Deciding Membership in a Class of Polyhedra

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Abstract. Parameterized linear systems allow for modelling and reasoning over classes of polyhedra. Collections of squares, rectangles, polytopes, and so on can readily be defined by means of linear systems with parameters in constant terms. In this paper, we consider the membership problem of deciding whether a given polyhedron belongs to the class defined by a parameterized linear system. As an example, we are interested in questions such as: "does a given polytope belong to the class of hypercubes?" We show that the membership problem is $\mathbb{NP}$-complete, even when restricting to the 2-dimensional plane. Despite the negative result, the constructive proof allows us to devise a concise decision procedure using constraint logic programming over the reals, namely CLP($\mathbb{R}$), which searches for a characterization of all instances of a parameterized system that are equivalent to a given polyhedron.

1 Introduction

Linear systems of inequalities over real numbers are basic tools for representing and reasoning over polyhedral sets. They have been extensively adopted in several fields of artificial intelligence, including geometric reasoning [1], constraint (logic) programming [9], robot motion planning [8], computer vision [2], resource planning [23], pattern recognition and classification [18], expert systems [13], and many others. The intuitive meaning of a linear system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ is the polyhedral set of its solutions. As an example, the points belonging to a rectangle of length 2 and height 3 can be described as the solutions $(x_1\, x_2)$ of the inequalities $0 \leq x_1 \leq 2, 0 \leq x_2 \leq 3$. Consider now the simple, yet intriguing, problem of defining a *class* of polyhedral sets. Recalling the previous example, the class of points belonging to a rectangle of length a and height b greater or equal than a can be described as the solutions $(x_1\, x_2)$ of the inequalities $0 \leq x_1 \leq a, 0 \leq x_2 \leq b, a \leq b$, where a and b are parameters. A minimalist approach at extending the expressive power of linear systems consists then of admitting parameters among the constant terms. A parameterized linear system over the reals is a system of linear inequalities $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{M} \cdot \mathbf{r}$ where variables in $\mathbf{r}$ are *parameters*. The intended meaning of a parameterized linear system is then a class of polyhedra over variables in $\mathbf{x}$, each obtained by instantiating the parameters $\mathbf{r}$. In the previous example, a class of rectangles with one vertex at the origin and the sides parallel to the axes was defined. Notice that translations can be readily expressed, e.g., $a \leq x_1 \leq b, c \leq x_2 \leq d$ denotes rectangles with a vertex at a generic point $(a\, b)$. However, rotations cannot be expressed, since it would require parameters to appear as coefficients of variables. In this sense, the extension is "minimalist".

Parameterized linear systems trace back to the late 60's in the context of (multi-)parametric linear programming, where the objective is to optimize a parameterized linear function over the solutions of a parameterized linear system. It has a solid theoretical basis and a wide range of applications [6, 11, 14], including hybrid parametric/stochastic programming, process planning under uncertainty, material design under uncertainty, model based control, multi-processor scheduling. Besides linear programming problems, parameterized linear systems have been adopted for symbolic dependence analysis in computing a parallel schedule [12], fracture mechanics and engineering [19], hardware verification [3], and type systems for constraint programming [16].

In this paper, we investigate the decision problem of checking whether the sets of solutions of a given linear system belongs to the class of polyhedra defined by a parameterized linear system. Stated otherwise, we consider deciding whether the set of solutions of a linear system can be obtained by instantiating parameters in a given parameterized linear system. More interestingly, we aim at deriving all or any of such parameter instances. Example questions of the type above include: "Does this linear system denotes an axis-parallel square?" and "Which parameter instances do yield the same set of solutions of this linear system?" We show that the membership decision problem is $\mathbb{NP}$-complete, even when restricting to polytopes over the two dimensional plane. The proof that the problem is in $\mathbb{NP}$ is constructive. A concise decision procedure is devised using constraint logic programming over the reals, namely CLP($\mathbb{R}$), which enumerates all instances of the parameterized linear system that are equivalent to the polyhedron of a given linear system.

The rest of this paper is organized as follows. First, we recall in Section 2 some basic notation and results on linear system and polyhedra. After defining the decision problem in Section 3, we first tackle the sub-problem of inclusion in Section 4, and then the full decision problem in Section 5. A CLP($\mathbb{R}$) implementation of the decision procedure is presented in Section 6. Finally, we summarize the contributions of the paper in the conclusions.

2 Background

We adhere to standard notation of linear algebra [17]. $\mathbb{R}$ is the set of real numbers. Small bold letters ($\mathbf{a}, \mathbf{b}, \ldots$) denote column vectors, while capital bold letters ($\mathbf{A}, \mathbf{B}, \ldots$) denote matrices. $\mathbf{0}$ (resp., $\mathbf{1}$) is a column vector with all elements equal to 0 (resp., 1). The transposed vector of $\mathbf{a}$ is denoted by $\mathbf{a}^T$. The inner product is denoted by $\mathbf{a}^T \cdot \mathbf{b}$. $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ denotes a system of linear inequalities over the variables in $\mathbf{x}$, also called a *linear system*. We assume that the dimensions of vectors and matrices in inner products and linear systems are of the appropriate size. A *polyhedron* is the set of solution points of a linear system: $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}) = \{\mathbf{x}_0 \in \mathbb{R}^{|\mathbf{x}|} \mid \mathbf{A} \cdot \mathbf{x}_0 \leq \mathbf{b}\}$. Polyhedra are convex sets. A linear program:

$$\max \mathbf{c}^T \cdot \mathbf{x}$$
$$\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$$

