Computational complexities of inclusion queries over polyhedral sets

Pavlos Eirinakis *
LDCSEE
West Virginia University
Morgantown, WV
pavlos.eirinakis@mail.wvu.com

Salvatore Ruggieri
Dipartimento di Informatica
Universita' di Pisa
Pisa, Italy
ruggieri@di.unipi.it

K. Subramani *
LDCSEE
West Virginia University
Morgantown, WV
ksmani@csee.wvu.edu

Piotr Wojciechowski *
LDCSEE
West Virginia University
Morgantown, WV
pwojciec@mix.wvu.edu

Abstract

In this paper we discuss the computational complexities of procedures for inclusion queries over polyhedral sets. The polyhedral sets that we consider occur in a wide range of applications, ranging from logistics to program verification. The goal of our study is to establish boundaries between hard and easy problems in this context.

1 Introduction

Extending linear constraints by admitting parameters allows for more abstract problem modelling and reasoning. However, how does the basic notion of constraint entailment extend in the presence of parameters? Does its computational complexity become harder or intractable? In this paper, we define entailment of parametric linear constraints and investigate its computational complexity.

Let us provide some intuition. We adhere to standard notation of linear algebra (Schrijver 1987). $\Re$ is the set of real numbers. Small capital letters ($\mathbf{a}$, $\mathbf{b}$, ...) denote column vectors, while capital letters ($\mathbf{A}$, $\mathbf{B}$, ...) denote matrices. $\mathbf{0}$ is the column vector with all elements equal to 0. a_i denotes the i^{th} element in $\mathbf{a}$. $\mathbf{a}^T$ denotes the transposed vector of $\mathbf{a}$. $\cdot$ denotes the inner product.

Consider two linear constraints (or, equivalently, two linear systems of inequalities) over the reals $c_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ and $c_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}$. c_1 entails c_2 if every solution of c_1 is a solution of c_2, or, in logic formula, if $\forall \mathbf{x} \ \ [c_1 \rightarrow c_2]$ is true in the domain of the reals. Entailment for linear constraints is a polynomial time decision problem (Subramani 2009).

A parameterized linear constraint over the reals is a system of linear inequalities $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ where variables in $\mathbf{s}$ are called *parameters*. We say that a parameterized linear constraint $cp_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ entails $cp_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ if for every parameter instance $\mathbf{s}_0$ of $\mathbf{s}$ there exists an instance $\mathbf{r}_0$ of $\mathbf{r}$ such that the linear system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}_0$ entails $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0$. Note that cp_1 will be referred to as the entailing polyhedral set, while cp_2 as the entailed polyhedral set.

The rest of the paper is organized as follows. Section 2 discusses basic notions that will be used and the contribution of the paper. Section 3 details the motivation and related work. The complexities of different classes of entailment is investigated in Sections 4, 5, and 6. Section 7 concludes the paper.

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2 Background and Contribution

A simple (unquantified) linear system is denoted as:

$$\exists \mathbf{x} \in \Re \ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \tag{1}$$

We assume that the dimensions of vectors and matrices in inner products and linear systems are of the appropriate size. Observe that System (1) represents a convex, polyhedral set (Schrijver 1987).

An equivalent formulation of linear systems is provided in terms of logic formulas over the reals. A *primitive linear constraint* is an expression $a_1 \cdot x_1 + \ldots a_n \cdot x_n \leq a_0$, where $a_0, \ldots, a_n$ are constants in $\Re$ and $x_1, \ldots, x_n$ are variables. We will also use the inner product form by rewriting it as $\mathbf{c}^T \cdot \mathbf{x} \leq \alpha$. A *linear constraint* c is a conjunction of primitive constraints. Conjunction is syntactically represented either by "$\wedge$" or by a comma. Inequalities $\mathbf{c}^T \cdot \mathbf{x} \geq \alpha$ and equalities $\mathbf{c}^T \cdot \mathbf{x} = \alpha$ can be reduced to linear constraints.

A system of linear inequalities $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{B} \cdot \mathbf{s}$ is a parameterized linear system over $\Re$; variables in $\mathbf{s}$ are referred to as *parameters*. The semantics of a parameterized linear system is then a collection of linear systems over variables in $\mathbf{x}$, each obtained by instantiating the parameters $\mathbf{s}$. The solutions of parameterized linear systems are modeled by the notion of parameterized polyhedra (Loechner and Wilde 1997).

Quantified Linear Programming (QLP) was introduced in (Subramani 2007). A *quantified linear program* is a linear system in which the program variables can be either existentially quantified or universally quantified. A typical quantified linear program will have the form:

$$\exists x_1 \ \forall y_1 \in [l_1, u_1] \ldots \exists x_n \ \forall y_n \in [l_n, u_n] \qquad \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}$$

where, possibly, x_1 and/or y_n are empty, and l_i, u_i are lower and upper bounds in the reals for $y_i, i = 1, \ldots, n$. An example of a quantified linear program is the following:

$$\exists x_1 \ \forall y_1 \in [1,4] \ \exists x_2 \qquad \begin{array}{r} x_1 + y_1 + x_2 \geq 4 \\ 3x_1 - 5y_1 \leq -5 \end{array}$$

Two special cases of the QLP problem have also been studied in (Subramani 2007). In particular, the **E**-QLP problem, which is of the form $\exists \mathbf{x}\ \forall \mathbf{y} \in [\mathbf{l}, \mathbf{u}]\ \mathbf{A}\mathbf{x} \leq \mathbf{b}$ (shown to be in **P**), and the **F**-QLP problem, which is of the form $\forall \mathbf{y} \in [\mathbf{l}, \mathbf{u}]\ \exists \mathbf{x}\ \mathbf{A}[\mathbf{x}\ \mathbf{y}] \leq \mathbf{b}$ (shown to be **coNP-complete**). Note that $\mathbf{y} \in [\mathbf{l}, \mathbf{u}]$ denotes $l_i \leq y_i \leq u_i, \forall i = 1, \ldots, n$.

