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A Complexity Perspective on Entailment of Parameterized Linear Constraints

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Abstract Extending linear constraints by admitting parameters allows for more abstract problem modeling and reasoning. A lot of focus has been given to conducting research that demonstrates the usefulness of parameterized linear constraints and implementing tools that utilize their modeling strength. However, there is no approach that considers basic theoretical tools related to such constraints that allow for reasoning over them. Hence, in this paper we introduce *satisfiability with respect to polyhedral sets* and *entailment* for the class of parameterized linear constraints. In order to study the computational complexities of these problems, we relate them to classes of quantified linear implications. The problem of satisfiability with respect to polyhedral sets is then shown to be co-$\mathbb{NP}$ hard. The entailment problem is also shown to be co-$\mathbb{NP}$ hard in its general form. Nevertheless, we characterize some subclasses for which this problem is in $\mathbb{P}$. Furthermore, we examine a weakening and a strengthening extension of the entailment problem. The weak entailment problem is proved to be $\mathbb{NP}$ complete. On the other hand, the strong entailment problem is shown to be co-$\mathbb{NP}$ hard.

1 Introduction

Linear constraints can be extended by admitting parameters, which allow for more abstract problem modeling and reasoning. As an example, the points belonging to a rectangle of length 2 and height 3 can be described as the

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solutions $(x_1\, x_2)$ of the inequalities $0 \leq x_1 \leq 2, 0 \leq x_2 \leq 3$. More abstractly, the points belonging to a rectangle of length a and height b can be described as the solutions $(x_1\, x_2)$ of the inequalities $0 \leq x_1 \leq a, 0 \leq x_2 \leq b$. However, how do the basic notions of constraint satisfiability and entailment extend in the presence of parameters? Do their computational complexities become harder or intractable? In this paper, we answer precisely those two questions. We define satisfiability and entailment of parameterized linear constraints and investigate their computational complexities.

Let us provide some basic intuitions. Consider two linear constraints (or, equivalently, two linear systems of inequalities) over the reals $c_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ and $c_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}$. We say that c_1 entails c_2 if every solution of c_1 is a solution of c_2. In formal logic, c_1 entails c_2 if $\forall \mathbf{x}\, [c_1 \to c_2]$ is true in the domain of the reals. Entailment for linear constraints is a polynomial time decision problem. It reduces to showing that for every inequality $\mathbf{c}^T \cdot \mathbf{x} \leq d$ in c_2, the linear programming problem:

$$\max \mathbf{c}^T \cdot \mathbf{x}$$
$$\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$$

is either infeasible or its solution is bounded by d. Since linear programming problems are solvable in polynomial time, as shown first by Khachiyan (1979), the conclusion readily follows.

A parameterized linear constraint over the reals is a system of linear inequalities $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ where variables in $\mathbf{s}$ are *parameters*. The intended meaning of a parameterized linear constraint is a collection of linear constraints over variables in $\mathbf{x}$, each obtained by instantiating the parameters $\mathbf{s}$. We say that a parameterized linear constraint $cp_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ entails $cp_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{P} \cdot \mathbf{s}$ if for every parameter instance $\mathbf{s}_0$ of $\mathbf{s}$ there exists an instance $\mathbf{r}_0$ of $\mathbf{r}$ such that the (non-parameterized) linear system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}_0$ entails $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0 + \mathbf{P} \cdot \mathbf{s}_0$. Notice that $\mathbf{r}$ occurs only in cp_2, while $\mathbf{s}$ may occur in both. Intuitively, entailment is a generalization of inclusion: for every instance of cp_1 obtained by fixing $\mathbf{s}$ to $\mathbf{s}_0$, we can find at least one instance of cp_2, obtained by possibly fixing additional parameters $\mathbf{r}$ to $\mathbf{r}_0$, that admits at least the same set of solutions.

As an example, consider the two parameterized linear systems:

$$\begin{array}{ccc} & \quad & x_1 + x_2 \leq d \\ 0 \leq x_1 \leq a & & 0 \leq x_1 \\ 0 \leq x_2 \leq b & & 0 \leq x_2 \\ c \leq x_3 \leq c+2 & & c \leq x_3 \leq c+2 \\ (cp_1) & & (cp_2) \end{array}$$

In cp_1, x_1 and x_2 vary from 0 to a and b respectively, while x_3 assumes any value in a range of width 2 (delimited by c and $c+2$). For fixed a and b, every solution (x_1, x_2, x_3) of cp_1 is such that $x_1 + x_2 \leq a + b$. By setting $d = a + b$, we have that any instance of cp_1 entails some instance of cp_2. Notice that parameters in cp_1, namely a, b and c, are universally quantified over, while parameters that appear only in cp_2, namely d, are existentially quantified over. Hence, this entailment problem is equivalent to the following implication:

$$\forall a\ \forall b\ \forall c\ \exists d\ \forall x_1\ \forall x_2\ \forall x_3\ \ [cp_1 \rightarrow cp_2]$$

In this paper, we introduce entailment and a notion of satisfiability with respect to polyhedral sets as basic tools for reasoning over parameterized linear constraints. In their general formulation, both problems are shown to be co-$\mathbb{NP}$ hard. We characterize some tractable instances constructively, namely by providing algorithms for checking satisfiability and entailment. A weakening and a strengthening variant of the entailment problem are also investigated.

The rest of this paper is organized as follows. Section 2 discusses the preliminaries and notations that will be used for the rest of the paper. Section 3 details the motivation for our work, our contributions, and related work. Satisfiability is dealt with in Section 4. Entailment is defined in Section 5, and its complexity investigated in Section 6. Some tractable instances are characterized in Section 7. Extensions of entailment are considered in Section 8, including a weaker and a stronger notion of entailment. Finally, we summarize our contributions and discuss avenues for future research in Section 9.

2 Background and Notation

2.1 Linear Systems and Constraints

We adhere to standard notation of linear algebra (Schrijver, 1987). $\Re$ is the set of real numbers. Small bold letters ($\mathbf{a}$, $\mathbf{b}$, ...) denote column vectors, while capital bold letters ($\mathbf{A}$, $\mathbf{B}$, ...) denote matrices. $\mathbf{0}$ is the column vector with all elements equal to 0 and $\mathbf{1}$ is the column vector with all elements equal to 1. Also, a_i denotes the i^{th} element in $\mathbf{a}$, while the transposed vector of $\mathbf{a}$ is denoted by $\mathbf{a}^T$. The inner product is denoted by $\mathbf{a^T} \cdot \mathbf{b}$.

$\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ denotes a system of linear inequalities over the variables in $\mathbf{x}$, also called a *linear system*. We assume that the dimensions of vectors and matrices in inner products and linear systems are of the appropriate size. A *polyhedron* is the set of solution points of a linear system: $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}) = \{\mathbf{x}_0 \in \Re^{|\mathbf{x}|} \mid \mathbf{A} \cdot \mathbf{x}_0 \leq \mathbf{b}\}$. Polyhedra are convex sets. A linear system is *satisfiable* if its polyhedron is non-empty. A linear program:

$$\begin{aligned} &\max \mathbf{c}^T \cdot \mathbf{x} \\ &\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \end{aligned}$$

is the problem of finding $\max\{\mathbf{c}^T \cdot \mathbf{x}_0 \mid \mathbf{x}_0 \in Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})\}$. If the polyhedron $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})$ is empty, the problem is said to be *infeasible*. If the linear function $\mathbf{c}^T \cdot \mathbf{x}$ is unbounded, the problem is said to be *unbounded*.

An equivalent formulation of linear systems is provided in terms of logic formulas over the reals. A *primitive linear constraint* is an expression $a_1\, x_1 +$

$\ldots + a_n\, x_n \le a_0$, where $a_0, \ldots, a_n$ are constants in $\Re$ and $x_1, \ldots, x_n$ are variables. We will also use the inner product form by rewriting it as $\mathbf{c}^T \cdot \mathbf{x} \le \alpha$. A *linear constraint* c is a conjunction of primitive constraints. Conjunction is syntactically represented either by "$\wedge$" or by a comma. Inequalities $\mathbf{c}^T \cdot \mathbf{x} \ge \alpha$ and equalities $\mathbf{c}^T \cdot \mathbf{x} = \alpha$ can be reduced to linear constraints. Any linear system can be represented as a linear constraint and vice-versa. So, we will use the two notions interchangeably throughout the paper. A *difference constraint* is a primitive linear constraint of the form $x_i - x_j \le a_{ij}$, while a linear system which consists only of difference constraints is referred to as *difference constraints system.*

2.2 Parameterized Linear Constraints

A *parameterized linear system* over the reals is a system *cp* of linear inequalities $\mathbf{A} \cdot \mathbf{x} \le \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ where variables in $\mathbf{s}$ are *parameters.* A *parameterized primitive linear constraint* is an expression $a_1\, x_1 + \ldots + a_i\, x_i \le a_0 + n_1 s_1 + \ldots + n_j s_j$ where $a_0, \ldots, a_i$ and $n_1, \ldots, n_j$ are constants in $\Re$, $x_1, \ldots, x_i$ are variables and $s_1, \ldots, s_j$ are parameters. A *parameterized linear constraint cp* is a conjunction of parameterized primitive linear constraints. The intended meaning of a parameterized linear system is then a collection of linear systems over variables in $\mathbf{x}$, each obtained by instantiating the parameters $\mathbf{s}$. The notion of parameterized polyhedra from Loechner and Wilde (1997) models the solutions of parameterized linear systems and is presented below.

Definition 1 A *parameterized polyhedron* is a collection of polyhedra defined by fixing the value for parameters in a parameterized system of linear inequalities: $Sol(\mathbf{A} \cdot \mathbf{x} \le \mathbf{b} + \mathbf{N} \cdot \mathbf{s}, \mathbf{s}_0) = \{\mathbf{x}_0 \in \Re^{|\mathbf{x}|} \mid \mathbf{A} \cdot \mathbf{x}_0 \le \mathbf{b} + \mathbf{N} \cdot \mathbf{s_0}\}$, where $\mathbf{s}_0 \in \Re^{|\mathbf{s}|}$ is an instance of the parameters $\mathbf{s}$.

Note that $Sol()$ now corresponds to the set of solution points of a parameterized linear system with respect to a specific assignment to parameters.