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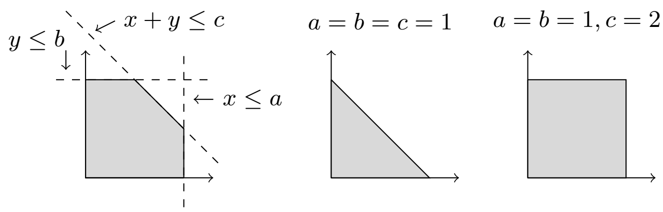

Figure 1. A parameterized linear system and two of its instances.

is the problem of finding $\max\{\mathbf{c}^T \cdot \mathbf{x}_0 \mid \mathbf{x}_0 \in Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})\}$. If the polyhedron $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})$ is empty, the problem is said to be *infeasible*. If the linear function $\mathbf{c}^T \cdot \mathbf{x}$ is unbounded, the problem is said to be *unbounded*.

An equivalent formulation of linear systems is provided in terms of logic formulas over the reals. A *primitive linear constraint* is an expression $a_1 x_1 + \ldots a_n x_n \leq a_0$, where $a_0, \ldots, a_n$ are constants in $\mathbb{R}$ and $x_1, \ldots, x_n$ are variables. We will also use the inner product form by rewriting it as $\mathbf{c}^T \cdot \mathbf{x} \leq d$. A *linear constraint* c is a conjunction of primitive constraints. Conjunction is syntactically represented either by "$\wedge$" or by a comma. Inequalities $\mathbf{c}^T \cdot \mathbf{x} \geq d$ and equalities $\mathbf{c}^T \cdot \mathbf{x} = d$ can be reduced to linear constraints. Any linear system can be represented as a linear constraint and vice-versa. So, we will use the two notions interchangeably throughout the paper.

Consider two linear systems $P_1 : \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ and $P_2 : \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}$. The problem of checking whether every solution of P_1 is a solution of P_2, namely whether $Sol(P_1) \subseteq Sol(P_2)$, can also be expressed as the first-order sentence $\forall \mathbf{x}\ [P_1 \rightarrow P_2]$. This decision problem can be solved in polynomial time [20]. It suffices to show that for every primitive constraint $\mathbf{c}_i^T \cdot \mathbf{x} \leq d_i$ in P_2 (where $\mathbf{c}_i$ is the vector corresponding to the i^{th} row of $\mathbf{C}$), the following linear programming problem is either infeasible or its solution is bounded by d_i:

$$\max \mathbf{c}_i^T \cdot \mathbf{x}$$
$$\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$$

The conclusion follows from the fact that linear programming problems are solvable in polynomial time [10] and there is a finite number of primitive constraints in P_2. Formally, the following result holds.

Theorem 1 ([20]) *The decision problem for the class of formulas* $\forall \mathbf{x}\ [\mathbf{A}\mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}]$ *is in* $\mathbb{P}$.

A *parameterized linear system* (or, equivalently, a parameterized linear constraint) over the reals is a system of linear inequalities $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{M} \cdot \mathbf{r}$ where variables in $\mathbf{r}$ are *parameters*. We will use meta-variables a, b, c, for parameters and x, y, z for variables. The intended meaning of a parameterized linear system is then a collection of linear systems over variables in $\mathbf{x}$, each obtained by instantiating the parameters $\mathbf{r}$. The notion of *parameterized polyhedra* from [12] models the solutions of parameterized linear systems.

Definition 2.1 *A* parameterized polyhedron *is a collection of polyhedra defined by fixing the value of parameters in a parameterized linear system:* $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{M} \cdot \mathbf{r}, \mathbf{r}_0) = \{\mathbf{x}_0 \in \mathbb{R}^{|\mathbf{x}|} \mid \mathbf{A} \cdot \mathbf{x}_0 \leq \mathbf{b} + \mathbf{M} \cdot \mathbf{r_0}\}$*, where* $\mathbf{r}_0 \in \mathbb{R}^{|\mathbf{r}|}$ *is an instance of the parameters* $\mathbf{r}$.

The $Sol()$ function now returns the set of solution points of a parameterized linear system for a specific assignment to parameters.

Example 2.1 *Figure 1 shows the parameterized linear system* $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ *defined as* $0 \leq x \leq a, 0 \leq y \leq b, x + y \leq c$*, where* x, y *are variables and* a, b, c *are parameters. Several types of polytopes can be obtained as special cases: rectangled-isosceles triangles by setting* a, b, c *to a same value (*1 *in the figure), squares by setting* a, b *to a same value* v*, and* $c \geq 2v$ *(*$v = 1$ *in the figure), rectangles by setting* $c \geq a + b$*, and, in addition, some classes of right trapezoids, and some classes of irregular pentagons.*

The expressive power of parameterized linear system is, however, limited by the fact that parameters can only appear in constant terms. As an example, the class of rectangled triangles cannot be defined because it would require inequalities with parameters appearing as coefficients of variables. For instance, the inequality $dx + y \leq c$ in the place of $x + y \leq c$ in Figure 1 would allow for defining hypotenuses with any angle of inclination.

