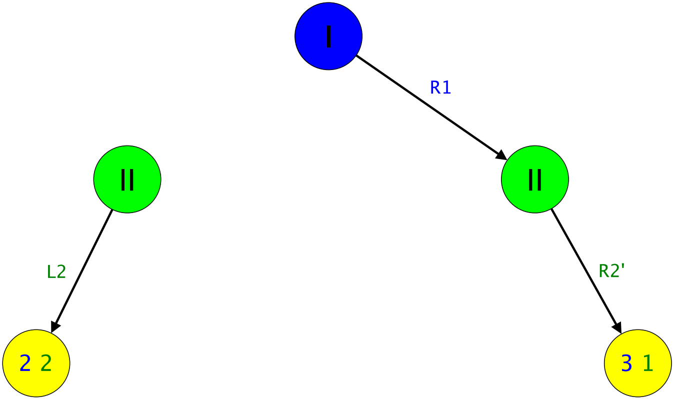

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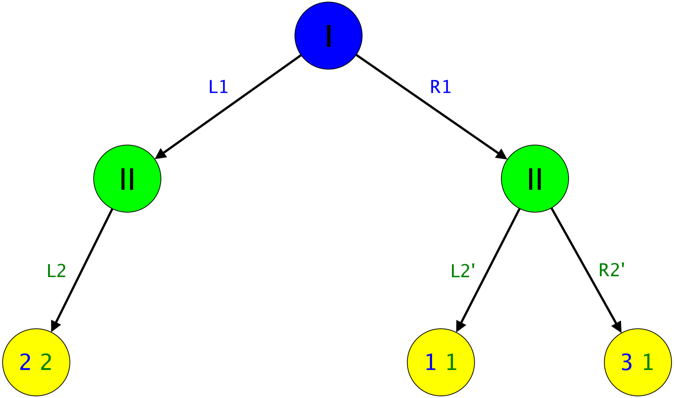

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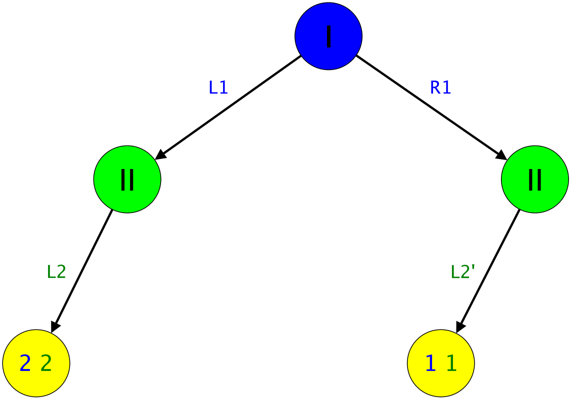

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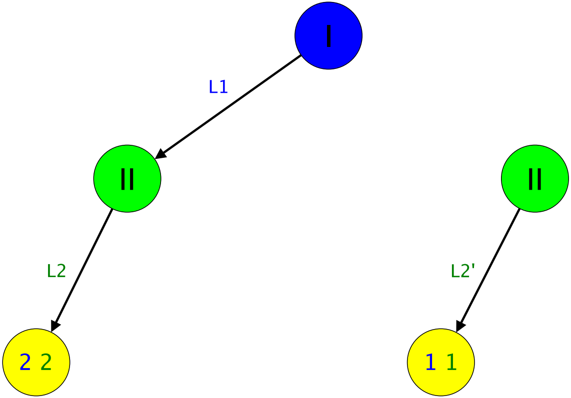

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Stackelberg duopoly: indivisible commodity

$x_i \in \mathbb{Z}_+$ units of the commodity to be produced $\longrightarrow$ at most $(T-c)-1$

$$u_i(x_1, x_2) = x_i \max\{T - (x_1 + x_2), 0\} - c x_i$$

1 = leader   2 = follower

Example: $T = 10$, $c = 3$

I/II	⓪	①	②	③	④	⑤	⑥
⓪	(0,0)	(0,6)	(0,10)	(0,12)	(0,12)	(0,10)	(0,6)
①	(6,0)	(5,5)	(4,8)	(3,9)	(2,8)	(1,5)	(0,0)
②	(10,0)	(8,4)	(6,6)	(4,6)	(2,4)	(0,0)	(-2,-6)
③	(12,0)	(9,3)	(6,4)	(3,3)	(0,0)	(-3,-5)	(-6,-12)
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the leader anticipates the follower's responses

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Example: $T = 10$, $c = 3$

I/II	⓪	①	②	③	④	⑤	⑥
⓪				(0,12)	(0,12)		
①				(3,9)			
②			(6,6)	(4,6)			
③			(6,4)				
④		(8,2)	(4,2)				
⑤		(5,1)					
⑥	(6,0)	(0,0)					

the leader anticipates the follower's responses

Stackelberg duopoly: indivisible commodity

$x_i \in \mathbb{Z}_+$ units of the commodity to be produced $\longrightarrow$ at most $(T-c)-1$

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