2.3 Quantified Linear Programs

A *quantified linear program*, introduced by Subramani (2007), is a linear constraint in which variables can be either existentially or universally quantified. The general form is an alternation of existential and universal quantifiers:

$$\exists \mathbf{x}_1\, \forall \mathbf{x}_2 \in [\mathbf{l}_2, \mathbf{u}_2]\ \ldots \exists \mathbf{x}_{n-1}\, \forall \mathbf{x}_n \in [\mathbf{l}_n, \mathbf{u}_n]\ \ \mathbf{A} \cdot \mathbf{x} \le \mathbf{b}$$

where $\mathbf{x}_1 \ldots \mathbf{x}_n$ is a partition of $\mathbf{x}$ with, possibly, $\mathbf{x}_1$ and/or $\mathbf{x}_n$ empty, and $\mathbf{l}_{2i}$, $\mathbf{u}_{2i}$ are lower and upper bounds in the reals for $\mathbf{x}_{2i}, i = 1, \ldots, n/2$. Note that existentially quantified variables are not bounded in the quantifier string; this is because any such bounds can be placed within the linear constraint. However, this is not the case for universally quantified variables, since their values are only restricted by the bounds on the quantifier string. Hence, if those bounds were to be put in the linear constraint, the query would then be trivially falsified.

We will make use of the following result from (Subramani, 2007), where from now on validity and satisfiability of a formula is to be intended in the domain of the reals. In particular, a linear system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ is satisfiable if and only if its existential closure $\exists \mathbf{x}\ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ holds in the domain of the reals.

Theorem 1 *The validity problem for the class of quantified linear programs of the form* $\forall \mathbf{x}_1 \in [\mathbf{l_1}, \mathbf{u_1}]\ \ \exists \mathbf{x}_2\ \ \mathbf{A}_1 \cdot \mathbf{x}_1 + \mathbf{A}_2 \cdot \mathbf{x}_2 \leq \mathbf{b}$ *is co-*$\mathbb{NP}$ *complete.*

2.4 Quantified Linear Implications

The notion of quantified linear implications was introduced in (Eirinakis et al, 2012). Quantified linear implications extend quantifications to implications of linear constraints by considering formulas of the form:

$$\exists \mathbf{x}_1\ \forall \mathbf{x}_2\ \ \ldots \exists \mathbf{x}_{n-1}\ \forall \mathbf{x}_n\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}]$$

where $\mathbf{x}_1 \ldots \mathbf{x}_n$ is a partition of $\mathbf{x}$ with, possibly, $\mathbf{x}_1$ and/or $\mathbf{x}_n$ empty.

Let us introduce a nomenclature to succinctly specify the problems under consideration. We use a triple $\langle T, Q, R \rangle$ to denote the number T of quantifier alternations, the first quantifier Q, and a $(T+1)$-character string R, which specifies for each quantified set of variables (in the order they appear in the quantifier string) whether they may participate in the **L**eft, in the **R**ight, or in **B**oth sides of the implication. For instance, $\langle 1, \exists, \mathbf{RB} \rangle$ indicates a problem of the form:

$$\exists \mathbf{x}_1\ \forall \mathbf{x}_2\ \ [\mathbf{A} \cdot \mathbf{x}_2 \leq \mathbf{b} \rightarrow \mathbf{C}_1 \cdot \mathbf{x}_1 + \mathbf{C}_2 \cdot \mathbf{x}_2 \leq \mathbf{d}]$$

We will make use of the following result from Subramani (2009), which formalizes the decision procedure for entailment of non-parameterized linear constraints provided in the introduction.

Theorem 2 *The validity problem for the class of quantified linear implications of the form* $\forall \mathbf{x}\ \ [\mathbf{A}\mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}]$*, i.e., for* $\langle 0, \forall, \mathbf{B} \rangle$*, is in* $\mathbb{P}$*.*

3 Motivation, Contributions and Related Work

3.1 Motivation

Parameterized linear constraints are an extension of linear systems that can be traced back to the late 60's in the context of (multi-)parametric linear programming, where the objective is to optimize a parameterized linear or quadratic function over the solutions of a parameterized linear system. Parametric linear programming is closely related to sensitivity analysis of linear programs, where one is interested in determining how "sensitive" the optimal solution is to changes in variable values. Parametric linear programming, instead, provides a complete map of the optimal solution in the space of the parameter values. Parametric linear programming has a solid theoretical basis (Willner, 1967; Murty, 1980; Gal, 1995; Pistikopoulos et al, 2007b) and a wide range of applications, including hybrid parametric/stochastic programming, process planning under uncertainty, material design under uncertainty, model based control, multi-processor scheduling (Kvasnica, 2009; Pistikopoulos et al, 2007a).

Besides linear programming problems, parameterized linear systems are recently gaining interest from several research communities. They have been adopted for: symbolic dependence analysis in computing a parallel schedule (Loechner and Wilde, 1997), fracture mechanics and engineering (Ioakimidis, 2000; Solares and Chaves, 2008), hardware verification (Cachera and Morin-Allory, 2005), type systems for constraint programming (Ruggieri and Mesnard, 2010), real-time scheduling (Subramani, 2002, 2005a), and loop invariant inference in program verification (Colón and Sankaranarayanan, 2011).

The explicit calculation of the solutions of a parameterized system of inequalities, represented in a Minkowski's sum of rays and parameterized vertices, can be computed by a generalization of the double description method, as described in Loechner and Wilde (1997). The `polylib` library (Loechner, 2011) implements the approach and provides basic set-oriented primitives for manipulating the solutions of parameterized linear systems. In the context of parametric linear programming, the Multi-Parametric Toolbox (Kvasnica, 2009; Kvasnica et al, 2011) solves parametric linear and quadratic programming problems by explicitly computing the optimization function over the solutions of a parameterized linear system.

While the above research and tools demonstrate the usefulness of parameterized linear constraints, to the best of our knowledge, there is no approach that considers basic tools for reasoning over them, and in particular entailment. As an example, linear constraint-based languages and systems (such as the constraint logic programming systems CLP($\mathcal{R}$), ECLiPSe, Sictus Prolog, SWI Prolog) could be readily extended to reason about parameterized linear constraints by using satisfiability and entailment as building blocks. It is then of utmost importance to understand and characterize the computational complexities of the satisfiability and entailment problems for parameterized linear constraints. This basic issue motivates our work.

3.2 Contributions

The entailment problem for parameterized linear constraints is equivalent to the validity problem for the $\langle 2, \forall, \mathbf{BRB} \rangle$ class, namely to formulas:

$$\forall \mathbf{s} \ \exists \mathbf{r} \ \forall \mathbf{x} \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{P} \cdot \mathbf{s}] \qquad (1)$$

The principal contributions of the paper are briefly presented below:

1. The validity problem for the $\langle 2, \forall, \mathbf{BRB}\rangle$ class is polynomial-time reducible to the validity problem for the $\langle 2, \forall, \mathbf{LRB}\rangle$ class and vice-versa (Lemma 2).
2. Query (1) (and hence the entailment problem) is co-$\mathbb{NP}$ hard (Theorem 4).
3. Query (1) is co-$\mathbb{NP}$ hard even if $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ is a difference constraints system (Corollary 2).
4. Query (1) is co-$\mathbb{NP}$ hard even if the set of variables $\mathbf{x}$ is empty (Theorem 5).
5. Query (1) is in $\mathbb{P}$ under any of the following restrictions:
   (a) no primitive linear constraint contains both variables $\mathbf{x}$ and parameters $\mathbf{s}$ (Theorem 6). This includes the case in which there are no parameters $\mathbf{s}$, or, stated otherwise, the class $\langle 1, \exists, \mathbf{RB}\rangle$ (Corollary 3),
   (b) for every parameter in $\mathbf{r}$, its coefficients in $\mathbf{M}$ are all non-negative or all non-positive (Lemma 4),
   (c) $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{P} \cdot \mathbf{s}$ consists of a single equality (Lemma 5),
   (d) $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{P} \cdot \mathbf{s}$ can be partitioned into two systems for which entailment is in $\mathbb{P}$ and with no overlapping parameters in $\mathbf{r}$ (Lemma 6).

A weakening and a strengthening extension of the entailment problem are discussed in Section 8. The weak entailment problem is equivalent to the validity problem for the $\langle 1, \exists, \mathbf{BB}\rangle$ class, while the strong entailment problem is defined by replacing implication with equivalence in query (1). We show that:

6. The weak entailment problem is $\mathbb{NP}$ complete (Theorem 7 and Theorem 8);
7. The strong entailment problem is at least as hard as the entailment problem, and therefore co-$\mathbb{NP}$ hard (Theorem 9).

3.3 Related Work

Polynomial time complexity for the satisfiability of linear systems was firstly achieved by Khachiyan (1979), with further improvements by Karmarkar (1984) and Vaidya (1987). Entailment of non-parameterized linear systems is in $\mathbb{P}$, as discussed intuitively in the introduction; the formal proof is due to Subramani (2009). Unfortunately, a direct extension of that procedure to parameterized linear systems by using parametric linear programming does not work. This is because the optimal solution of a parametric linear programming problem is a piecewise affine function defined over a polyhedral partition of the feasible parameters (Borrelli et al, 2003). Such a function can be exponentially large in the size of the linear program, even if there is only one parameter (Murty, 1980). The notion of quantified linear implications was first discussed in (Eirinakis et al, 2012). The general class of quantified propositional combinations of linear inequalities is known as the *theory of real numbers with addition*. Sontag (1985) shown that the sub-class of such formulas starting with an existential quantifier and with $k-1$ quantifier alternations is log-complete in Σ_k^P.

Quantified formulas over the reals can be solved by quantifier elimination methods (Dolzmann et al, 1998b; Weispfenning, 1988). Although its complexity has been improved several times (Renegar, 1992; Basu et al, 1996), in the worst case, quantifier elimination is doubly exponential in the number of quantifier alternations and exponential in the number of variables (Davenport and Heintz, 1988; Weispfenning, 1988). Nevertheless, approaches efficient-in-practice have been proposed and successfully applied to theorem proving and program verification. We mention partial cylindrical algebraic decomposition by Collins and Hong (1991), provided in the QEPCAD/QEPCAD-B systems (Brown, 2003), and virtual substitution of test terms by Dolzmann et al (1998a), provided in the REDLOG system (Dolzmann and Sturm, 1997), which is specialized for low-degree polynomials. Finally, we mention the RSolver algorithm by Ratschan (2006), whose implementation is publicly available (Ratschan, 2011).

The computational complexities of satisfiability and entailment in presence of quantifiers have been considered also in related domains (without parameters). Efficient solvers have been proposed for quantified boolean formulas (Le Berre et al, 2004; Giunchiglia et al, 2004; Pulina and Tacchella, 2009; Egly et al, 2009). Satisfiability of quantified linear systems is covered by Subramani (2007). The integer case is detailed in (Subramani, 2005b). The use of quantifiers has also been extended in the context of constraint satisfaction problems (Bodirsky and Chen, 2009; Stynes and Brown, 2009), with Gottlob et al (2005) investigating computatonal complexity. A framework for solving constraint satisfaction problems is presented by Goldsztejn et al (2009). An extension of satisfiability with preferences is examined in Di Rosa et al (2010). Entailment of binary and multi-valued propositional logics is investigated by Cadoli and Schaerf (1996). Subramani (2005b) covers entailment of linear systems over integers.