3 Problem Statement

The membership problem we are interested in consists of deciding whether the polyhedron of a given linear system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ belongs to the class of polyhedra defined by a given parameterized linear system $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$. A formal statement follows.

Membership problem. Given a linear system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ and a parameterized linear system $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$, decide whether there exists $\mathbf{r}_0$ such that:

$$Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}) = Sol(\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}, \mathbf{r}_0) \tag{1}$$

The problem can be readily re-stated as to decide whether the following first order logic formula holds over the domain of the reals:

$$\exists \mathbf{r}\ \forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \leftrightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}] \tag{2}$$

Quantified logic formulas over the reals can be solved by quantifier elimination methods [4, 15, 21]. In this paper, however, we are interested in characterizing the exact computational complexity of the class of formulas (2). In addition to the decision problem, we are also interested in the search problem of characterizing which parameter instances $\mathbf{r}_0$ satisfy condition (1).

4 Deciding Inclusion

Let us start considering the "$\rightarrow$" side of formula (2):

$$\exists \mathbf{r}\ \forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}] \tag{3}$$

This implication can be read as whether there exist parameter values $\mathbf{r_0}$ such that $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}) \subseteq Sol(\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}, \mathbf{r}_0)$, namely a form of inclusion of polyhedra. [5, Theorem 4.1] firstly showed that (3) can be checked in polynomial time. For completeness, we report here a (slightly revised) proof that will be used later on for deriving a CLP($\mathbb{R}$) implementation of the decision procedure.

Theorem 2 ([5]) *The problem of deciding whether (3) holds is in* $\mathbb{P}$.

Proof. Let P_1 denote $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ and P_2 denote $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$. The satisfiability of P_1 (i.e., whether $Sol(P_1) \neq \emptyset$) can be checked in polynomial time [10]. If P_1 is unsatisfiable, (3) is always true. Assume now it is satisfiable. We build a linear program on $\mathbf{r}$ as follows. For every row $\mathbf{c}_i^T \cdot \mathbf{x} \leq d_i + \mathbf{m}_i^T \cdot \mathbf{r}$ in P_2, let k_i be the solution of the linear program:

$$\max\ \mathbf{c}_i^T \cdot \mathbf{x}$$
$$\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$$

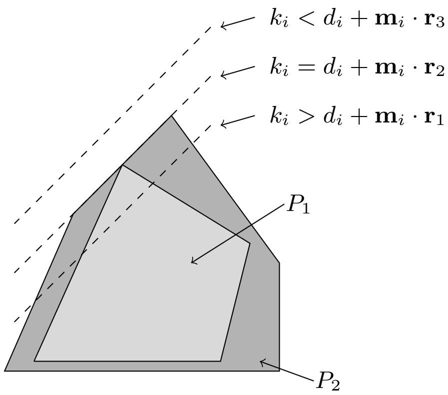

Figure 2. Intersecting ($\mathbf{r}_1$), incident ($\mathbf{r}_2$), and external ($\mathbf{r}_3$) hyperplanes at the variation of parameters in $\mathbf{r}$ (see the proof of Theorem 2 for the notation).

If the linear program is unbounded, there cannot be any variable instance $\mathbf{r}_0$ of $\mathbf{r}$ such that $\mathbf{c}_i^T \cdot \mathbf{x} \leq d_i + \mathbf{m}_i^T \cdot \mathbf{r}_0$ for all $\mathbf{x}$ such that $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$. Thus, (3) is false. Assume now that all k_i's are finite and let $\mathbf{k}$ be the vector of all k_i's. We claim that $\forall \mathbf{x} \quad [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0]$ holds iff:

$$\mathbf{k} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0 \tag{4}$$

Intuitively, this means that each hyperplane of $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0$ is either incident to some extreme point of $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})$ or external to $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})$, e.g., see Figure 2.

Let us show the claim.

If part. Let $\mathbf{r_0}$ be such that $\mathbf{k} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r_0}$. Since by construction of $\mathbf{k}$, we have that $\forall \mathbf{x} \;\; [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{k}]$ holds, by transitivity the following holds $\forall \mathbf{x} \;\; [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r_0}]$.

Only-if part. Consider a row $\mathbf{c}_i^T \cdot \mathbf{x} \leq d_i + \mathbf{m}_i^T \cdot \mathbf{r}$ of P_2. By hypothesis, we have $\forall \mathbf{x} \quad [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{c}_i^T \cdot \mathbf{x} \leq d_i + \mathbf{m}_i^T \cdot \mathbf{r}_0]$, which implies:

$$\max\{\mathbf{c}_i^T \cdot \mathbf{x} \mid \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\} = k_i \leq d_i + \mathbf{m}_i^T \cdot \mathbf{r}_0$$

By definition of $\mathbf{k}$, we conclude $\mathbf{k} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0$.

As a consequence of our claim, the set of $\mathbf{r}$'s that make the existential quantification (3) hold is characterized by the solutions of:

$$\mathbf{k} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} \tag{5}$$

Hence, (3) holds iff the linear system (5) is satisfiable. Since calculating $\mathbf{k}$ and checking satisfiability of (5) are solvable in polynomial time, we conclude that checking (3) is a problem in $\mathbb{P}$. □

Notice that the proof provides a constructive way of determining the parameter instance $\mathbf{r}_0$ for which $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}) \subseteq Sol(\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}, \mathbf{r}_0)$ holds.