We extend the quantification to implications of linear systems. A typical *quantified linear implication* has the following form:

$$\exists x_1\ \forall y_1\ \exists x_2\ \forall y_2 \ldots \exists x_n\ \forall y_n \quad \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b} \quad \rightarrow \quad \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \leq \mathbf{d}$$

where, possibly, x_1 and/or y_n are empty. Let us introduce a nomenclature to succinctly specify the problems under consideration. We use a triple $\langle A, Q, R \rangle$ to denote the number A of quantifier alternations, the first quantifier Q , and an $(A+1)$-character string which specifies for each quantified set of variables whether they possibly appears **L**eft, **R**ight, or on **B**oth sides of the implication. For instance, $\langle 1, \exists, \mathbf{RB} \rangle$ indicates a problem of the form:

$$\exists \mathbf{y}\ \forall \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \leq \mathbf{d}]$$

A quantified linear implication is said to *hold* if it is true as a first-order sentence over the real numbers.

The principal contributions of this paper are briefly presented below:

- Problem $\langle 1,\ \exists,\ \mathbf{RB} \rangle$ is in **P** (Section 4).
- Problem $\langle 1,\ \exists,\ \mathbf{LB} \rangle$ is **NP-hard** (Section 5).
- Problem $\langle 2,\ \forall,\ \mathbf{LRB} \rangle$ is **coNP-hard** (Section 6).
- Problem $\langle 2,\ \forall,\ \mathbf{LRB} \rangle$ is reducible to problem $\langle 2,\ \forall,\ \mathbf{BRB} \rangle$ and vice versa (Section 6).
- A special case of problem $\langle 2,\ \forall,\ \mathbf{LRB} \rangle$ is presented which is still **coNP-hard** (Section 6.1).
- Several sub-classes of problem $\langle 2,\ \forall,\ \mathbf{LRB} \rangle$ are shown to be in **P** (Section 6.2).

3 Motivation and Related Work

Parameterized linear systems are an extension of linear systems that trace back to late 60's in the context of (multi-)parametric linear programming, where the objective is to optimize a parameterized linear or quadratic function over the solutions of a parameterized linear system. Parametric linear programming differentiates from sensitivity analysis, which provides solutions in the neighborhood of the nominal value of the varying parameters, by providing a complete map of the optimal solution in the space of the varying parameters. Parametric linear programming has a solid theoretical basis (see (Gal 1995; Pistikopoulos, Georgiadis, and Dua 2007b)), and a wide range of applications, including hybrid parametric/stocastic programming, process planning under uncertainty, material design under uncertainty, model based control, multi-processor scheduling (see (Kvasnica 2009; Pistikopoulos, Georgiadis, and Dua 2007a)).

Besides linear programming problems, parameterized systems are recently gaining interest from several other research communities. They have been adopted for: symbolic dependence analysis in computing a parallel schedule ((Loechner and Wilde 1997)); fracture mechanics and engineering ((Ioakimidis 2000; Solares and Chaves 2008)); hardware verification ((Cachera and Morin-Allory 2005)); type systems for constraint programming ((Ruggieri and Mesnard 2010)).

The explicit calculation of the solutions of a parameterized system of inequalities, represented in a Minkowski's sum of rays and parameterized vertices, can be computed by a generalization of the double description method, as described in (Loechner and Wilde 1997). The `polylib` library (Loechner 2011) implements the approach, and it provides basic set-oriented primitives for manipulating the solutions of parameterized linear systems. In the context of parametric linear programming, The Multi-Parametric Toolbox ((Kvasnica 2009; Kvasnica et al. 2011)) solves parametric linear and quadratic programming problems by explicitly computing the optimization function over the solutions of a parameterized linear system.

While the above research and tools demonstrate the usefulness of parameterized linear systems, to the best of our knowledge, there is no approach that consider primitives for reasonings over them, and in particular entailment. As an example, linear constraint-based languages and systems (such as constraint logic programming systems CLP($\mathcal{R}$), ECLiPSe, Sictus Prolog, SWI Prolog) could be readily extended to reason on parameterized linear systems by using satisfiability and entailment as building blocks. It is then of outermost importance to understand and characterize computational complexity of the entailment problem for parameterized linear constraints. This basic issue motivates our work.

Polynomial time complexity for the satisfiability of linear systems was firstly achieved by (Khachiyan 1979), with further improvements by (Karmarkar 1984) and (Vaidya 1987). Entailment of non-parameterized linear systems is in $\mathbf{P}$. A formal proof of the intuition provided in the introduction is due to (Subramani 2009). Unfortunately, a direct extension of the procedure to parameterized linear system by using parametric linear programming does not work. In fact, the optimal solution of a parametric linear programming problem is a piecewise affine function defined over a polyhedral partition of the feasible parameters (Borrelli, Bemporad, and Morari 2003). Such a function can be exponentially large in the size of the linear program, even if there is only one parameter (Murty 1980).

Computational complexity of satisfiability and entailment has been considered also in close domains (without parameters). Satisfiability of quantified linear systems is covered by (Subramani 2007). The integer case is detailed in (Subramani 2005). Quantified constraint satisfaction problems are investigated by (Gottlob, Greco, and Scarcello 2005). Entailment of binary and multi-valued propositional logics is investigated by (Cadoli and Schaerf 1996). (Subramani 2005) covers entailment of linear systems over integers.

Quantified logic formulas over the reals can be solved by quantifier elimination methods (see (Dolzmann, Sturm, and Weispfenning 1998b; Weispfenning 1988)). Although its complexity has been improved several times ((Renegar 1992; Basu, Pollack, and Roy 1996)), in the worst case quantifier elimination is doubly exponential in the number of quantifier alternation and exponential in the number of variables (see (Davenport and Heintz 1988; Weispfenning 1988)). Nevertheless, efficient-in-practice approaches have been proposed and successfully applied to theorem proving and program verification. We mention partial cylindrical algebraic decomposition by (Collins and Hong 1991) and provided in the QEPCAD/QEPCAD-B systems ((Brown 2003)); virtual substitution of test terms by (Dolzmann, Sturm, and Weispfenning 1998a) provided in the REDLOG system ((Dolzmann and Sturm 1997)), which is specialized for low-degree polynomials; and, finally, the RSolver algorithm by (Ratschan 2006) whose implementation is publicly available ((Ratschan 2011)).