4 The Satisfiability Problem

Let us introduce the definition of satisfiability for parameterized linear constraints, which, intuitively, holds when there is an instance of the parameters with a non-empty set of solutions.

Definition 2 A parameterized linear constraint $cp = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ is satisfiable if there exists $\mathbf{s}_0 \in \Re^{|\mathbf{s}|}$ such that $Sol(cp, \mathbf{s}_0) \neq \emptyset$.

Stated otherwise, cp is satisfiable if there exists a parameter instance $\mathbf{s}_0$ such that $cp[\mathbf{s} \leftarrow \mathbf{s_0}]$ is satisfiable, where $cp[\mathbf{s} \leftarrow \mathbf{s_0}]$ is the linear constraint obtained by instantiating the parameters $\mathbf{s}$ with real values $\mathbf{s_0}$. Satisfiability in presence of parameters can be reduced to satisfiability of (non-parameterized) linear constraints. Such a reduction is well-known in the context of parametric linear programming (Gal, 1995).

Lemma 1 *The problem of checking the satisfiability of parameterized linear constraints is in* $\mathbb{P}$.

Proof It is readily checked that $cp = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ is satisfiable if and only if $\exists \mathbf{s}\ \exists \mathbf{x}\ cp$ holds in the domain of the reals. That is, if and only if cp

is satisfiable as a linear constraint over the space of variables plus parameters $(\mathbf{x}, \mathbf{s})$. The conclusion follows since the satisfiability of linear constraints is in $\mathbb{P}$ (Khachiyan, 1979). $\square$

Example 1 Consider variables x_1, x_2 and parameters s_1, s_2, and the following parameterized linear constraint:

$$cp = (x_1 + x_2 \leq s_1) \ \wedge \ (x_1 - x_2 \geq s_2) \ \wedge \ (s_1 \leq 1) \ \wedge \ (s_2 \geq 0)$$

A solution of *cp* as a non-parameterized linear system over the space of variables plus parameters is: $s_1 = s_2 = x_1 = x_2 = 0$. Hence, there is an instance of the parameters $(s_1 = s_2 = 0)$ with a non-empty set of solutions $(x_1 = x_2 = 0)$.

The space of parameter instances $\mathbf{s}$ with $Sol(cp, \mathbf{s}) \neq \emptyset$ is called *feasibility domain*. It turns out to be a polyhedron; in fact, it can be obtained by simply eliminating all variables $\mathbf{x}$ from *cp*. A strengthening of satisfiability consists then of checking whether every parameter in a given polyhedral set admits a solution, i.e., whether the polyhedral set is included in the feasibility domain. Let us introduce satisfiability with respect to polyhedral sets of parameters.

Definition 3 A parameterized linear constraint $cp = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ is satisfiable with respect to (the solutions of) $\mathbf{P}\cdot\mathbf{s} \leq \mathbf{d}$ if for every $\mathbf{s}_0 \in Sol(\mathbf{P}\cdot\mathbf{s} \leq \mathbf{d})$ we have $Sol(cp, \mathbf{s}_0) \neq \emptyset$.

That is, satisfiability with respect to polyhedral sets can be expressed in terms of quantified linear implications as follows:

$$\forall \mathbf{s} \ \exists \mathbf{x} \ \ [\mathbf{P} \cdot \mathbf{s} \leq \mathbf{d} \rightarrow \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}]$$

Remark 1 The class of problems of satisfiability with respect to polyhedral sets is equivalent to the validity problem for the class of formulas $\langle 1, \forall, \mathbf{BR} \rangle$.

Satisfiability with respect to polyhedral sets is a co-$\mathbb{NP}$ hard problem.

Theorem 3 *The problem of checking satisfiability with respect to polyhedral sets of parameters is co-$\mathbb{NP}$ hard.*

Proof We reduce a co-$\mathbb{NP}$ complete problem to the problem of satisfiability with respect to polyhedral sets of parameters. Let $\mathbf{P} \cdot \mathbf{s} \leq \mathbf{d}$ be $\mathbf{l} \leq \mathbf{s} \leq \mathbf{h}$, where $\mathbf{l}, \mathbf{u} \in \Re^{|\mathbf{s}|}$. Definition 3 reduces to showing:

$$\forall \mathbf{s} \ \ \exists \mathbf{x} \ \ [\mathbf{l} \leq \mathbf{s} \leq \mathbf{h} \rightarrow \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}]$$

which, in turn, is equivalent to: $\forall \mathbf{s} \in [\mathbf{l}, \mathbf{h}] \ \ \exists \mathbf{x} \ \ \mathbf{A}\cdot\mathbf{x} - \mathbf{N}\cdot\mathbf{s} \leq \mathbf{b}$. By Theorem 1, the validity problem for this class of formulas is co-$\mathbb{NP}$ complete. $\square$

Example 2 Consider Example 1. By Fourier-Motzkin elimination of x_1 and x_2, *cp* is equivalent to the system $(s_2 + x_2 \leq x_1 \leq s_1 - x_2) \wedge (2x_2 \leq s_1 - s_2) \wedge (s_1 \leq 1) \ \wedge \ (s_2 \geq 0)$. Hence, the feasibility domain of *cp* is $(s_1 \leq 1) \ \wedge \ (s_2 \geq 0)$.

5 The Entailment Problem

The entailment problem for parameterized linear constraints is formulated next.

Definition 4 We say that $cp_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ entails $cp_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{P} \cdot \mathbf{s}$ if for every $\mathbf{s}_0 \in \Re^{|\mathbf{s}|}$ there exists $\mathbf{r}_0 \in \Re^{|\mathbf{r}|}$ such that $Sol(cp_1, \mathbf{s}_0) \subseteq Sol(cp_2, (\mathbf{s}_0\, \mathbf{r}_0))$.

We call cp_1 the *entailing constraint* and cp_2 the *entailed constraint*. Intuitively, parameters in $\mathbf{s}$ may occur both in cp_1 and in cp_2, and are universally quantified. Parameters in $\mathbf{r}$ occur only in cp_2 and they are existentially quantified.

Example 3 Let x be a variable and r a parameter. The problem of deciding whether cp_1 entails $cp_2 : (x = r)$ consists of deciding whether x is equal to a constant value once the parameters in cp_1 are fixed. Similarly, cp_1 entails $cp_2 : (x \leq r)$ means that x is upper bounded once the parameters in cp_1 are fixed, while cp_1 entails $cp_2 : (r \leq x \leq r + 1)$ means that x has a variability range of width at most 1. These observations allow for reasoning over problems with uncertainty or imprecision. As an example, assume that a Celsius temperature (represented by variable x_c) is known with an approximation of at most ± 2 degrees, and that we are interested in checking its approximation in Fahrenheit scale (the Fahrenheit temperature is represented by variable x_f). The constraint $cp_1 : (s - 2 \leq x_c \leq s + 2) \ \wedge \ (x_f = 9/5 x_c + 32)$ models the approximation $(s - 2 \leq x_c \leq s + 2)$ and the conversion rule $(x_f = 9/5 x_c + 32)$. Then, cp_1 entails $cp_2 : (r - 3.6 \leq x_f \leq r + 3.6)$ if the approximation on the Fahrenheit scale is at most ± 3.6 degrees, where s and r are the parameters of the entailing and the entailed constraint respectively.

An equivalent formulation in terms of quantified linear implications follows immediately.

Remark 2 The class of entailment problems is equivalent to the validity problem for the class of formulas $\langle 2, \forall, \mathbf{BRB} \rangle$.

Stated otherwise, proving that $cp_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ entails $cp_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{P} \cdot \mathbf{s}$ is equivalent to showing the validity of the formula:

$$\forall \mathbf{s} \ \exists \mathbf{r} \ \forall \mathbf{x} \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{P} \cdot \mathbf{s}] \tag{2}$$

Example 4 Consider the following parameterized linear constraints:

$$\begin{aligned} cp_1 &: (-1 + s_1 \leq x_1 + x_2 \leq 1 + 2s_1) \ \wedge \ (-1 + s_1 \leq x_1 - x_2 \leq 1 + 3s_1) \\ cp_2 &: (-1 - s_1 \leq x_1 \leq 1 + 2r_1 + s_1) \ \wedge \ (-1 - r_1 \leq x_2 \leq 1 + r_1 + 2s_1) \end{aligned}$$

Figure 1 presents the entailing (cp_1) and the entailed constraint (cp_2) for parameter values $s_1 = r_1 = 0$. Note that for these specific values cp_1 is included

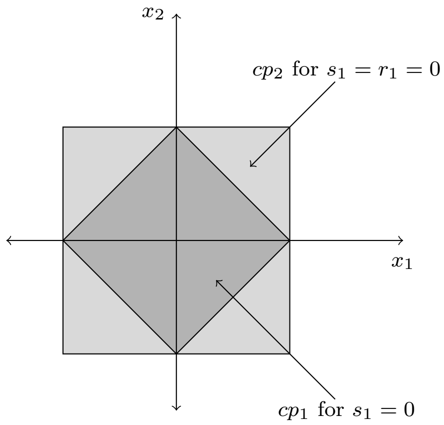

Fig. 1 cp_1 entails cp_2 from Example 4.

in cp_2. However, in order to check whether cp_1 entails cp_2, we need to check the validity of the following $\langle 2, \forall, \mathbf{BRB} \rangle$ instance:

$$\forall s_1 \, \exists r_1 \, \forall x_1 \, \forall x_2$$
$$\begin{array}{lcl} x_1 + x_2 \geq -1 + s_1 & & x_1 \geq -1 - s_1 \\ x_1 + x_2 \leq 1 + 2s_1 & \rightarrow & x_1 \leq 1 + 2r_1 + s_1 \\ x_1 - x_2 \geq -1 + s_1 & & x_2 \geq -1 - r_1 \\ x_1 - x_2 \leq 1 + 3s_1 & & x_2 \leq 1 + r_1 + 2s_1 \end{array}$$

Formulating the entailment problem as a quantified linear implication problem is very useful in proofs. For instance, the following result shows that, as one would expect, entailing an always false constraint boils down to unsatisfiability. We code the always false logic predicate as $0 = 1$.