Example 4.1 *Consider the the parameterized linear system $0 \leq x \leq a, 0 \leq y \leq b, x + y \leq c$ from Example 2.1. Let us apply the procedure of the theorem to the linear system $P = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ defined as $0 \leq x, 0 \leq y, x + y \leq 1$. First of all, we compute $\mathbf{k} = (k_1\ k_2\ k_3\ k_4\ k_5)$ where:*

$$k_1 = max\{-x \mid P\} \quad k_2 = max\{-y \mid P\}$$
$$k_3 = max\{x \mid P\} \quad k_4 = max\{y \mid P\} \quad k_5 = max\{x + y \mid P\}$$

It turns out that $k_1 = 0, k_2 = 0, k_3 = 1, k_4 = 1, k_5 = 1$. The instances of the parameterized linear system that include $Sol(P)$ are then characterized by the solutions of the following linear system over the parameters:

$$0 \leq 0, 0 \leq 0, 1 \leq a, 1 \leq b, 1 \leq c$$

Notice that not all of them lead to the polyhedron $Sol(P)$. As shown in Figure 1, $a = b = c = 1$ leads to $Sol(P)$, but $a = b = 1, c = 2$ leads to a stricly larger polyhedron.

5 Deciding Equivalence

Consider now our main problem (2). By moving from inclusion to equivalence, the problem becomes $\mathbb{NP}$-hard.

Theorem 3 *The problem of deciding whether (2) holds is $\mathbb{NP}$-hard.*

Proof. Let us reduce the 3SAT problem, which is $\mathbb{NP}$-complete, to formulas (2). Let $\phi = \wedge_{i=1...n}\phi_i$ be a boolean formula in conjunctive normal form, with ϕ_i being a disjunction of at most 3 literals over the boolean variables $a_1, \ldots, a_k$. We build a formula (2) as follows. Let $x_1, \ldots, x_n$ be variables, one for each clause in ϕ; and $r_1, \ldots, r_k$ be parameters, one for each boolean variable in ϕ. The system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ is $0 \leq x_1 \leq 1, \ldots, 0 \leq x_n \leq 1$ – namely a hypercube. The parameterized system $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ is the set of inequalities $0 \leq x_i \leq 1 + r_m$ for every positive literal a_m appearing in a clause ϕ_i; and $0 \leq x_i \leq 2 - r_m$ for every negated literal $\overline{a_m}$ appearing in a clause ϕ_i. We claim that ϕ is satisfiable iff (2) holds.

If part. Assume that $\forall \mathbf{x} \quad [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \leftrightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}]$ holds for parameter $\mathbf{r} = (r_1 \ldots r_k)$. We define an assignment to boolean variables as follows: we set a_m to true if parameter r_m is 0, and a_m to false otherwise. Consider now a clause ϕ_i. Since the corresponding inequalities in the RHS of the implication must lead to $0 \leq x_i \leq 1$, there exists a bound, $1 + r_m$ or $2 - r_m$, equal to 1, and all other bounds greater or equal to 1. If $1 + r_m = 1$ then $r_m = 0$; if $2 - r_m = 1$ then $r_m = 1$. In the former case, there exists a positive literal a_m in ϕ_i which evaluates to true (since $r_m = 0$); in the latter case, there exists a negative literal $\overline{a_m}$ in ϕ_i which, again, evaluates to true (since $r_m \neq 0$). Therefore, every ϕ_i is satisfied by our assignment, which then makes ϕ satisfiable.

Only-if part. Assume ϕ satisfiable by some assignment. We fix the value of a parameter r_m to 0 if the boolean variable a_m is true in the assignment, and to 1 otherwise. The formula $\forall \mathbf{x} \quad [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \leftrightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}]$ holds iff every x_i is constrained to $0 \leq x_i \leq 1$ in the RHS for $i = 1, \ldots, n$. Since ϕ_i is true, we have that there exists at least a literal in ϕ_i, say a_m (resp., $\overline{a_m}$), that evaluates to true. Hence, $0 \leq x_i \leq 1 + r_m$ (resp., $0 \leq x_i \leq 2 - r_m$) evaluates to $0 \leq x_i \leq 1$. The literals in ϕ_i that evaluate to false lead to inequalities $0 \leq x_i \leq 2$, hence they do not affect the upper bound of x_i. Summarizing, every variable x_i in the RHS is constrained to range over $0 \leq x_i \leq 1$, hence the LHS and the RHS denote the same polyhedron. Therefore, (2) holds. □

Notice that the proof actually shows that deciding whether a hypercube belongs to the class defined by a parameterized linear system is an $\mathbb{NP}$-hard problem.