4 Problem $\langle 1,\ \exists,\ \mathbf{RB} \rangle$

First, we analyze the complexity of a quantified linear implication of the form $\langle 1,\ \exists,\ \mathbf{RB} \rangle$, i.e., problems of the form:

$$\exists \mathbf{r}\ \forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}] \tag{2}$$

This implication can be read as whether there exist parameter instances $\mathbf{r}$ such that the instance of the *parameterized* polyhedron $\mathbf{C}{\cdot}\mathbf{x} \leq \mathbf{d}{+}\mathbf{M}{\cdot}\mathbf{r}$ includes the *non-parameterized* one $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$. We will show that this problem is in $\mathbf{P}$. The following observation is crucial to our proof.

Lemma 4.1 *The set* $R = \{\mathbf{r} : \forall\ \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}]\}$ *is convex.*

Proof: Assume otherwise and consider the set $H = \{\mathbf{x} : \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\}$, and the set $K_{\mathbf{r}} = \{\mathbf{x} : \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}\}$. Let $\mathbf{r_1}, \mathbf{r_2} \in R$ and let $\mathbf{r_3}$ be a convex combination of $\mathbf{r_1}$ and $\mathbf{r_2}$, i.e., for some $0 \leq \alpha \leq 1$, $\mathbf{r_3} = \alpha\mathbf{r_1} + (1-\alpha)\mathbf{r_2}$. Further, assume that $\mathbf{r_3} \notin R$, thus we have that $H \subseteq K_{\mathbf{r_1}}$ and $H \subseteq K_{\mathbf{r_2}}$ but $H \not\subseteq K_{\mathbf{r_3}}$. Thus there exists an $\mathbf{x_0}$ such that $\mathbf{x_0} \in H$ (which also implies that $\mathbf{x_0} \in K_{\mathbf{r_1}}$ and $\mathbf{x_0} \in K_{\mathbf{r_2}}$), while $\mathbf{x_0} \notin K_{\mathbf{r_3}}$.

Let us consider the linear program (LP): $\mathbf{C} \cdot \mathbf{x_0} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ with respect to $\mathbf{r}$. Since $\mathbf{x_0} \in K_{\mathbf{r_1}}$ and $\mathbf{x_0} \in K_{\mathbf{r_2}}$ we have that $\mathbf{r_1}$ and $\mathbf{r_2}$ must satisfy this LP. However, since the solution space to an LP is convex, $\mathbf{r_3}$ must also solve this LP, which contradicts the assumption that $\mathbf{r} \notin R$. Hence, R is convex. □

The next theorem completes our proof.

Theorem 4.1 *The problem* $\langle 1,\ \exists,\ \mathbf{RB} \rangle$ *is in* $\mathbf{P}$.

Proof: We want to show that quantified linear implications of the form (2) are in $\mathbf{P}$.

First, note that if $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ is infeasible, the formula is always true.

Otherwise, we build an LP on $\mathbf{r}$ as follows. For every row $\mathbf{c_i}$ of $\mathbf{C}$, let d'_i be the solution to the LP:

$$\begin{gathered}\max\ \mathbf{c_i}^T \cdot \mathbf{x} \\ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\end{gathered}$$

Let d_i be the i^{th} element of $\mathbf{d}$. Then, if d'_i is infinite, there is no value of $\mathbf{r}$ for which

$$\mathbf{c_i}^T \cdot \mathbf{x} \leq d_i + \mathbf{m_i}^T \cdot \mathbf{r}$$

is true for all $\mathbf{x}$ that satisfy $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$. Thus, if any of the d'_i are infinite, System (2) is infeasible.

Consider now the case in which all the d'_i are finite; let vector $\mathbf{g} = \mathbf{d'} - \mathbf{d}$. We create the following LP:

$$\mathbf{M} \cdot \mathbf{r} \geq \mathbf{g} \tag{3}$$

If this LP is feasible, there exists a vector $\mathbf{r'}$ such that $\mathbf{M} \cdot \mathbf{r'} + \mathbf{d} \geq \mathbf{d'}$. Consider the i^{th} constraint in this system, i.e., $\mathbf{m_i}^T \cdot \mathbf{r'} + d_i \geq d'_i$. By construction

$$\forall \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{c_i}^T \cdot \mathbf{x} \leq d'_i]$$

Thus, we have that

$$\forall \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{c_i}^T \cdot \mathbf{x} \leq \mathbf{m_i}^T \cdot \mathbf{r'} + d_i]$$

Since this holds for each constraint, we have that

$$\forall \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{M} \cdot \mathbf{r'} + \mathbf{d}]$$

Hence, System (2) holds if System (3) is feasible.

We will show that if all d'_is are finite, System (3) is always feasible (and thus System (2) holds). Recall that for system (2) to hold, either $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ is infeasible or we can construct vector $\mathbf{d'}$ and there exists a vector $\mathbf{r'}$ such that

$$\forall \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{M} \cdot \mathbf{r'} + \mathbf{d}]$$

Let us consider the i^{th} constraint of $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{M} \cdot \mathbf{r'} + \mathbf{d}$, namely $\mathbf{c_i}^T \cdot \mathbf{x} \leq \mathbf{m_i}^T \cdot \mathbf{r'} + \mathbf{d}$. Then we have that

$$\forall \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{c_i}^T \cdot \mathbf{x} \leq \mathbf{m_i}^T \cdot \mathbf{r'} + \mathbf{d}]$$

Thus

$$\max_{\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}} (\mathbf{c_i}^T \cdot \mathbf{x}) \leq \mathbf{m_i}^T \cdot \mathbf{r'} + d_i$$

But, by construction, $d'_i = \max_{\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}} (\mathbf{c_i}^T \cdot \mathbf{x})$ and so $d'_i \leq \mathbf{m_i}^T \cdot \mathbf{r'} + d_i$. Since this holds for every constraint, System (3) is always satisfiable.