Corollary 1 *Let* $cp = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$. *Then* cp *entails* $0 = 1$ *if and only if* cp *is unsatisfiable.*

Proof By Remark 2, cp entails $0 = 1$ if and only if $\forall \mathbf{s} \, \forall \mathbf{x} \, [cp \rightarrow 0 = 1]$. The latter can be rewritten as $\neg \exists \mathbf{s} \, \exists \mathbf{x} \, cp$, i.e., as the complement of the satisfiability of cp. $\square$

The next result shows that, when considering the computational complexity of the entailment problem, we can restrict ourselves to entailed constraints cp_2 without any parameter appearing in cp_1, namely cp_2 of the form $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$, i.e., with $\mathbf{P} = \mathbf{0}$. Stated otherwise, we can restrict ourselves to the validity problem for formulas in the class $\langle 2, \forall, \mathbf{LRB} \rangle$:

$$\forall \mathbf{s} \;\; \exists \mathbf{r} \;\; \forall \mathbf{x} \;\; [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}] \tag{3}$$

Lemma 2 *The validity problems for the classes of formulas* $\langle 2, \forall, \mathbf{BRB} \rangle$ *and* $\langle 2, \forall, \mathbf{LRB} \rangle$ *are polynomial-time reducible to each other.*

Proof Every formula in $\langle 2, \forall, \mathbf{LRB} \rangle$ is also in $\langle 2, \forall, \mathbf{BRB} \rangle$. To show the opposite, note that formula (2) holds if and only if the following holds:

$$\forall \mathbf{s}\ \exists \mathbf{r}\ \forall \mathbf{x}', \mathbf{x}\ \ [\mathbf{x}' = \mathbf{s} \wedge \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \rightarrow \mathbf{C} \cdot \mathbf{x} - \mathbf{P} \cdot \mathbf{x}' \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}] \quad (4)$$

More specifically, note that $\mathbf{s}$ has been replaced by the newly introduced $\mathbf{x}'$ in the right hand side of formula (4). Nevertheless, the variables in $\mathbf{x}'$ are constrained to assume the values of the parameters in $\mathbf{s}$, since $\mathbf{x}' = \mathbf{s}$ has been added in the left hand side of formula (4) (note that the existence of $\mathbf{x}' = \mathbf{s}$ only restricts the values of $\mathbf{x}'$ and not of $\mathbf{s}$). Hence, although $\mathbf{s}$ has been removed from the right hand side, the entailed constraint still depends on the values of $\mathbf{s}$ exactly as in (2). Therefore, if formula (2) holds, then the corresponding formula (4) also holds. The opposite is shown similarly.

Since formula (4) is of the form (3), and it is clearly obtained from formula (2) in linear time, we have that $\langle 2, \forall, \mathbf{BRB} \rangle$ reduces in polynomial time to $\langle 2, \forall, \mathbf{LRB} \rangle$. □

Example 5 Recall the $\langle 2, \forall, \mathbf{BRB} \rangle$ instance of Example 4. It can be transformed into an equivalent $\langle 2, \forall, \mathbf{LRB} \rangle$ instance by introducing variable x_1', which replaces parameter s_1 in the entailed constraint:

$$\forall s_1\ \exists r_1\ \forall x_1\ \forall x_2\ \forall x_1'$$

$$\begin{array}{lcl} x_1 + x_2 \geq -1 + s_1 & & x_1 + x_1' \geq -1 \\ x_1 + x_2 \leq 1 + 2s_1 & & x_1 - x_1' \leq 1 + 2r_1 \\ x_1 - x_2 \geq -1 + s_1 & \rightarrow & x_2 \geq -1 - r_1 \\ x_1 - x_2 \leq 1 + 3s_1 & & x_2 - 2x_1' \leq 1 + r_1 \\ x_1' = s_1 & & \end{array}$$

6 Complexity of Entailment

The entailment problem is, in general, at least as hard as any problem in co-$\mathbb{NP}$.

Theorem 4 *The entailment problem for parameterized linear constraints is co-$\mathbb{NP}$ hard.*

Proof We reduce a co-$\mathbb{NP}$ complete problem to the validity problem for $\langle 2, \forall, \mathbf{LRB} \rangle$ formulas. The result then follows by Remark 2 and Lemma 2. Let $\mathbf{l}, \mathbf{u} \in \Re^{|\mathbf{s}|}$. By setting $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ in (3) to $\mathbf{x} = \mathbf{s} \wedge \mathbf{l} \leq \mathbf{s} \leq \mathbf{h}$, we have the following instance:

$$\forall \mathbf{s}\ \ \exists \mathbf{r}\ \ \forall \mathbf{x}\ \ [\mathbf{x} = \mathbf{s} \wedge \mathbf{l} \leq \mathbf{s} \leq \mathbf{h} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}]$$

which is equivalent to: $\forall \mathbf{s} \in [\mathbf{l}, \mathbf{h}]\ \exists \mathbf{r}\ \ \mathbf{C} \cdot \mathbf{s} - \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}$. By Theorem 1, the validity problem for this class of formulas is co-$\mathbb{NP}$ complete. □

It is worth noting from the proof of the above theorem that hardness for the entailment problem holds even under rather restrictive conditions on the entailing constraint cp_1, namely that for every parameter instance $\mathbf{s}$:

- $Sol(cp_1, \mathbf{s})$ is non-empty if and only if $\mathbf{s}$ belongs to a hypercube ($\mathbf{l} \leq \mathbf{s} \leq \mathbf{h}$), and,
- if $Sol(cp_1, \mathbf{s})$ is non-empty, then it is singleton (since $\mathbf{x} = \mathbf{s}$).

Next, we examine a special case of the entailment problem in which the entailing constraint consists only of difference constraints. This case will also be shown to be co-$\mathbb{NP}$ hard. Its importance lies in the interesting structural properties of difference constraints systems and in their applicability in various fields, e.g., program verification.

Corollary 2 *The entailment problem is co-$\mathbb{NP}$ hard, even if the entailing constraint is a system of difference constraints.*

Proof Consider a quantified linear program of the form

$$\forall \mathbf{z} \in [\mathbf{l}, \mathbf{u}] \; \exists \mathbf{y} \; \mathbf{C} \cdot \mathbf{z} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{y} \tag{5}$$

Recall that the problem of checking the validity of such a class of formulas is co-$\mathbb{NP}$ complete (Theorem 1). We reduce this problem to an instance of the form (3) with $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ being restricted to a difference constraints system as follows:

1. Add the constraints $x_1 - s_1 \leq 0$ and $s_1 - x_1 \leq 0$ to $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$.
2. For each element z_i of $\mathbf{z}$, add the constraints:
   - $x_{i+1} - s_{i+1} \leq 0$,
   - $s_{i+1} - x_{i+1} \leq 0$,
   - $s_{i+1} - s_1 \leq u_i$, and
   - $s_{i+1} - s_1 \geq l_i$

   to $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$.
3. Let vector $\mathbf{x}'$ be such that $x'_i = x_{i+1} - x_1$.
4. Let vector $\mathbf{s}'$ be such that $s'_i = s_{i+1} - s_1$.
5. Create the instance $\forall \mathbf{s} \; \exists \mathbf{r} \; \forall \mathbf{x} \; [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \rightarrow \mathbf{C} \cdot \mathbf{x}' \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}]$.

In this instance we have that $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ can only be satisfied if $\mathbf{x} = \mathbf{s}$ and $l_i \leq s_{i+1} - s_1 = s'_i \leq u_i$. Thus, for the constructed instance to hold, $\mathbf{C} \cdot \mathbf{x}' \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ must also hold when $\mathbf{x}' = \mathbf{s}'$ and $l_i \leq s'_i \leq u_i$. By replacing $\mathbf{x}'$ with $\mathbf{s}'$ in this system of primitive constraints, we get that $\forall \mathbf{s}' \in [\mathbf{l}, \mathbf{u}] \; \exists \mathbf{r} \;\; \mathbf{C} \cdot \mathbf{s}' \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ must also hold.

Similarly, if the constructed instance does not hold, there must exist an $\mathbf{s}'$ such that the following implication holds:

$$\begin{aligned} \forall \mathbf{r} \; \exists \mathbf{x} \quad & \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}' \\ & \mathbf{C} \cdot \mathbf{x}' \not\leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} \end{aligned}$$

Recall that $l_i \leq s'_i \leq u_i$ and that the only value of $\mathbf{x}$ which can satisfy $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}'$ is $\mathbf{x} = \mathbf{s}'$. Hence, we must have that $\forall \mathbf{r} \; \mathbf{C} \cdot \mathbf{s}' \not\leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$, which means that $\forall \mathbf{s}' \in [\mathbf{l}, \mathbf{u}] \; \exists \mathbf{r} \; \mathbf{C} \cdot \mathbf{s}' \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ does not hold.

Thus, we can reduce the quantified linear program (5) to an instance of (3), where $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ is restricted to difference constraints. The co-$\mathbb{NP}$ hardness result follows by Remark 2 and Theorem 1. □

In the proofs of the previous two results, the entailed constraint cp_2 is restricted not to contain parameters that appear in cp_1, namely parameters in $\mathbf{s}$. If this restriction is relaxed, the following case of co-$\mathbb{NP}$ hardness can be established.

Theorem 5 *The entailment problem is co-$\mathbb{NP}$ hard even if the set of variables $\mathbf{x}$ is empty.*

Proof For $\mathbf{x}$ empty, formula (2) reduces to:

$$\forall \mathbf{s} \ \exists \mathbf{r} \ [-\mathbf{N} \cdot \mathbf{s} \leq \mathbf{b} \rightarrow -\mathbf{M} \cdot \mathbf{r} \leq \mathbf{d} + \mathbf{P} \cdot \mathbf{s}]$$

which is a $\langle 1, \forall, \mathbf{BR} \rangle$ formula. By Remark 1 and Theorem 3, the validity problem for this class of formulas is a co-$\mathbb{NP}$ hard problem. □

Intuitively, the proof consists of observing that, in absence of variables $\mathbf{x}$, the entailment problem boils down to the problem of satisfiability with respect to polyhedral sets, where parameters $\mathbf{r}$ in the entailed constraint play the role of variables.

7 Tractable Instances

In this section we study sufficient conditions for tractability of the entailment problem. The results of Section 6 show that deciding whether cp_1 entails cp_2 is intractable even for very restricted instances of cp_1. The problem becomes tractable as soon as we prevent any connection between variables $\mathbf{x}$ and parameters $\mathbf{s}$ in cp_1 and cp_2.