Example 5.1 *Consider the formula $\phi = (a_1 \vee \overline{a_2} \vee a_3) \wedge (\overline{a_1} \vee a_2)$. Using the reduction from the proof of Theorem 2, ϕ is satisfiable iff the following formula holds:*

$$\exists r_1, r_2, r_3\ \forall x_1\ \forall x_2$$

$$\left.\begin{array}{l} 0 \leq x_1 \leq 1 \\ 0 \leq x_2 \leq 1 \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} 0 \leq x_1, 0 \leq x_2 \\ x_1 \leq 1 + r_1, x_1 \leq 2 - r_2, \\ x_1 \leq 1 + r_3 \\ x_2 \leq 2 - r_1, x_2 \leq 1 + r_2 \end{array}\right.$$

Boolean variables a_1, a_2, a_3 are mapped into parameters r_1, r_2, r_3 with the intuitive reading that a_i is true iff $r_i = 0$. The RHS parameterized system denotes the unit square, i.e., the polyhedron of the LHS system, iff x_1 and x_2 have at least an upper bound of 1 and all other upper bounds greater or equal than 1. For x_1, this means that at least one among $1+r_1$, $2-r_2$ and $1+r_3$ must be 1. Thus, $r_1 = 0$ or $r_2 = 1$ or $r_3 = 0$ must holds, which means that at least one among a_1, $\overline{a_2}$ and a_3 must be true. This is precisely the semantics of disjunction.

Let us concentrate now on showing that the problem is in $\mathbb{NP}$. First, we recall from [7] the notions of redundant inequalities and irredundant linear systems. We write $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \setminus \{\mathbf{c}^T \cdot \mathbf{x} \leq d\}$ to denote the linear system obtained by removing the inequality $\mathbf{c}^T \cdot \mathbf{x} \leq d$ from $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$.

Definition 5.1 *An inequality $\mathbf{c}^T \cdot \mathbf{x} \leq d$ from a system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ is called* redundant *if $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}) = Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \setminus \{\mathbf{c}^T \cdot \mathbf{x} \leq d\})$.*

A subset $\hat{\mathbf{A}} \cdot \mathbf{x} \leq \hat{\mathbf{b}}$ of $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ is called minimal *if it contains no redundant inequality and $Sol(\hat{\mathbf{A}} \cdot \mathbf{x} \leq \hat{\mathbf{b}}) = Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})$.*

Intuitively, redundant inequalities can be removed from a linear system without altering its set of solutions. Notice that deciding whether an inequality is redundant is a polynomial time problem. In fact, it boils down to checking that $\forall \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \setminus \{\mathbf{c}^T \cdot \mathbf{x} \leq d\} \rightarrow \mathbf{c}^T \cdot \mathbf{x} \leq d$ holds, which is a problem in $\mathbb{P}$ by Theorem 1. Therefore, checking that a given subset $\hat{\mathbf{A}} \cdot \mathbf{x} \leq \hat{\mathbf{b}}$ is minimal is in $\mathbb{P}$ as well – as it consists of checking that all inequalities in it are irredundant and that $\forall \mathbf{x}\ [\hat{\mathbf{A}} \cdot \mathbf{x} \leq \hat{\mathbf{b}} \rightarrow \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}]$ holds. Also, notice that a linear system may contain one or more minimal subsets.

Let us show now that our main problem is $\mathbb{NP}$-complete.

Theorem 4 *The problem of deciding whether (2) holds is $\mathbb{NP}$-complete.*

Proof. We only have to show that the problem is in $\mathbb{NP}$.

Let $\mathbf{k}$ be defined as in the proof of Theorem 2. We claim that $\forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \leftrightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0]$ holds iff there exists a minimal subset $\hat{\mathbf{C}} \cdot \mathbf{x} \leq \hat{\mathbf{k}}$ of $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k}$ such that:

$$\mathbf{k} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0 \tag{6}$$

$$\hat{\mathbf{k}} \geq \hat{\mathbf{d}} + \hat{\mathbf{M}} \cdot \mathbf{r}_0 \tag{7}$$

and the following holds:

$$\forall \mathbf{x}\ [\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k} \rightarrow \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}] \tag{8}$$

where the system in (7) denotes the subset of inequalities of $\mathbf{k} \geq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0$ corresponding to the rows of $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k}$ that are in $\hat{\mathbf{C}} \cdot \mathbf{x} \leq \hat{\mathbf{k}}$.

If part. As shown in the proof of Theorem 2, (6) implies $\forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0]$. Let us show the "$\leftarrow$" side. Assume $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0$. Since $\hat{\mathbf{C}} \cdot \mathbf{x} \leq \hat{\mathbf{d}} + \hat{\mathbf{M}} \cdot \mathbf{r}$ is a subset of inequalities of $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$, we have that $\hat{\mathbf{C}} \cdot \mathbf{x} \leq \hat{\mathbf{d}} + \hat{\mathbf{M}} \cdot \mathbf{r}_0$ holds as well. Then, by (7), we have that $\hat{\mathbf{C}} \cdot \mathbf{x} \leq \hat{\mathbf{k}}$ holds as well. Since $\hat{\mathbf{C}} \cdot \mathbf{x} \leq \hat{\mathbf{k}}$ is an irredundant subset of $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k}$, we have that also $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k}$ holds. By (8), we conclude $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$.