To complete the proof, note that in order to decide System (2), we first solve $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ (which is in P). If it is infeasible, System (2) holds. If it is feasible, we then solve at most n LPs $\max_{\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}} \mathbf{c_i}^T \cdot \mathbf{x}, i = 1, 2, ..., n$, where n is the number of constraints in $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$. If all such LPs have a finite solution, System (2) holds; otherwise, it does not hold. Since n is finite and each LP is solvable in $\mathbf{P}$, this part is also in $\mathbf{P}$. The result follows. □

5 Problem $\langle 1,\ \exists,\ \mathbf{LB} \rangle$

Next, we examine quantified linear implications of the form $\langle 1,\ \exists,\ \mathbf{LB} \rangle$, i.e., problems of the form:

$$\exists \mathbf{r}\ \ \forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{r} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}] \tag{4}$$

We will show that such implications are **NP-Hard** by proving that 3SAT can be reduced to it.

Theorem 5.1 *The problem* $\langle 1,\ \exists,\ \mathbf{LB} \rangle$ *is* **NP-Hard**

Proof: We want to show that quantified linear implications of the form (4) are **NP-hard**.

Given a 3SAT instance ϕ, we wish to produce a corresponding implication of the form $\langle 1,\ \exists,\ \mathbf{LB} \rangle$ which holds if and only if ϕ is satisfiable. Let E represent the set of constraints $\mathbf{A} \cdot \mathbf{x} \le \mathbf{b} + \mathbf{N} \cdot \mathbf{r}$ and F the set of constraints $\mathbf{C} \cdot \mathbf{x} \le \mathbf{d}$. Consider an instance of 3SAT in Conjunctive Normal Form with n variables $y_i, i \in \{1, ..., n\}$ and m clauses $\phi_j, j = \{1, ..., m\}$, i.e., a 3SAT formula $\phi = \phi_1 \wedge \phi_2 \wedge ... \wedge \phi_m$ on the literals $\{y_1, \bar{y}_1, y_2, \bar{y}_2, ..., y_n, \bar{y}_n\}$. We reduce it to an implication of the form $\langle 1, \ \exists, \ \mathbf{LB} \rangle$ as follows:

For each variable x_i in the instance of 3SAT we add an existentially quantified variable r_i and a universally quantified variable x_i. We also add the constraints $x_i \le r_i$ and $x_i \le 1 - r_i$ to E and the constraint $x_i \le 0$ to F. These constraints are equivalent to $x \le \min(r_i, 1 - r_i) \rightarrow x_i \le 0$. This forces $r_i \ge 1$ or $r_i \le 0$.

For each clause ϕ_j in the instance of 3SAT, we add the universally quantified variable x_{n+j} and the constraint $x_{n+j} \ge 1$ to F. Then, depending on the form of the clause ϕ_j, we do one of the following:

1. If $\phi_j = (y_i, y_k, y_l)$, we add the constraint $x_{n+j} \ge r_i + r_k + r_l$ to E.
2. If $\phi_j = (y_i, y_k, \bar{y_l})$, we add the constraint $x_{n+j} \ge r_i + r_k + (1 - r_l)$ to E.
3. If $\phi_j = (y_i, \bar{y_k}, \bar{y_l})$, we add the constraint $x_{n+j} \ge r_i + (1 - r_k) + (1 - r_l)$ to E.
4. If $\phi_j = (\bar{y_i}, \bar{y_k}, \bar{y_l})$, we add the constraint $x_{n+j} \ge (1 - r_i) + (1 - r_k) + (1 - r_l)$ to E.

It is obvious that the resultant implication is of the form $\langle 1, \ \exists, \ \mathbf{LB} \rangle$. For example, the following instance of 3SAT

$$(x_1, x_2, \bar{x_3}), (\bar{x_2}, x_3, \bar{x_4})$$

becomes

$$\begin{array}{rcr} \multicolumn{3}{l}{\exists r_1 \, \exists r_2 \, \exists r_3 \, \exists r_4 \, \forall x_1 \, \forall x_2 \, \forall x_3 \, \forall x_4 \, \forall x_5 \, \forall x_6} \\ x_1 \le r_1, \ x_1 \le 1 - r_1 & & x_1 \le 0 \\ x_2 \le r_2, \ x_2 \le 1 - r_2 & & x_2 \le 0 \\ x_3 \le r_3, \ x_3 \le 1 - r_3 & \rightarrow & x_3 \le 0 \\ x_4 \le r_4, \ x_4 \le 1 - r_4 & & x_4 \le 0 \\ x_5 \ge r_1 + r_2 + (1 - r_3) & & x_5 \ge 1 \\ x_6 \ge (1 - r_2) + r_3 + (1 - r_4) & & x_6 \ge 1 \end{array}$$

Consider an assignment y to the clause set ϕ which satisfies ϕ_j for each $j \in 1, ..., m$. This means that at least one of the literals in each ϕ_j is true. Without loss of generality, assume that y_k is in ϕ_j and $y_k = \mathbf{true}$. In the resultant $\langle 1, \ \exists, \ \mathbf{LB} \rangle$ instance, we set $r_k = 1$. This forces x_{n+j} to be at least 1, in order to satisfy its corresponding constraint (see 1-4 above); for example, $x_{n+j} \ge r_i + r_k + (1 - r_l)$. Hence, whenever an assignment satisfies ϕ_j, then also $x_{n+j} \ge 1$, thus satisfying the corresponding constraint of the set of constraints F. Moreover, since $r_i \in \{0, 1\}$, we have $x_i \le 0$ for all $i \in \{1, ..., n\}$, thus satisfying the corresponding constraint of F ($x_i \le 0$). Therefore, if ϕ is satisfiable, the resultant $\langle 1, \ \exists, \ \mathbf{LB} \rangle$ implication holds.

On the other hand, consider an assignment y' to the clause set ϕ which does not satisfy ϕ_j (and hence ϕ). This means that the constraint corresponding to ϕ_j, for example $x_{n+j} \ge r_i + r_k + (1 - r_l)$, will set $x_{n+j} \ge 0$. But then the resultant $\langle 1, \ \exists, \ \mathbf{LB} \rangle$ implication does not hold, since there exists x_{n+j}, with $0 \le x_{n+j} < 1$, such that E is feasible and F is infeasible.