Theorem 6 *Let $cp_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \mathbf{N} \cdot \mathbf{s} \leq \mathbf{n}$ and $cp_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}, \mathbf{P} \cdot \mathbf{s} \leq \mathbf{m}$. The problem of deciding whether cp_1 entails cp_2 is in $\mathbb{P}$.*

Proof By Remark 2, the problem consists of showing the validity of:

$$\forall \mathbf{s} \ \exists \mathbf{r} \ \forall \mathbf{x} \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \wedge \mathbf{N} \cdot \mathbf{s} \leq \mathbf{n} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} \wedge \mathbf{P} \cdot \mathbf{s} \leq \mathbf{m}] \tag{6}$$

By Lemma 1, the satisfiability of cp_1 can be checked in polynomial time. If cp_1 is unsatisfiable, (6) is always true. Assume now it is satisfiable. The validity of:

$$\forall \mathbf{s} \ [\mathbf{N} \cdot \mathbf{s} \leq \mathbf{n} \rightarrow \mathbf{P} \cdot \mathbf{s} \leq \mathbf{m}]$$

can be checked in polynomial time by Theorem 2, since it consists of entailment of non-parameterized constraints over the variables $\mathbf{s}$. If the formula above is false, then (6) is false (simply consider any solution of $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ together with

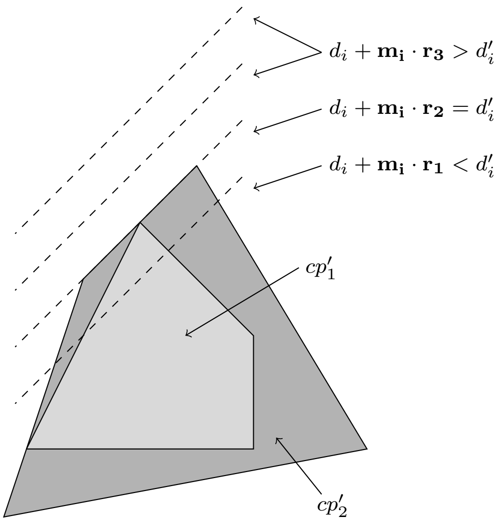

Fig. 2 Let $cp'_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ and $cp'_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$. cp'_1 entails cp'_2 if and only if $\mathbf{d} + \mathbf{M} \cdot \mathbf{r} \geq \mathbf{d'}$, with $d'_i = \max\{\mathbf{c}_i^T \cdot \mathbf{x}_0 | \mathbf{x_0} \in Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})\}$

the instance of $\mathbf{s}$ that falsifies the formula). Assume that the formula above is true. Then (6) clearly reduces to:

$$\exists \mathbf{r} \ \ \forall \mathbf{x} \ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}] \tag{7}$$

Let cp'_2 be the implied system $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$. We build a linear program on $\mathbf{r}$ as follows. For every row $\mathbf{c}_i^T \cdot \mathbf{x} \leq d_i + \mathbf{m}_i^T \cdot \mathbf{r}$ in cp'_2, let d'_i be the solution of the linear program:

$$\begin{aligned} &\max \ \mathbf{c}_i^T \cdot \mathbf{x} \\ &\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \end{aligned}$$

If the linear program is unbounded, there cannot be any parameter instance $\mathbf{r}_0$ of $\mathbf{r}$ such that $\mathbf{c}_i^T \cdot \mathbf{x} \leq d_i + \mathbf{m}_i^T \cdot \mathbf{r}_0$ for every $\mathbf{x}$. Thus, (7) is false.

Assume now that all d'_is are finite and let $\mathbf{d'}$ be the vector of all d'_is. We claim that (7) holds if and only if the following linear system on $\mathbf{r}$ is feasible:

$$\mathbf{d} + \mathbf{M} \cdot \mathbf{r} \geq \mathbf{d'} \tag{8}$$

Intuitively, this means that there exists some value of $\mathbf{r}$, say $\mathbf{r}_0$, for which each hyperplane of $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0$ is either incident to some extreme point of $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ or not in $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})$ (e.g., see Figure 2, where $\mathbf{r_1}, \mathbf{r_2}$, and $\mathbf{r_3}$ are different values of $\mathbf{r}$).

If part. Let $\mathbf{r_0}$ be such that $\mathbf{d} + \mathbf{M} \cdot \mathbf{r_0} \geq \mathbf{d'}$. Since by construction of $\mathbf{d'}$, we have that $\forall \mathbf{x} \ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d'}]$ holds, by transitivity the following holds:

$$\forall \mathbf{x} \ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r_0}]$$

By re-introducing existential quantifiers on $\mathbf{r}$, we have (7).

Only-if part. By hypothesis, there exists $\mathbf{r}_0$ such that

$$\forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0]$$

Let us show that $\mathbf{r}_0$ is a solution of (8). Consider a row $\mathbf{c}_i^T \cdot \mathbf{x} \leq d_i + \mathbf{m}_i^T \cdot \mathbf{r}$ of cp_2'. By definition of $\mathbf{r}_0$, the following holds: $\forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{c}_i^T \cdot \mathbf{x} \leq d_i + \mathbf{m}_i^T \cdot \mathbf{r}_0]$, which implies:

$$d_i + \mathbf{m}_i^T \cdot \mathbf{r}_0 \geq \max\{\mathbf{c}_i^T \mathbf{x} \mid \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\}$$

By definition of $\mathbf{d}'$, we conclude $\mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0 \geq \mathbf{d}'$, hence (8) is satisfiable. □

As a special case of this result, if $\mathbf{s}$ is empty, i.e., if cp_1 has no parameter, then the resulting problem is in $\mathbb{P}$. By Remark 2, this can be re-stated as the fact that $\langle 1, \exists, \mathbf{RB} \rangle$ is in $\mathbb{P}$.

Corollary 3 *Let* $cp_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ *and* $cp_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$. *The problem of deciding whether* cp_1 *entails* cp_2 *is in* $\mathbb{P}$.

Example 6 Consider the linear system cp_1 which has no parameters:

$$cp_1 : (x_1 \leq 2)\ \wedge\ (x_1 + x_2 \leq 3)\ \wedge\ (x_2 \leq 5)$$

and the parameterized linear constraint cp_2:

$$cp_2 : (x_1 \leq r_1)\ \wedge\ (x_1 - 3x_2 \leq 1 - 2r_1)\ \wedge\ (7x_1 + 4x_2 \leq -5 + 3r_1)$$

To check whether cp_1 entails cp_2, we first solve the following three linear programs:

$$\begin{array}{lll}
\max\ x_1 = d_1' & \max\ x_1 - 3x_2 = d_2' & \max\ 7x_1 + 4x_2 = d_3' \\
x_1 \leq 2 & x_1 \leq 2 & x_1 \leq 2 \\
x_1 + x_2 \leq 3 & x_1 + x_2 \leq 3 & x_1 + x_2 \leq 3 \\
x_2 \leq 5 & x_2 \leq 5 & x_2 \leq 5
\end{array}$$

If any of these linear programs is unbounded, then cp_1 does not entail cp_2. On the other hand, cp_1 entails cp_2 if and only if the following system is feasible:

$$\begin{aligned} r_1 &\geq d_1' \\ 1 - 2r_1 &\geq d_2' \\ -5 + 3r_1 &\geq d_3' \end{aligned}$$

It is worth noting that the solution of such a system provides us with the instances $\mathbf{r}_0$ for which cp_1 entails $cp_2[\mathbf{r} \leftarrow \mathbf{r}_0]$ as non-parameterized constraints.

An *interval constraint* on a parameter s is a constraint of the form $l_s \leq s \leq u_s$. By Theorem 6, the case where the parameters $\mathbf{s}$ in the entailing and in the entailed constraints appear only in interval constraints is also in $\mathbb{P}$.

Example 7 Consider the following parameterized linear systems:

$$cp_1 : (0 \leq x_1 + x_2 \leq 3) \ \wedge \ (1 \leq s_1 \leq 4)$$
$$cp_2 : (x_1 \geq 1 + 2r_1) \ \wedge \ (x_1 + x_2 \leq r_1) \ \wedge \ (0 \leq s_1 \leq 5)$$

In both of them, the parameter s_1 appears only in interval constraints. By Theorem 6, checking whether cp_1 entails cp_2 can be reduced to checking the validity of a $\langle 0, \forall, \mathbf{B} \rangle$ formula:

$$\forall s_1 \ \ [1 \leq s_1 \leq 4 \rightarrow 0 \leq s_1 \leq 5]$$

and a $\langle 1, \exists, \mathbf{RB} \rangle$ formula:

$$\exists r_1 \ \forall x_1 \ \forall x_2 \ \ [0 \leq x_1 + x_2 \leq 3 \rightarrow x_1 \geq 1 + 2r_1 \ \wedge \ x_1 + x_2 \leq r_1]$$

Let us concentrate now on syntactic restrictions on cp_2. The next result shows tractability when cp_2 has no additional parameters compared to cp_1, namely when $\mathbf{r}$ is empty.

Lemma 3 *Let $cp_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ and $cp_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{P} \cdot \mathbf{s}$. The problem of deciding whether cp_1 entails cp_2 is in $\mathbb{P}$.*

Proof By Remark 2, the problem consists of checking the validity of:

$$\forall \mathbf{s} \ \forall \mathbf{x} \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{P} \cdot \mathbf{s}]$$

This is an instance of $\langle 0, \forall, \mathbf{B} \rangle$ over the space of variables plus parameters $(\mathbf{x}, \mathbf{s})$. By Theorem 2, it can be checked in polynomial time. □

Example 8 Consider the following parameterized linear constraints:

$$cp_1 : (x_1 \geq 3 - s_1) \ \wedge \ (x_1 + 2x_2 \leq s_1 + s_2) \ \wedge \ (x_2 - 5x_1 \geq 3 - s_1 + 3s_2)$$
$$cp_2 : (2x_1 + 3x_2 \geq 3) \ \wedge \ (3x_1 - 4x_2 \leq s_2 - 2s_1) \ \wedge \ (x_1 + 6x_2 \leq 7 + 2s_1)$$

To check whether cp_1 entails cp_2, it suffices to check whether the following $\langle 0, \forall, \mathbf{B} \rangle$ instance holds over the space of variables plus parameters:

$$\forall s_1 \ \forall s_2 \ \forall x_1 \ \forall x_2$$
$$\begin{array}{lcl} x_1 + s_1 \geq 3 & & 2x_1 + 3x_2 \geq 3 \\ x_1 + 2x_2 - s_1 - s_2 \leq 0 & \rightarrow & 3x_1 - 4x_2 - s_2 + 2s_1 \leq 0 \\ x_2 - 5x_1 + s_1 - 3s_2 \geq 3 & & x_1 + 6x_2 - 2s_1 \leq 7 \end{array}$$

Lemma 3 generalizes as follows.

Lemma 4 *Let $cp_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ and $cp_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{P} \cdot \mathbf{s}$. If every column of $\mathbf{M}$ has only non-negative values or only non-positive values, the problem of deciding whether cp_1 entails cp_2 is in $\mathbb{P}$.*

Proof By Lemma 2, we can assume, without loss of generality, that $\mathbf{P} = \mathbf{0}$. Hence, the entailment problem consists of showing the validity of (3) in the case where $\mathbf{M}$ has only non-negative values or only non-positive values. Also, we can assume that every inequality in cp_2 contains at least a non-zero coefficient for a parameter in $\mathbf{r}$, or, stated otherwise, that no row in $\mathbf{M}$ is $\mathbf{0}$. In fact, if such a row $\mathbf{c}_i^T \cdot \mathbf{x} \leq d_i$ exists, the validity of the implication:

$$\forall \mathbf{s} \ \ \forall \mathbf{x} \ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \rightarrow \mathbf{c}_i^T \cdot \mathbf{x} \leq d_i]$$

can be checked in polynomial time by Theorem 2, hence such a row can be removed from cp_2. We will reduce our problem into an equivalent polynomial-time solvable problem in two steps.