Only-if part. Assume that $\forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \leftrightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0]$ holds. As shown in the proof of Theorem 2, $\forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0]$ implies (6). *A fortiori*, our hypothesis implies (6) as well. Also, we claim that:

$$\forall \mathbf{x}\ [\hat{\mathbf{C}} \cdot \mathbf{x} \leq \hat{\mathbf{k}} \leftrightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{k} \leftrightarrow \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}] \tag{9}$$

holds for any minimal subset $\hat{\mathbf{C}} \cdot \mathbf{x} \leq \hat{\mathbf{k}}$ of $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k}$. The $\leftarrow$ implications are immediate by definition of $\mathbf{k}$. Consider the $\rightarrow$ implications. Suppose $\hat{\mathbf{C}} \cdot \mathbf{x} \leq \hat{\mathbf{k}}$ holds. By re-introducing the redundant inequalities, we have that $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k}$ holds as well. By (6), we then have $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0$, which, by our hypothesis, allows us to conclude $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$. An immediate consequence of (9) is that (8) holds. Finally, let us show (7). Let $P = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0$, and $\hat{P} = \hat{\mathbf{C}} \cdot \mathbf{x} \leq \hat{\mathbf{d}} + \hat{\mathbf{M}} \cdot \mathbf{r}_0$ be any minimal subset of P. For every $\mathbf{c}_i^T \cdot \mathbf{x} \leq \mathbf{d}_i + \mathbf{m}_i^T \cdot \mathbf{r}_0$ in $\hat{P}$, we have that there exists $\mathbf{x}_0$ in $Sol(\hat{P}) = Sol(P)$ such that $\mathbf{c}_i^T \cdot \mathbf{x}_0 = \mathbf{d}_i + \mathbf{m}_i^T \cdot \mathbf{r}_0$ – otherwise, the inequality would be redundant and then $\hat{P}$ would not be a minimal subset. Since by hypothesis $Sol(P) = Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})$, by definition of k_i we have then $k_i \geq \mathbf{c}_i^T \cdot \mathbf{x}_0 = \mathbf{d}_i + \mathbf{m}_i^T \cdot \mathbf{r}_0$. Summarizing, $\hat{\mathbf{k}} \geq \hat{\mathbf{d}} + \hat{\mathbf{M}} \cdot \mathbf{r}_0$, i.e., (7). However, we still have to show that $\hat{\mathbf{C}} \cdot \mathbf{x} \leq \hat{\mathbf{k}}$ is a minimal subset of $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k}$. We observe that $\hat{\mathbf{k}} \geq \hat{\mathbf{d}} + \hat{\mathbf{M}} \cdot \mathbf{r}_0$ and (6) imply $\hat{\mathbf{k}} = \hat{\mathbf{d}} + \hat{\mathbf{M}} \cdot \mathbf{r}_0$, and then $\hat{P}$ is equal to $\hat{\mathbf{C}} \cdot \mathbf{x} \leq \hat{\mathbf{k}}$. Therefore, $\hat{\mathbf{C}} \cdot \mathbf{x} \leq \hat{\mathbf{k}}$ is a minimal subset of P. Since $Sol(P) = Sol(\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k})$ (by 9 and our hypothesis), $\hat{\mathbf{C}} \cdot \mathbf{x} \leq \hat{\mathbf{k}}$ is a minimal subset of $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k}$ as well.

Summarizing, we can devise a procedure that first computes $\mathbf{k}$, which takes polynomial time, and then non-deterministically generates a subset $\hat{\mathbf{C}} \cdot \mathbf{x} \leq \hat{\mathbf{k}}$ of $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k}$ and checks that it is minimal and that the following conditions hold:

$$\mathbf{k} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}, \hat{\mathbf{k}} \geq \hat{\mathbf{d}} + \hat{\mathbf{M}} \cdot \mathbf{r} \text{ is satisfiable}$$

$$\forall \mathbf{x}\ [\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k} \rightarrow \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}] \text{ holds.}$$

Since minimality checking and the two conditions above are polynomial time problems, the devised procedure is in $\mathbb{NP}$. □

Despite the negative result, the proof provides us with a constructive procedure for deciding equivalence.

Example 5.2 *Consider again Examples 2.1 and 4.1. We have already computed $\mathbf{k} = (0\ 0\ 1\ 1\ 1)$. Therefore $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k}$ is:*

$$0 \leq x, 0 \leq y, x \leq 1, y \leq 1, x + y \leq 1$$

It is readily checked that $\forall \mathbf{x}\ [\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k} \rightarrow \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}]$ holds, since $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ is $0 \leq x, 0 \leq y, x+y \leq 1$. Also, condition $\mathbf{k} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ has been already computed:

$$1 \leq a, 1 \leq b, 1 \leq c \tag{10}$$

We are left with finding out a minimal subset of $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k}$. Inequalities $x \leq 1$ and $y \leq 1$ are clearly redundant (see Figure 1). Therefore, $\hat{\mathbf{C}} \cdot \mathbf{x} \leq \hat{\mathbf{k}}$ defined as:

$$0 \leq x, 0 \leq y, x + y \leq 1$$

is minimal – and, actually, it is the unique minimal subset. The system $\hat{\mathbf{k}} \geq \hat{\mathbf{d}} + \hat{\mathbf{M}} \cdot \mathbf{r}$ from the proof of Theorem 4 is then:

$$0 \geq 0, 0 \geq 0, 1 \geq c$$

which, together with (10) leads to conclude that the parameterized linear system is equivalent to $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ if and only if:

$$1 \leq a, 1 \leq b, c = 1$$

We now show that the hardness result holds even when restricting to the 2-dimensional plane. In the following, we make use of a system in 2 variables with n irredundant inequalities of the form $\alpha_i x + \beta_i y \leq \gamma_i$, for some constants $\alpha_i, \beta_i, \gamma_i$ with $i = 1 \ldots n$. An example of such a system is the one with solution points belonging to a regular n-gon (see Figure 3).