Now consider the case that the resultant $\langle 1, \ \exists, \ \mathbf{LB} \rangle$ implication holds. Since each r_i satisfies $r_i \ge 1$ or $r_i \le 0$, the constraints 1-4 given above on x_{n+j} are equivalent to forcing each one of the $r_i \ge 1$ (thereby setting y_i to true) and thus the ϕ_j clauses to be true, and hence ϕ to be true. If, on the other hand, the implication does not hold, then at least one of the constraints 1-4 sets $x_{n+j} \ge 0$, allowing all literals of clause ϕ_j to be **false**, which also makes ϕ_j (and hence ϕ) **false**.

Thus, the 3SAT problem can be reduced to the resultant implication, which is of the form $\langle 1, \ \exists, \ \mathbf{LB} \rangle$. $\square$

6 Problem $\langle 2, \ \forall, \ \mathbf{LRB} \rangle$

We will now focus on the problems of the form $\langle 2, \ \forall, \ \mathbf{LRB} \rangle$, i.e., problems of the form:

$$\forall \mathbf{s} \ \exists \mathbf{r} \ \forall \mathbf{x} \ [\mathbf{A} \cdot \mathbf{x} \le \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \rightarrow \mathbf{C} \cdot \mathbf{x} \le \mathbf{d} + \mathbf{M} \cdot \mathbf{r}] \quad (5)$$

Theorem 6.1 *The problem* $\langle 2, \ \forall, \ \mathbf{LRB} \rangle$ *is* **coNP-hard**.

Proof: We want to show that quantified linear implications of the form (5) are **coNP-hard**.

Consider an instance of the **F**-QLP problem:

$$\forall \mathbf{s} \in [\mathbf{l}, \mathbf{u}] \ \exists \mathbf{r} \ \mathbf{C} \cdot \mathbf{s} - \mathbf{M} \cdot \mathbf{r} \le \mathbf{d}$$

The problem of checking validity of such a class of formulas is **coNP-complete** (Subramani 2007). This problem will be reduced to a problem of the form (5) as follows: For each s_i of $\mathbf{s}$, add x_i such that $\mathbf{s} = \mathbf{x}$. We create the following equivalent implication using elementary logical properties:

$$\forall \mathbf{s} \ \exists \mathbf{r} \ \forall \mathbf{x} \ \ [(\mathbf{x} = \mathbf{s} \wedge \mathbf{G} \cdot \mathbf{s} \le \mathbf{h}) \rightarrow \mathbf{C} \cdot \mathbf{x} \le \mathbf{d} + \mathbf{M} \cdot \mathbf{r}]$$

where $\mathbf{G} \cdot \mathbf{s} \le \mathbf{h}$ captures $\mathbf{s} \in [\mathbf{l}, \mathbf{u}]$. It is easy to see that the initial **F**-QLP problem is satisfied if and only if the resultant $\langle 2, \ \forall, \ \mathbf{LRB} \rangle$ problem is satisfied: If there exists $\mathbf{s}$, with $\mathbf{s} \in [\mathbf{l}, \mathbf{u}]$, such that $\mathbf{x} = \mathbf{s}$ is included in $\mathbf{C} \cdot \mathbf{x} \le \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ (i.e., $\mathbf{C} \cdot \mathbf{s} \le \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ is satisfied for $\mathbf{s} \in [\mathbf{l}, \mathbf{u}]$), then the corresponding **F**-QLP will be satisfied as well. On the other hand, if the **F**-QLP is satisfied, then since $\mathbf{x} = \mathbf{s}$ the corresponding implication will also be satisfied. The result follows. $\square$

We show that solving an instance of the problem $\langle 2, \ \forall, \ \mathbf{LRB} \rangle$ is equivalent to solving the corresponding instance of the problem $\langle 2, \ \forall, \ \mathbf{BRB} \rangle$.

Theorem 6.2 *The validity problems for the classes of formulas* $\langle 2, \forall, \mathbf{BRB} \rangle$ *and* $\langle 2, \forall, \mathbf{LRB} \rangle$ *are polynomial-time reducible to each other.*

Proof: Every instance of the form $\langle 2, \forall, \mathbf{LRB} \rangle$ is also in $\langle 2, \forall, \mathbf{BRB} \rangle$; that is, an instance of the form (5) can be rewritten as:

$$\forall \mathbf{s} \ \exists \mathbf{r} \ \forall \mathbf{x} \ [\mathbf{A} \cdot \mathbf{x} \le \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \rightarrow \mathbf{C} \cdot \mathbf{x} \le \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{0} \cdot \mathbf{s}]$$

which is clearly of the form $\langle 2, \forall, \mathbf{BRB} \rangle$.

On the other hand, consider a problem $\langle 2, \forall, \mathbf{BRB} \rangle$, i.e., a problem of the form:

$$\forall \mathbf{s} \ \exists \mathbf{r} \ \forall \mathbf{x} \ [\mathbf{A} \cdot \mathbf{x} \le \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \rightarrow \mathbf{C} \cdot \mathbf{x} \le \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{P} \cdot \mathbf{s}]$$

It is readily checked that it holds if and only if the following formula holds:

$$\forall \mathbf{s}\ \exists \mathbf{r}\ \forall \mathbf{x}\ \forall \mathbf{x}'$$
$$(\mathbf{A}\cdot\mathbf{x} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s} \wedge \mathbf{x}' = \mathbf{s}) \rightarrow \mathbf{C}\cdot\mathbf{x} - \mathbf{P}\cdot\mathbf{x}' \le \mathbf{d} + \mathbf{M}\cdot\mathbf{r}$$

where the newly introduced variables included in vector $\mathbf{x}'$ are constrained to assume the values of the parameters in $\mathbf{s}$. Since the formula above is of the form (5), and it is clearly obtained in linear time, we have that $\langle 2, \forall, \mathbf{BRB}\rangle$ reduces in polynomial time to $\langle 2, \forall, \mathbf{LRB}\rangle$. □

6.1 A coNP-hard Special Case

A *difference constraint* is a constraint of the form $x_i - x_j \le b_{ij}$. We present a special case of problem $\langle 2, \forall, \mathbf{LRB}\rangle$ which is still **coNP-hard**, although the entailing polyhedral set consists only of difference constraints.