The first step of the reduction is the following. Let $\mathbf{M}'$ be obtained by replacing every non-zero value in $\mathbf{M}$ with 1. We claim that (3) holds if and only if

$$\forall \mathbf{s} \ \ \exists \mathbf{r}' \ \ \forall \mathbf{x} \ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M}' \cdot \mathbf{r}'] \tag{9}$$

holds. For a column $\mathbf{m}_i$ in $\mathbf{M}$, let m_i^l be the maximum non-zero value in $\mathbf{m}_i$ and m_i^s be the minimum non-zero value in it (if $\mathbf{m}_i$ contains only zeros, then the parameter $\mathbf{r}_i$ never appears, hence $\mathbf{m}_i$ can be discarded from $\mathbf{M}$). By assumption, $sgn(m_i^l) = sgn(m_i^s)$, where $sgn()$ is the sign function.

Only-if part. For fixed $\mathbf{s}$, let $\mathbf{r}$ be such that (3) holds. We set $r_i' = m_i^l r_i$ if $r_i \geq 0$ and $sgn(m_i^l) = 1$; $r_i' = m_i^s r_i$ if $r_i \geq 0$ and $sgn(m_i^l) = -1$; $r_i' = m_i^s r_i$ if $r_i < 0$ and $sgn(m_i^l) = 1$; $r_i' = m_i^l r_i$ if $r_i < 0$ and $sgn(m_i^l) = -1$. With this settings, for every $a\, r_i$ appearing in $\mathbf{M} \cdot \mathbf{r}$, we have $a\, r_i \leq r_i'$, hence $\mathbf{M} \cdot \mathbf{r} \leq \mathbf{M}' \cdot \mathbf{r}'$. This implies $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d} + \mathbf{M}' \cdot \mathbf{r}'$, hence (9) holds.

If part. For fixed $\mathbf{s}$, let $\mathbf{r}'$ be such that (9) holds. We set $r_i = r_i'/m_i^s$ if $r_i' \geq 0$ and $sgn(m_i^l) = 1$; $r_i = r_i'/m_i^l$ if $r_i' \geq 0$ and $sgn(m_i^l) = -1$; $r_i = r_i'/m_i^l$ if $r_i' < 0$ and $sgn(m_i^l) = 1$; $r_i = r_i'/m_i^s$ if $r_i' < 0$ and $sgn(m_i^l) = -1$. With this settings, for every r_i' appearing in $\mathbf{M}' \cdot \mathbf{r}'$, we have $r_i' \leq a r_i$ for any non-zero coefficient a appearing in $\mathbf{M} \cdot \mathbf{r}$, hence $\mathbf{M}' \cdot \mathbf{r}' \leq \mathbf{M} \cdot \mathbf{r}$. This implies $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M}' \cdot \mathbf{r}' \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$, hence (3) holds as well.

We have shown that the problem can be reduced to an equivalent problem of the form (9). The second step of the reduction follows.

We will show that (9) holds if and only if $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ is unsatisfiable or the following holds:

$$\exists \mathbf{r}'' \ \ \forall \mathbf{x} \ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{0} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M}' \cdot \mathbf{r}''] \tag{10}$$

Only-if part. If $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ is unsatisfiable, we are done. Otherwise, let $\mathbf{s}_0$ be such that there exists $\mathbf{x}_0$ with $\mathbf{A} \cdot \mathbf{x}_0 \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}_0$. By hypothesis, there exists $\mathbf{r}'$ such that $\forall \mathbf{x} \ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}_0 \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M}' \cdot \mathbf{r}']$ holds. This implies (actually, it is equivalent to) $\mathbf{d} + \mathbf{M}' \cdot \mathbf{r}' \geq max\{\mathbf{C} \cdot \mathbf{x} \mid \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}_0\}$. By well-known results on linear programs (see, e.g., Corollary 3.1 from Murty (1983)), the property of (un)boundedness for the satisfiable linear system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}_0$ holds if and only if it holds for its cone $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{0}$. Hence there exists $k \geq 0$ such that $\mathbf{d} + \mathbf{M}' \cdot \mathbf{r}' + k\mathbf{1} \geq max\{\mathbf{C} \cdot \mathbf{x} \mid \mathbf{A} \cdot \mathbf{x} \leq \mathbf{0}\}$, where

k is an appropriate bounding constant. Due to the form of $\mathbf{M}'$ (namely, all coefficients are either 0 or 1, and no row is $\mathbf{0}$), this implies that for $\mathbf{r}'' = \mathbf{r}' + k\mathbf{1}$, we can conclude $\mathbf{d} + \mathbf{M}' \cdot \mathbf{r}'' \geq \mathbf{d} + \mathbf{M}' \cdot \mathbf{r}' + k\mathbf{1} \geq max\{\mathbf{C} \cdot \mathbf{x} \mid \mathbf{A} \cdot \mathbf{x} \leq \mathbf{0}\}$, i.e., that (10) holds.

If part. If $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ is unsatisfiable, (9) clearly holds. Otherwise, let $\mathbf{s}_0$ be such that $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}_0$ is satisfiable. Again by Murty (1983)[Corollary 3.1], the assumption that there exists $\mathbf{r}''$ such that $\mathbf{d} + \mathbf{M}' \cdot \mathbf{r}'' \geq max\{\mathbf{C} \cdot \mathbf{x} \mid \mathbf{A} \cdot \mathbf{x} \leq \mathbf{0}\}$ implies that, for some $k \geq 0$, we have $\mathbf{d} + \mathbf{M}' \cdot \mathbf{r}'' + k\mathbf{1} \geq max\{\mathbf{C} \cdot \mathbf{x} \mid \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}_0\}$. Due to the form of $\mathbf{M}'$ (namely, all coefficients are either 0 or 1, and no row is $\mathbf{0}$), this implies that for $\mathbf{r}' = \mathbf{r}'' + k\mathbf{1}$, we can conclude $\mathbf{d} + \mathbf{M}' \cdot \mathbf{r}' \geq \mathbf{d} + \mathbf{M}' \cdot \mathbf{r}'' + k\mathbf{1} \geq max\{\mathbf{C} \cdot \mathbf{x} \mid \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s_0}\}$, i.e., $\exists \mathbf{r}' \ \forall \mathbf{x} \ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}_0 \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M}' \cdot \mathbf{r}']$ holds. Since this is true for any $\mathbf{s}_0$, we conclude that (9) holds.

Summarizing, we have reduced our original problem to showing the feasibility of cp_1, which is in $\mathbb{P}$ by Lemma 1, and to checking the validity of (10), which is in $\mathbb{P}$ by Corollary 3. □

Example 9 Consider the following parameterized linear constraints:

$$cp_1 : (x_1 + 3x_2 \leq 1 - s_1) \ \wedge \ (2x_1 - x_2 \leq 2s_1)$$

$$cp_2 : (2x_1 \leq 3 + 5r_1) \ \wedge \ (3x_1 - 4x_2 \leq 2r_1 - 5r_2 - s_1)$$

By Lemma 2, the problem of checking whether cp_1 entails cp_2 is equivalent to the validity problem for the following $\langle 2, \forall, \mathbf{LRB} \rangle$ formula:

$$\forall s_1 \ \exists r_1 \ \exists r_2 \ \forall x_1 \ \forall x_2 \ \forall x_3$$
$$\begin{array}{lcl} x_1 + 3x_2 \leq 1 - s_1 & & 2x_1 \leq 3 + 5r_1 \\ 2x_1 - x_2 \leq 2s_1 & \rightarrow & 3x_1 - 4x_2 + x_3 \leq 2r_1 - 5r_2 \\ x_3 = s_1 & & \end{array}$$

Note that r_1 has only non-negative coefficients and r_2 has only non-positive coefficients in the right hand side linear system. Therefore, we can replace all the (non-zero) coefficients of r_1 and r_2 with 1 and solve the equivalent problem:

$$\forall s_1 \ \exists r_1 \ \exists r_2 \ \forall x_1 \ \forall x_2 \ \forall x_3$$
$$\begin{array}{lcl} x_1 + 3x_2 \leq 1 - s_1 & & 2x_1 \leq 3 + r_1 \\ 2x_1 - x_2 \leq 2s_1 & \rightarrow & 3x_1 - 4x_2 + x_3 \leq r_1 + r_2 \\ x_3 = s_1 & & \end{array}$$

This instance will hold if and only if either cp_1 is unsatisfiable or if the following $\langle 1, \exists, \mathbf{RB} \rangle$ formula holds:

$$\exists r_1 \ \exists r_2 \ \forall x_1 \ \forall x_2 \ \forall x_3$$
$$\begin{array}{lcl} x_1 + 3x_2 \leq 0 & & 2x_1 \leq 3 + r_1 \\ 2x_1 - x_2 \leq 0 & \rightarrow & 3x_1 - 4x_2 + x_3 \leq r_1 + r_2 \\ x_3 = 0 & & \end{array}$$

Notice that the occurrence of the parameter s_1 in the entailed constraint could have been removed at once, since it is first replaced by the new variable x_3, which is then assumed equal to 0.

The previous result states tractability when existentially quantified parameters play the role of upper or lower bounds, but not both. As a consequence, equalities involving them are not admitted. The next result shows that entailing *single* equalities can be checked in polynomial time.

Lemma 5 *If cp_2 consists of a single equality, the problem of deciding whether cp_1 entails cp_2 is in $\mathbb{P}$.*

Proof By Lemma 2, we can assume, without loss of generality, that no parameter of cp_1 appears in cp_2. The entailment problem consists of showing the validity of (3). Let $cp_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$, and cp_2 be $\mathbf{c}^T \cdot \mathbf{x} = d + \mathbf{m}^T \cdot \mathbf{r}$.

If $\mathbf{m} = \mathbf{0}$, then the result follows by Lemma 3.

By Lemma 1, the satisfiability of cp_1 can be checked in polynomial time. If it is unsatisfiable, the entailment is always true. Assume now that cp_1 is satisfiable. It can be transformed in polynomial time (see Greenberg, 1996) into an equivalent system $\mathbf{A}^= \cdot \mathbf{x} = \mathbf{b}^= + \mathbf{N}^= \cdot \mathbf{s}, \mathbf{A}^+ \cdot \mathbf{x} \leq \mathbf{b}^+ + \mathbf{N}^+ \cdot \mathbf{s}$ such that there exist $\mathbf{s}_0$ and $\mathbf{x}_0$ with $\mathbf{A}^= \cdot \mathbf{x}_0 \leq \mathbf{b}^= + \mathbf{N}^= \cdot \mathbf{s}_0$ and $\mathbf{A}^+ \cdot \mathbf{x}_0 < \mathbf{b}^+ + \mathbf{N}^+ \cdot \mathbf{s}_0$. The system $cp_1^=$, defined as $\mathbf{A}^= \cdot \mathbf{x} = \mathbf{b}^= + \mathbf{N}^= \cdot \mathbf{s}$, is called the set of *implicit equalities* of cp_1, and $(\mathbf{x}_0\, \mathbf{s}_0)$ is called an *inner point* (over the space of variables plus parameters). We claim that cp_1 entails cp_2 if and only if $cp_1^=$ entails cp_2.