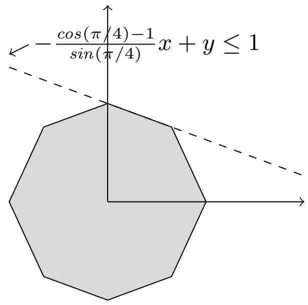

Figure 3. The regular 8-gon, an equiangular and equilateral polygon inscribed in the unit circle.

```
:- use_module(library(clpr)).

dec_inc(P, ParP) :-
4     copy_term(P-ParP, CopyP-CopyParP),
      tell_cs(CopyP),
6     max(CopyParP, ParP, Leq),
      tell_cs(Leq).

max([], [], []).
10 max([E =< _|Ps], [_ =< P1|P1s], [K=<P1|Ls]) :-
      sup(E, K),
12    max(Ps, P1s, Ls).

14 tell_cs([]).
tell_cs([C|Cs]) :-
16    {C},
      tell_cs(Cs).
```

Figure 4. Inclusion decision procedure in SWI-Prolog.

Theorem 5 *The problem of deciding whether (2) holds is* $\mathbb{NP}$*-complete even when restricting to 2 variables.*

Proof. The reduction of the 3SAT problem in the proof of Theorem 3 can be re-stated using only 2 variables as follows. The system $\mathbf{A}\cdot\mathbf{x} \leq \mathbf{b}$ is any system in 2 variables with n irredundant inequalities of the form $\alpha_i x + \beta_i y \leq \gamma_i$, with $i = 1 \ldots n$. The parameterized system $\mathbf{C}\cdot\mathbf{x} \leq \mathbf{d}+\mathbf{M}\cdot\mathbf{r}$ includes, for $i = 1 \ldots n$: an inequality $\alpha_i x+\beta_i y \leq \gamma_i + r_m$ for every positive literal a_m appearing in a clause ϕ_i; and an inequality $\alpha_i x + \beta_i y \leq \gamma_i + (1 - r_m)$ for every negated literal $\overline{a_m}$ appearing in a clause ϕ_i. The assumption that the n inequalities are irredundant allows us to conclude that some $\gamma_i + r_m$ or $\gamma_i + (1-r_m)$ must be equal to γ_i in any parameter instance that leads to the LHS polyhedron. The rest of the proof mimics the proof of Theorem 3. □

From the proof of Theorem 4, it is readily checked that a sufficient condition for concluding (2) is to check that $\mathbf{k} = \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ is satisfiable, and $\forall\mathbf{x}\ [\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k} \rightarrow \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}]$ holds, where $\mathbf{k}$ is defined as in the proof of the theorem. Since satisfiability of linear systems of equations is a problem in $\mathbb{P}$ and since checking the $\forall$ formula above is also in $\mathbb{P}$ (by Theorem 1), the condition above can be checked in polynomial time.

Example 5.3 *Consider the parameterized linear system* $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ *denoting axis-parallel hypercubes:*

$$r_i \leq x_i \leq s_i, \text{ for } i = 1 \ldots n$$

Since every inequality is irredundant, the only minimal subset of $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k}$ *is itself. As a consequence, the sufficient condition above is also necessary. Thus, we are in the position to conclude that deciding whether a linear system denotes an axis-parallel hypercube is a problem in* $\mathbb{P}$.

More generally, since the decision procedure consists of a search in the lattice of minimal subsets of a linear system, the membership problem for a sub-class of parameterized linear systems is in $\mathbb{P}$ if, for such a sub-class, the size of the lattice above can be polynomially bounded.

6 Decision Procedures in CLP($\mathbb{R}$)

The Constraint Logic Programming (CLP) Scheme defines a family of languages, CLP($\mathbb{C}$), that are parametric in the constraint domain $\mathbb{C}$. We consider here CLP($\mathbb{R}$), namely the constraint domain of linear inequalities over the reals [9]. CLP systems supporting CLP($\mathbb{R}$) include ECLiPSe, SICStus Prolog, SWI-Prolog, and many others. The decision procedures for inclusion (Theorem 2) and equivalence (Theorem 4) can be concisely coded in CLP($\mathbb{R}$).

Figure 4 shows the core implementation of the inclusion procedure. The code runs under SWI-Prolog [22]. Systems of linear inequalities are represented as lists of inequalities. The top-level predicate `dec_inc/2` is given a linear system P (i.e., $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$) and a parameterized linear system $ParP$ (i.e., $\mathbf{C}\cdot\mathbf{x} \leq \mathbf{d}+\mathbf{M}\cdot\mathbf{r}$). It makes a copy of its arguments (using the built-in `copy_term/2`). Then the inequalities in *CopyP* are asserted. The predicate `tell_cs/1` adds to the constraint store all inequalities passed as an argument. Then the predicate `max/3` computes the maximum[2] values of the copy of $\mathbf{C} \cdot \mathbf{x}$ over the constraint store (where the copy of $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ has been already asserted), which means calculating $\mathbf{k}$. `max/3` assumes in the second argument $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ and then it returns in the third argument $\mathbf{k} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$. Finally, the clause of `dec_inc/2` asserts the inequalities $\mathbf{k} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$, thus constraining the values of $\mathbf{r}$ to those for which inclusion holds.