Theorem 6.3 *Problem (5) is* **coNP-hard** *even if* $\mathbf{A}\cdot\mathbf{x} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s}$ *is restricted to difference constraints.*

Proof: Consider an instance of the $\mathbf{F}$-QLP problem:

$$\forall \mathbf{z} \in [\mathbf{l}, \mathbf{u}]\ \exists \mathbf{y}\ \mathbf{C}\cdot\mathbf{z} \le \mathbf{d} + \mathbf{M}\cdot\mathbf{y}$$

Recall that the problem of checking validity of such a class of formulas is **coNP-complete** (Subramani 2007). We can reduce this problem to an instance of the form (5) with $\mathbf{A}\cdot\mathbf{x} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s}$ being restricted to difference constraints as follows:

1. Add the constraints $x_1 - s_1 \le 0$ and $s_1 - x_1 \le 0$ to $\mathbf{A}\cdot\mathbf{x} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s}$.
2. For each element z_i of $\mathbf{z}$, add the constraints $x_{i+1} - s_{i+1} \le 0$, $s_{i+1} - x_{i+1} \le 0$, $s_{i+1} - s_1 \le u_i$, and $s_{i+1} - s_1 \ge l_i$ to $\mathbf{A}\cdot\mathbf{x} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s}$.
3. Let vector $\mathbf{x}'$ be such that $x'_i = x_{i+1} - x_1$.
4. Let vector $\mathbf{s}'$ be such that $s'_i = s_{i+1} - s_1$.
5. Create the instance $\forall \mathbf{s}\ \exists \mathbf{r}\ \forall \mathbf{x}\ [\mathbf{A}\cdot\mathbf{x} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s} \rightarrow \mathbf{C}\cdot\mathbf{x}' \le \mathbf{d} + \mathbf{M}\cdot\mathbf{r}]$.

In this instance, we have that $\mathbf{A}\cdot\mathbf{x} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s}$ can only be satisfied if $\mathbf{x} = \mathbf{s}$ and $l_i \le s_{i+1} - s_1 = s'_i \le u_i$. Thus, for the constructed instance to be satisfiable, $\mathbf{C}\cdot\mathbf{x}' \le \mathbf{d} + \mathbf{M}\cdot\mathbf{r}$ must also hold when $\mathbf{x}' = \mathbf{s}'$ and $l_i \le s'_i \le u_i$. By replacing $\mathbf{x}'$ with $\mathbf{s}'$ in this system of constraints, we get that $\forall \mathbf{s}' \in [\mathbf{l}, \mathbf{u}]\ \exists \mathbf{r}\ \ \mathbf{C}\cdot\mathbf{s}' \le \mathbf{d} + \mathbf{M}\cdot\mathbf{r}$ must also be satisfiable.

Similarly, if the constructed instance is unsatisfiable, there must exist an $\mathbf{s}$ such that

$$\forall \mathbf{r}\ \exists \mathbf{x}\ [\mathbf{A}\cdot\mathbf{x} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s} \rightarrow \mathbf{C}\cdot\mathbf{x}' \not\le \mathbf{d} + \mathbf{M}\cdot\mathbf{r}]$$

Recall that $l_i \le s'_i \le u_i$ and the only value of $\mathbf{x}$ which can satisfy $\mathbf{A}\cdot\mathbf{x} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s}$ is $\mathbf{x} = \mathbf{s}$. Hence, we must have that $\forall \mathbf{r}\ \ \mathbf{C}\cdot\mathbf{s}' \not\le \mathbf{d} + \mathbf{M}\cdot\mathbf{r}$, which means that $\forall \mathbf{s}' \in [\mathbf{l}, \mathbf{u}]\ \exists \mathbf{r}\ \mathbf{C}\cdot\mathbf{s}' \le \mathbf{d} + \mathbf{M}\cdot\mathbf{r}$ is also unsatisfiable.

Thus we can reduce $\mathbf{F}$-QLP to an instance of this special case of $\langle 2, \forall, \mathbf{LRB}\rangle$, which means that even when $\mathbf{A}\cdot\mathbf{x} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s}$ is restricted to difference constraints, $\langle 2, \forall, \mathbf{LRB}\rangle$ is still **coNP-hard**. □

6.2 Polynomially Solvable Special Cases

The proof of Theorem 6.1 shows that problem (5) is intractable even for very restricted instances of the entailing polyhedral set, such as the case in which it is simply consisting of lower and upper bounds on parameters. On the other hand, Theorem 4.1 shows that if there is no parameter in the entailing polyhedral set, then the problem is in $\mathbf{P}$. It is then natural to concentrate on syntactic restrictions of the entailed polyhedral set. An immediate tractable case is when the entailed polyhedral set has no parameters at all, namely when $\mathbf{r}$ is empty.

Lemma 6.1 *Problem (5) is in* $\mathbf{P}$ *if there is no constraint in the parameters of* $\mathbf{r}$.

Proof: Under the hypothesis, problem (5) reduces to:

$$\forall \mathbf{s}\ \forall \mathbf{x}\ \ [\mathbf{A}\cdot\mathbf{x} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s} \rightarrow \mathbf{C}\cdot\mathbf{x} \le \mathbf{d}]$$

which turns out to be an instance of $\langle 0, \forall, \mathbf{B}\rangle$, which is in $\mathbf{P}$ (Subramani 2009). □

This result is generalized as follows.

Lemma 6.2 *Problem (5) is in* $\mathbf{P}$ *if no column of* $\mathbf{M}$ *has both positive and negative values.*

Proof: Let cp_1 denote the left-hand side and cp_2 the right-hand side of the implication. Hence, $cp_1 = \mathbf{A}\cdot\mathbf{x} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s}$ and $cp_2 = \mathbf{C}\cdot\mathbf{x} \le \mathbf{d} + \mathbf{M}\cdot\mathbf{r} + \mathbf{P}\cdot\mathbf{s}$.