The if part is immediate. As for the only-if part, we first observe that, due to the form of cp_2, we have that cp_1 entails cp_2 if and only if for every $\mathbf{s}_1$, the set $\{\mathbf{c}^T \cdot \mathbf{x}' \mid \mathbf{x}' \in Sol(cp_1, \mathbf{s}_1)\}$ is either empty or a singleton. A similar observation holds for $cp_1^=$ in the place of cp_1. We show the only-if part by contraposition. Assume that there exist $\mathbf{s}_1$ and $\mathbf{x}_1, \mathbf{x}_2 \in Sol(cp_1^=, \mathbf{s}_1)$ with $\mathbf{c}^T \cdot \mathbf{x}_1 \neq \mathbf{c}^T \cdot \mathbf{x}_2$. By convexity of cp_1 as a linear system over the space of variables plus parameters, we can consider the segments between the inner point $(\mathbf{x}_0\, \mathbf{s}_0)$, and the external points $(\mathbf{x}_1\, \mathbf{s}_1)$ and $(\mathbf{x}_2\, \mathbf{s}_1)$. There exists $0 < \delta \leq 1$ such that, for $\mathbf{x}'_1 = \mathbf{x}_1\delta + \mathbf{x}_0(1-\delta)$, $\mathbf{x}'_2 = \mathbf{x}_2\delta + \mathbf{x}_0(1-\delta)$, and $\mathbf{s}'_1 = \mathbf{s}_1\delta + \mathbf{s}_0(1-\delta)$, we have $\mathbf{x}'_1, \mathbf{x}'_2 \in Sol(cp_1, \mathbf{s}'_1)$. Since $\mathbf{c}^T \cdot \mathbf{x}_1 \neq \mathbf{c}^T \cdot \mathbf{x}_2$ implies $\mathbf{c}^T \cdot \mathbf{x}'_1 \neq \mathbf{c}^T \cdot \mathbf{x}'_2$ (due to $\delta > 0$), we have that cp_1 does not entail cp_2.

Our conclusion then follows since proving whether $cp_1^=$ entails cp_2 is solvable in polynomial time by applying Gauss-Jordan elimination of variables in $\mathbf{x}$ from $cp_1^=$ and then checking that the resulting expression $\mathbf{c}^T \cdot \mathbf{x}$ contains no free variable. □

Example 10 Consider $cp_1 : (x_1 \leq x_2 \leq s_1) \,\wedge\, (x_1 \geq s_1)$ and $cp_2 : (2x_2 = 5 + r_1)$. The set of implicit equalities of cp_1 is represented by $cp_1^= : (x_1 = x_2) \wedge (x_1 = s_1)$ (other equivalent systems are possible, of course). By Gauss-Jordan elimination of x_1 and x_2, we obtain $2x_2 = 2s_1$, namely the left hand side of the equality contains no free variables. Therefore, cp_1 entails cp_2. Moreover, by equating $2s_1 = 5 + r_1$, we can constructively find a parameter instance $r_1 = 2s_1 - 5$ for which the entailment holds.

The entailment problem can be divided into two or more parts by splitting the entailed constraint cp_2 under the assumption that the two or more parts do not share parameters not appearing in cp_1.

Lemma 6 *Let* $cp_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ *and* $cp_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}_1 + \mathbf{M}_1 \cdot \mathbf{r}_1 + \mathbf{P}_1 \cdot \mathbf{s}$*, and* $cp_3 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}_2 + \mathbf{M}_2 \cdot \mathbf{r}_2 + \mathbf{P}_2 \cdot \mathbf{s}$*, where* $\mathbf{r}_1$ *and* $\mathbf{r}_2$ *share no parameter. We have that* cp_1 *entails* $cp_2 \wedge cp_3$ *if and only if* cp_1 *entails* cp_2 *and* cp_1 *entails* cp_3*.*

Proof Since $\mathbf{r}_1$ and $\mathbf{r}_2$ share no parameter, by basic logical equivalence, the characterization of Remark 2:

$$\forall \mathbf{s} \;\; \exists \mathbf{r}_1, \mathbf{r}_2 \;\; \forall \mathbf{x} \;\; [cp_1 \rightarrow cp_2 \wedge cp_3]$$

holds if and only if both $\forall \mathbf{s} \;\; \exists \mathbf{r}_1 \;\; \forall \mathbf{x} \;\; [cp_1 \rightarrow cp_2]$, and $\forall \mathbf{s} \;\; \exists \mathbf{r}_2 \;\; \forall \mathbf{x} \;\; [cp_1 \rightarrow cp_3]$ hold. □

Notice that partitioning the parameterized primitive linear constraints in cp_2 on the basis of the parameters in $\mathbf{r}$ can be done in polynomial time. If each of the sub-problems obtained by partitioning cp_2 can be shown to be in $\mathbb{P}$, we can conclude that the overall problem is in $\mathbb{P}$.

Example 11 Consider cp_1 and cp_2 from Example 10, and $cp_3 = (x_1 + x_2 \leq r_2)$. Since r_2 has only non-negative coefficients in cp_3, we can apply the procedure from the proof of Lemma 4 to conclude that cp_1 entails cp_3, e.g., by setting $r_2 = 2s_1$. This fact together with Lemma 6 and the conclusion of Example 10 allow for stating that cp_1 entails $cp_2 \wedge cp_3$.

8 Variants of Entailment

In this section, we consider weaker and stronger variants of entailment and establish their computational complexities.

8.1 Weak Entailment

In the problem "cp_1 entails cp_2" we have considered whether for *every* parameter instance of cp_1 one can find an instance of cp_2 including it. A weakening of the problem consists of asking whether for *at least one* parameter instance of cp_1 one can find an instance of cp_2 including it.

Definition 5 We say that $cp_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ weakly entails $cp_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{P} \cdot \mathbf{s}$ if there exist $\mathbf{s}_0 \in \Re^{|\mathbf{s}|}$ and $\mathbf{r}_0 \in \Re^{|\mathbf{r}|}$ such that $Sol(cp_1, \mathbf{s}_0) \subseteq Sol(cp_2, (\mathbf{s}_0 \, \mathbf{r}_0))$.

Proving that $cp_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ weakly entails $cp_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{P} \cdot \mathbf{s}$ is equivalent to showing the validity of the $\langle 1, \exists, \mathbf{BB} \rangle$ formula:

$$\exists \mathbf{s}, \mathbf{r} \;\; \forall \mathbf{x} \;\; [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{P} \cdot \mathbf{s}] \qquad (11)$$

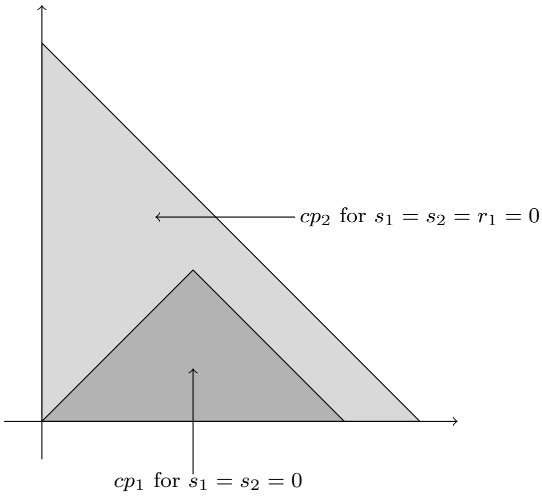

Fig. 3 cp_1 weakly entails cp_2 from Example 12.

Example 12 Consider the following parameterized linear constraints:

$$cp_1 : (x_1 \geq 2s_1) \ \wedge \ (x_1 - x_2 \leq s_1 + s_2) \ \wedge \ (x_1 + x_2 \leq 4 - s_1 - s_2)$$
$$cp_2 : (x_1 \geq s_1 + r_1) \ \wedge \ (x_2 \geq s_1 + s_2 + 2r_1) \ \wedge \ (x_1 + x_2 \leq 5 - s_2 + r_1)$$

The problem of checking whether cp_1 weakly entails cp_2 is equivalent to checking the validity of the following $\langle 1, \exists, \mathbf{BB} \rangle$ instance:

$$\exists s_1 \ \exists s_2 \ \exists r_1 \ \forall x_1 \ \forall x_2$$
$$\begin{array}{lcl} x_1 \geq 2s_1 & & x_1 \geq s_1 + r_1 \\ x_1 - x_2 \leq s_1 + s_2 & \rightarrow & x_2 \geq s_1 + s_2 + 2r_1 \\ x_1 + x_2 \leq 4 - s_1 - s_2 & & x_1 + x_2 \leq 5 - s_2 + r_1 \end{array}$$

Note that for $s_1 = s_2 = r_1 = 0$, cp_1 entails cp_2 (see Figure 3).

Theorem 7 *The weak entailment problem for parameterized linear constraints is $\mathbb{NP}$ hard.*

Proof Every formula in $\langle 1, \exists, \mathbf{LB} \rangle$ is also a formula in $\langle 1, \exists, \mathbf{BB} \rangle$. Hence, let us reduce the $\mathbb{NP}$ complete problem 3SAT to the validity problem for $\langle 1, \exists, \mathbf{LB} \rangle$ formulas:

$$\exists \mathbf{s} \ \ \forall \mathbf{x} \ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}] \tag{12}$$

Consider a 3SAT formula $\phi = \phi_1 \wedge \phi_2 \wedge ... \wedge \phi_m$ on the literals $\{y_1, \bar{y}_1, y_2, \bar{y}_2, \ldots, y_n, \bar{y}_n\}$. We reduce the satisfiability of ϕ to the validity problem for a formula as in (12) as follows. Let cp_1 be $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$, and cp_2 be $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}$.

For each variable y_i, we set a (existentially quantified) parameter s_i and a (universally quantified) variable x_i. We also add the constraints $x_i \leq s_i$ and $x_i \leq 1 - s_i$ to cp_1 and the constraint $x_i \leq 0$ to cp_2. For each clause ϕ_j in ϕ, we set a (universally quantified) variable x_{n+j} and add the constraint $x_{n+j} \geq 1$ to cp_2. Then, depending on the form number of negative literals in the clause ϕ_j, we do one of the following:

1. If $\phi_j = (y_i, y_k, y_l)$, we add $x_{n+j} = s_i + s_k + s_l$ to cp_1.
2. If $\phi_j = (y_i, y_k, \bar{y_l})$, we add $x_{n+j} = s_i + s_k + (1 - s_l)$ to cp_1.
3. If $\phi_j = (y_i, \bar{y_k}, \bar{y_l})$, we add $x_{n+j} = s_i + (1 - s_k) + (1 - s_l)$ to cp_1.
4. If $\phi_j = (\bar{y_i}, \bar{y_k}, \bar{y_l})$, we add $x_{n+j} = (1 - s_i) + (1 - s_k) + (1 - s_l)$ to cp_1.