Let P be `[0 =< X, 0 =< Y, X+Y =< 1]`, and $ParP$ be `[-X =< 0, X =< A, -Y =< 0, Y =< B, X+Y =< C]` the systems from Example 4.1. Here it is the answer of a query checking inclusion:

```
?- dec_inc(P, ParP).
{A>=1}, {B>=1}, {C>=1}.
```

Figure 5 shows the core implementation of the equivalence procedure. The top-level predicate `dec_eq/2` is given, again, a linear system P and a parameterized linear system $ParP$. It makes a copy of its arguments, then asserts the copy of the linear system *CopyP* and call a revised `max/4` predicate. As in the inclusion procedure, such predicate computes $\mathbf{k} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ (as the fourth argument). In addition, it also computes $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k}$ as the third argument. In order to do this, it needs as input both *CopyP* (as first argument) and *ParP* (as second argument). After the call to `max/4`, the clause defining `dec_eq/2` asserts the inequalities $\mathbf{k} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$, as in the inclusion procedure, then it checks that $\forall\mathbf{x}\ [\mathbf{C}\cdot\mathbf{x} \leq \mathbf{k} \rightarrow \mathbf{A}\cdot\mathbf{x} \leq \mathbf{b}]$ by calling `entails/2`. Such a predicate implements the procedure of Theorem 1 on a copy of its arguments: first, it asserts the LHS system of the implication, then it checks that the RHS system of the implication holds in the current constraint store by resorting to the SWI Prolog built-in `entailed/1`. The last condition to be checked from the proof of Theorem 4 requires to compute minimal subsets of $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k}$. This is done by the `minimal/4` predicate, which scans inequalities in $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{k}$ accumulating a subset in its first argument.

[2] The call of `sup/2` at line 11 fails if the expression E has no supremum in the current constraint store. In such a case, the overall call to `dec_inc/2` fails, which is consistent with the fact that the inclusion does not hold.

```
   :- use_module(library(clpr)).

   dec_eq(P, ParP) :-
4      copy_term(P-ParP, CopyP-CopyParP),
       tell_cs(CopyP),
6      max(CopyParP, ParP, CxK, Leq),
       tell_cs(Leq),
8      entails(CxK, P),
       minimal([], CxK, Leq, Eq),
10     tell_cs(Eq).

12 minimal(Sub, [], [], []) :-
       irredundant([], Sub).
14 minimal(Sub, [C|Ps], [K=<P|Leq], [K=P|Eq]) :-
       \+ entails(Sub, C),
16     minimal([C|Sub], Ps, Leq, Eq).
   minimal(Sub, [C|Ps], [_|Leq], Eq) :-
18     append(Sub, Ps, Rest),
       entails(Rest, [C]),
20     minimal(Sub, Ps, Leq, Eq).

22 irredundant(_, []).
   irredundant(Ss, [C|Cs]) :-
24     append(Ss, Cs, All),
       \+ entails(All, [C]),
26     irredundant([C|Ss], Cs).

28 entails(S, C) :-
       copy_term(S-C, S1-C1),
30     tell_cs(S1),
       is_entailed(C1).

   is_entailed([]).
34 is_entailed([C|Cs]) :-
       entailed(C),
36     is_entailed(Cs).

38 max([], [], [], []).
   max([E =< _|Ps], [E1 =< P1|P1s], [E1 =< K|Ls], [K=<P1|Es]) :-
40     sup(E, K),
       max(Ps, P1s, Ls, Es).
```

Figure 5. Equality decision procedure in SWI-Prolog.

Clauses at lines `14` and `17` implement the non-deterministic choice of including or not an inequality `C` in the subset. The clause at line `12` checks that the selected subset includes only irredundant inequalities by means of the `irredundant/2` predicate. `minimal/4` returns in the fourth argument the set of equalities $\hat{\mathbf{k}} = \hat{\mathbf{d}} + \hat{\mathbf{M}} \cdot \mathbf{r}$. These are asserted by the `dec_eq/3` clause in order to check whether they hold, and for which values of parameters in $\mathbf{r}$.

Let P be `[0 =< X, 0 =< Y, X+Y =< 1]`, and $ParP$ be `[-X =< 0, X =< A, -Y =< 0, Y =< B, X+Y =< C]` the systems from Example 5.2. Here it is the answer of a query checking equivalence:

```
?- dec_eq(P, ParP).
{A>=1}, {B>=1}, {C=1}.
```

7 Conclusion

Parameterized linear systems represent a "minimalist" extension of the expressive power of linear systems. It allows for defining classes of polyhedra by admitting parameters in the constant terms of inequalities. In this paper, we have investigated the problem of deciding whether the polyhedron of solutions of a linear system belongs to the class defined by a parameterized linear system. It allows us to code questions such as: (1) "does a hypercube belongs to the polyhedra defined by a parameterized linear system?"; or, (2) "does a regular n-gon belong to the polyhedra defined by a parameterized linear system over 2 variables?"; or, (3) "does a linear system denote a hypercube?". We have shown that the membership problem is $\mathbb{NP}$-complete. More precisely, answering questions (1) and (2) is $\mathbb{NP}$-complete. Our proof is constructive, providing us with an algorithm which, basically, consists of searching in a lattice of minimal subsets of a linear system. When such a lattice is polynomially bounded, the problem is in $\mathbb{P}$ – this is the case, for instance, of answering question (3). A concise implementation of the procedure has been devised in constraint logic programming over the reals.

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