Let $\mathbf{M}'$ be obtained by replacing every non-zero value in $\mathbf{M}$ with 1. We claim that (5) holds if and only if

$$\forall \mathbf{s}\ \exists \mathbf{r}'\ \forall \mathbf{x}\ [\mathbf{A}\cdot\mathbf{x} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s} \rightarrow \mathbf{C}\cdot\mathbf{x} \le \mathbf{d} + \mathbf{M}'\cdot\mathbf{r}'] \quad (6)$$

holds. For a column $\mathbf{m}_i$ in $\mathbf{M}$, let m_i^l be the maximum non-zero value in $\mathbf{m}_i$ and m_i^s be the minimum non-zero value in it (if $\mathbf{m}_i$ contains only zero's, then the parameter $\mathbf{r}_i$ never appears, hence $\mathbf{m}_i$ can be discarded from $\mathbf{M}$). By assumption, $sgn(m_i^l) = sgn(m_i^s)$, where $sgn()$ is the sign function.

(Only-if part) Fixed $\mathbf{s}$, let $\mathbf{r}$ be such that (5) holds. We set $r'_i = m_i^l r_i$ if $r_i \ge 0$ and $sgn(m_i^l) = 1$; $r'_i = m_i^s r_i$ if $r_i \ge 0$ and $sgn(m_i^l) = -1$; $r'_i = m_i^s r_i$ if $r_i < 0$ and $sgn(m_i^l) = 1$; $r'_i = m_i^l r_i$ if $r_i < 0$ and $sgn(m_i^l) = -1$. With this settings, for every ar_i appearing in $\mathbf{M}\cdot\mathbf{r}$, we have $ar_i \le r'_i$, hence $\mathbf{M}\cdot\mathbf{r} \le \mathbf{M}'\cdot\mathbf{r}'$. This implies $\mathbf{C}\cdot\mathbf{x} \le \mathbf{d} + \mathbf{M}\cdot\mathbf{r} \le \mathbf{d} + \mathbf{M}'\cdot\mathbf{r}'$, hence (6) holds as well.

(If part) Fixed $\mathbf{s}$, let $\mathbf{r}'$ be such that (6) holds. We set $r_i = r'_i/m_i^s$ if $r'_i \ge 0$ and $sgn(m_i^l) = 1$; $r_i = r'_i/m_i^l$ if $r'_i \ge 0$ and $sgn(m_i^l) = -1$; $r_i = r'_i/m_i^l$ if $r'_i < 0$ and $sgn(m_i^l) = 1$; $r_i = r'_i/m_i^s$ if $r'_i < 0$ and $sgn(m_i^l) = -1$. With this settings, for every r'_i appearing in $\mathbf{M}'\cdot\mathbf{r}'$, we have $r'_i \le ar_i$ for any non-zero coefficient a appearing in $\mathbf{M}\cdot\mathbf{r}$, hence $\mathbf{M}'\cdot\mathbf{r}' \le \mathbf{M}\cdot\mathbf{r}$. This implies $\mathbf{C}\cdot\mathbf{x} \le \mathbf{d} + \mathbf{M}'\cdot\mathbf{r}' \le \mathbf{d} + \mathbf{M}\cdot\mathbf{r}$, hence (5) holds as well.

We now claim that (6) holds if and only if $\mathbf{A}\cdot\mathbf{x} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s}$ is unsatisfiable (as a linear system in $\mathbf{x}, \mathbf{s}$) or the following holds:

$$\exists \mathbf{r}''\ \ \forall \mathbf{x}\ \ [\mathbf{A}\cdot\mathbf{x} \le \mathbf{0} \rightarrow \mathbf{C}\cdot\mathbf{x} \le \mathbf{d} + \mathbf{M}'\cdot\mathbf{r}''] \quad (7)$$

(Only-if part) If $\mathbf{A}\cdot\mathbf{x} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s}$ is unsatisfiable, we are done. Otherwise, let $\mathbf{s}_0$ be such that there exists $\mathbf{x}_0$ with $\mathbf{A}\cdot\mathbf{x_0} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s_0}$. By hypothesis, there exists $\mathbf{r}'$ such that $\forall \mathbf{x}\ \ [\mathbf{A}\cdot\mathbf{x} \le \mathbf{b} + \mathbf{N}\cdot\mathbf{s_0} \rightarrow \mathbf{C}\cdot\mathbf{x} \le$

$\mathbf{d} + \mathbf{M}' \cdot \mathbf{r}']$ holds. Due to the form of $\mathbf{M}'$ this is equivalent to $\mathbf{d} + \mathbf{M}' \cdot \mathbf{r}' \geq max\{\mathbf{C} \cdot \mathbf{x} \mid \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s_0}\}$. By well-known results on linear programs (see e.g., Corollary 3.1 from (Murty 1983)), the property of (un)boundedness for the non-empty linear system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s_0}$ holds iff it holds for its cone $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{0}$. Hence, there exists $\mathbf{r}''$ such that $\mathbf{d} + \mathbf{M}' \cdot \mathbf{r}'' \geq max\{\mathbf{C} \cdot \mathbf{x} \mid \mathbf{A} \cdot \mathbf{x} \leq \mathbf{0}\}$, i.e., (7) holds.

(If part) If $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ is unsatisfiable, (6) clearly holds. If it is satisfiable, let $\mathbf{s}_0$ be such that $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s_0}$ is satisfiable. Again by (Murty 1983)[Corollary 3.1], the assumption that there exists $\mathbf{r}''$ such that $\mathbf{d} + \mathbf{M}' \cdot \mathbf{r}'' \geq max\{\mathbf{C} \cdot \mathbf{x} \mid \mathbf{A} \cdot \mathbf{x} \leq \mathbf{0}\}$ implies that for some $\mathbf{r}'$, $\mathbf{d} + \mathbf{M}' \cdot \mathbf{r}' \geq max\{\mathbf{C} \cdot \mathbf{x} \mid \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s_0}\}$, i.e. $\exists \mathbf{r}' \ \forall \mathbf{x} \quad [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s_0} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M}' \cdot \mathbf{r}']$ holds. Since this is true for any $\mathbf{s}_0$, we conclude that (6) holds.

Summarizing, we have reduced our original problem to showing the feasibility of $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$, which is a polynomial time problem, and (7), which is in $\mathbf{P}$ by Theorem 4.1. □

An *interval constraint* on a variable x is a constraint of the form $l \leq x \leq u$. The case where the entailing polyhedral set includes only interval constraints is presented below.

Lemma 6.3 *Problem (5) is in* $\mathbf{P}$ *if each constraint in* $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ *is an interval constraint.*

Proof: Note that if constraints in $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ contain only one variable or only one parameter, then constraints involving only parameters from $\mathbf{s}$ can be ignored; this is because there is no interaction between the vectors $\mathbf{s}$ and $\mathbf{x}$ at all and, hence, the only influence such constraints can have on the problem is to make it trivially easy (if no $\mathbf{s}$ can satisfy $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$.)

Thus we have that each element x_i of $\mathbf{x}$ is simply bound by an interval $l_i \leq x_i \leq u_i$ which means that $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ is equivalent to $\mathbf{x} \in [\mathbf{l}, \mathbf{u}]$ and so $\forall \mathbf{s} \ \exists \mathbf{r} \ \forall \mathbf{x} \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}]$ is equivalent to saying $\exists \mathbf{r} \ \forall \mathbf{x} \in [\mathbf{l}, \mathbf{u}] \ \ \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ which is in P. □

Problem $\langle 2, \ \forall, \ \mathbf{LRB} \rangle$ also becomes tractable as soon as we prevent any connection between variables and parameters in the antecedent of the implication.

Lemma 6.4 *Problem (5) is in* $\mathbf{P}$ *if there is no constraint in the antecedent of the implication which includes both variables and parameters.*

Proof: Let again cp_1 denote the left-hand side and cp_2 the right-hand side of the implication. Then, $cp_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \mathbf{N} \cdot \mathbf{s} \leq \mathbf{n}$ and $cp_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$.

Hence, the problem consists of showing the validity of:

$$\forall \mathbf{s} \ \exists \mathbf{r} \ \forall \mathbf{x}$$
$$(\mathbf{N} \cdot \mathbf{s} \leq \mathbf{n} \ \wedge \ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}) \quad \rightarrow \quad \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} \quad (8)$$

If cp_1 is unsatisfiable, the formula is always true. Hence, assume it is satisfiable. We build a linear program on $\mathbf{r}$ as follows: For every row $\mathbf{c_i}^T \cdot \mathbf{x} \leq d_i + \mathbf{m_i}^T \cdot \mathbf{r}$ in cp_2, let d_i' be the solution of the linear program:

$$\max \ \mathbf{c_i}^T \mathbf{x}$$
$$\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$$

If the linear system is unbounded (i.e., if at least one of the two sub-systems is unbounded), there cannot be any parameter instance $\mathbf{u}$ of $\mathbf{r}$ such that $\mathbf{c_i}^T \cdot \mathbf{x} \leq d_i + \mathbf{m_i}^T \cdot \mathbf{u}$ for every $\mathbf{x}$ once $\mathbf{s}$ has been fixed to any solution of $\mathbf{N} \cdot \mathbf{s} \leq \mathbf{n}$. Thus, (8) is false.

Assume now that all d_i's are finite, and let $\mathbf{d}'$ be the vector of all d_i's. We claim that (8) holds iff the following linear system on $\mathbf{r}$ is feasible:

$$\mathbf{d} + \mathbf{M} \cdot \mathbf{r} \geq \mathbf{d}' \quad (9)$$

(If part) Let $\mathbf{u}$ be such that $\mathbf{d} + \mathbf{M} \cdot \mathbf{u} \geq \mathbf{d}'$. Since, by construction of $\mathbf{d}'$, we have that $\forall \mathbf{x} \ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}']$ holds, by transitivity the following also holds:

$$\forall \mathbf{x} \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{u}]$$

By reintroducing existential quantifiers on $\mathbf{r}$, we have:

$$\exists \mathbf{r} \ \forall \mathbf{x} \ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}]$$

which implies (8).

(Only-if part) Since cp_1 is assumed to be satisfiable, there must exists $\mathbf{s}_0$ such that $\mathbf{N} \cdot \mathbf{s}_0 \leq \mathbf{n}$. By hypothesis, there exists $\mathbf{r}_0$ such that

$$\forall \mathbf{x} \ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \wedge \mathbf{N} \cdot \mathbf{s}_0 \leq \mathbf{n} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0]$$

Since $\mathbf{N} \cdot \mathbf{s}_0 \leq \mathbf{n}$ is true, this boils down to:

$$\forall \mathbf{x} \ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0]$$

Let us show that $\mathbf{r}_0$ is a solution of (9). Consider a row $\mathbf{c_i}^T \cdot \mathbf{x} \leq d_i + \mathbf{m_i}^T \cdot \mathbf{r}$ of cp_2. By definition of $\mathbf{r}_0$, the following holds:

$$\forall \mathbf{x} \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{c_i}^T \cdot \mathbf{x} \leq d_i + \mathbf{m_i}^T \cdot \mathbf{r}_0]$$

which implies:

$$d + \mathbf{m_i}^T \cdot \mathbf{r}_0 \geq \max\{\mathbf{c_i}^T \mathbf{x} \mid \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\}$$

By definition of $\mathbf{d}'$, we conclude $\mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0 \geq \mathbf{d}'$, hence (9) is satisfiable. □

7 Conclusions

In this paper we introduced entailment for the class of parameterized linear constraints and studied its computational complexity. We focused on three different types of entailment, namely $\langle 1, \ \exists, \ \mathbf{RB} \rangle$ (which was shown in $\mathbf{P}$), $\langle 1, \ \exists, \ \mathbf{LB} \rangle$ (which was shown to be **NP-hard**), and $\langle 2, \ \forall, \ \mathbf{LRB} \rangle$ (which was shown to be **coNP-hard**). For the latter, we showed that it is reducible to the problem $\langle 2, \ \forall, \ \mathbf{BRB} \rangle$ (and vice versa). Also, we presented a special case which is still **coNP-hard** as well as several sub-classes for which the problem is in $\mathbf{P}$.

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