We claim that ϕ is satisfiable if and only if the formula $\exists \mathbf{s} \ \forall \mathbf{x} \ [cp_1 \rightarrow cp_2]$ is true.

Only-if part. Consider an assignment $y_1, \ldots, y_n$ which satisfies ϕ_j for each $j \in 1, ..., m$. We claim that by setting $s_i = 1$ when $y_i = \mathbf{true}$ and $s_i = 0$ otherwise, we have that $\forall \mathbf{x} \ [cp_1 \rightarrow cp_2]$ is true. Let $\mathbf{x}$ be such that cp_1 holds. Since ϕ_j is satisfiable, at least one of the literals in each ϕ_j is true. By the definition of s_i and the constraints (1-4) above, cp_1 forces $x_{n+j} \geq 1$. Also, since $s_i \in \{0, 1\}$, the constraints $x_i \leq s_i$ and $x_i \leq 1 - s_i$ in cp_1 clearly imply $x_i \leq 0$. Summarizing, all constraints in cp_2 are satisfied.

If part. Let $\mathbf{s}$ be such that $\forall \mathbf{x} \ [cp_1 \rightarrow cp_2]$ is true. We claim that for all $i \in [1, n]$, either $s_i \geq 1$ or $s_i \leq 0$. Otherwise, by choosing $x_i = min(s_i, 1 - s_i)$, we have that the constraints $x_i \leq s_i$ and $x_i \leq 1 - s_i$ in cp_1 are satisfied, while $x_i \leq 0$ in cp_2 is not, which contradicts the assumption on $\mathbf{s}$. We define an assignment by setting $y_i = \mathbf{true}$ if and only if $s_i \geq 1$. Let us show that such an assignment satisfies ϕ_j for each $j \in [1, m]$, hence ϕ is satisfied. Let $\mathbf{x}$ be such that cp_1 holds (notice that cp_1 is satisfiable by construction). Assume $\phi_j = (y_i, y_k, \bar{y}_l)$ (for the other clauses the reasoning is similar). Due to the constraint $x_{n+j} \geq 1$ in cp_2 and the constraints (1-4) above, we have that $s_i \geq 1$ or $s_k \geq 1$ or $s_l \leq 0$. By definition of our assignment, at least one literal among y_i, y_k and $\bar{y}_l$ is true, i.e., ϕ_j is satisfied. □

Notice that the proof of the theorem actually shows that it is $\mathbb{NP}$ hard to check whether there is some parameter instance of a parameterized constraint that entails a non-parameterized constraint. Stated otherwise, the validity problem for the class of formulas (12), i.e., for $\langle 1, \exists, \mathbf{LB} \rangle$, is $\mathbb{NP}$ hard.

Example 13 Consider the 3SAT instance:

$$(x_1, x_2, \bar{x_3}), (\bar{x_2}, x_3, \bar{x_4})$$

This instance can be reduced into the following $\langle 1, \exists, \mathbf{LB} \rangle$ problem (which is also a formula in $\langle 1, \exists, \mathbf{BB} \rangle$):

$$\begin{array}{rcl} \multicolumn{3}{l}{\exists s_1\ \exists s_2\ \exists s_3\ \exists s_4\ \forall x_1\ \forall x_2\ \forall x_3\ \forall x_4\ \forall x_5\ \forall x_6} \\ x_1 \le s_1,\ x_1 \le 1 - s_1 & & x_1 \le 0 \\ x_2 \le s_2,\ x_2 \le 1 - s_2 & & x_2 \le 0 \\ x_3 \le s_3,\ x_3 \le 1 - s_3 & \to & x_3 \le 0 \\ x_4 \le s_4,\ x_4 \le 1 - s_4 & & x_4 \le 0 \\ x_5 = s_1 + s_2 + (1 - s_3) & & x_5 \ge 1 \\ x_6 = (1 - s_2) + s_3 + (1 - s_4) & & x_6 \ge 1 \end{array}$$

We present next the $\mathbb{NP}$ completeness of weak entailment.

Theorem 8 *The problem of deciding whether cp_1 weakly entails cp_2 is $\mathbb{NP}$ complete.*

Proof Since $\mathbb{NP}$ hardness holds by Theorem 7, we have to show that the problem is in $\mathbb{NP}$. First of all, we can assume that cp_2 has no parameter. In fact, by adopting the same strategy of Lemma 2, we can transform the formula (11) into:

$$\exists \mathbf{s}, \mathbf{r}\ \ \forall \mathbf{x}, \mathbf{x}', \mathbf{x}''\ \ [\mathbf{A} \cdot \mathbf{x} \le \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \wedge \mathbf{x}' = \mathbf{s} \wedge \mathbf{x}'' = \mathbf{r} \to \mathbf{C} \cdot \mathbf{x} - \mathbf{P} \cdot \mathbf{x}' - \mathbf{M} \cdot \mathbf{x}'' \le \mathbf{d}]$$

which is of the form (12).

Let now $cp_1 = \mathbf{A} \cdot \mathbf{x} \le \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ and $cp_2 = \mathbf{C} \cdot \mathbf{x} \le \mathbf{d}$. We will show that the problem (12) is in $\mathbb{NP}$ by proving that if there exists an $\mathbf{s}_0$ such that

$$\forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \le \mathbf{b} + \mathbf{N} \cdot \mathbf{s}_0 \to \mathbf{C} \cdot \mathbf{x} \le \mathbf{d}] \tag{13}$$

then there exists such an $\mathbf{s}_0$ whose representation is polynomial in the size of $\mathbf{A}, \mathbf{b}, \mathbf{C}, \mathbf{d}$, and $\mathbf{N}$. Since (13) belongs to the class $\langle 0, \forall, \mathbf{B} \rangle$, which can be solved in polynomial time, if such a polynomial $\mathbf{s}_0$ exists, then it is a valid certificate for the problem (12). Assume that there exists an $\mathbf{s}_0$ such that (13) holds. Consider the problem

$$\forall \mathbf{s}\ \ \forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \le \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \to \mathbf{C} \cdot \mathbf{x} \le \mathbf{d}] \tag{14}$$

This is an instance of $\langle 0, \forall, \mathbf{B} \rangle$ which is in $\mathbb{P}$. If this implication is true, then (12) is true as well, and we are done. Assume it is false. Then there exists an $\mathbf{s}_2$ such that:

$$\exists \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \le \mathbf{b} + \mathbf{N} \cdot \mathbf{s}_2, \mathbf{C} \cdot \mathbf{x} \not\le \mathbf{d}]$$

Let $S_{\mathbf{s}}$ be the set of extreme points of $\mathbf{A} \cdot \mathbf{x} \le \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$. Thus we have that $S_{\mathbf{s}_0} \subseteq Sol(\mathbf{C} \cdot \mathbf{x} \le \mathbf{d})$ and $S_{\mathbf{s}_2} \not\subseteq Sol(\mathbf{C} \cdot \mathbf{x} \le \mathbf{d})$. For a constraint $\mathbf{c}_i^T \cdot \mathbf{x} \le d_i$ in $\mathbf{C} \cdot \mathbf{x} \le \mathbf{d}$, let $d_i'(\mathbf{s})$ be the solution of the linear program:

$$\begin{array}{c} \max\ \ \mathbf{c}_i^T \cdot \mathbf{x} - d_i \\ \mathbf{A} \cdot \mathbf{x} \le \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \end{array}$$

and let $d'(\mathbf{s}) = \max_i(d'_i(\mathbf{s}))$. By (13), we have $d'(\mathbf{s}_0) \leq 0$. By (14), we have $d'(\mathbf{s}_2) > 0$. Since $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ is a convex set, there exists $0 \leq \alpha \leq 1$ such that $d'(\mathbf{s}') = 0$ when $\mathbf{s}' = \alpha\mathbf{s}_0 + (1-\alpha)\mathbf{s}_2$. Thus, we have that $\forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}' \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}]$ and that for some i, $d'_i(\mathbf{s}') = 0$. Therefore, for some $\mathbf{x}' \in S_{\mathbf{s}'}$, we have that $\mathbf{C} \cdot \mathbf{x}' \leq \mathbf{d}$ and $\mathbf{c}_i^T \cdot \mathbf{x}' = d_i$. We can now create a linear system from the constraints in $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ and $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}$ for which $(\mathbf{x}', \mathbf{s}')$ is an extreme point as follows:

1. For each constraint $\mathbf{a}^T \cdot \mathbf{x} \leq b + \mathbf{n}^T \cdot \mathbf{s}$ of $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ such that $\mathbf{a}^T \cdot \mathbf{x}' = b + \mathbf{n}^T \cdot \mathbf{s}'$, add the constraint $\mathbf{a}^T \cdot \mathbf{x} = b + \mathbf{n}^T \cdot \mathbf{s}$.
2. Add the constraint $\mathbf{c}_i^T \cdot \mathbf{x} = d_i$
3. Add the remaining constraints from $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ and $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}$.

Since $\mathbf{s}'$ is part of an extreme point solution to this system, its representation is polynomial in the size of $\mathbf{A}, \mathbf{b}, \mathbf{C}, \mathbf{d}$, and $\mathbf{N}$. Therefore, we have our desired polynomial sized $\mathbf{s}_0 = \mathbf{s}'$ such that (13) holds. □

8.2 Strong Entailment

A strengthening of entailment is obtained by requiring that for every parameter instance, the set of solutions of the entailing constraint is the same (rather than it is included) as the set of solutions of the entailed constraint.

Definition 6 We say that $cp_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ strongly entails $cp_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{P} \cdot \mathbf{s}$ if for all $\mathbf{s}_0 \in \Re^{|\mathbf{s}|}$ there exists $\mathbf{r}_0 \in \Re^{|\mathbf{r}|}$ such that $Sol(cp_1, \mathbf{s}_0) = Sol(cp_2, (\mathbf{s}_0\, \mathbf{r}_0))$.

Proving that $cp_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ strongly entails $cp_2 = \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{P} \cdot \mathbf{s}$ is equivalent to showing the validity of the formula:

$$\forall \mathbf{s}\ \ \exists \mathbf{r}\ \ \forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \leftrightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{P} \cdot \mathbf{s}] \tag{15}$$

Example 14 Consider the parameterized linear constraints of Example 4. The problem of checking whether cp_1 strongly entails cp_2 is equivalent to checking the validity of the following formula:

$$\forall s_1\ \exists r_1\ \forall x_1\ \forall x_2$$
$$\begin{array}{lcl} x_1 + x_2 \geq -1 + s_1 & & x_1 \geq -1 - s_1 \\ x_1 + x_2 \leq 1 + 2s_1 & \leftrightarrow & x_1 \leq 1 + 2r_1 + s_1 \\ x_1 - x_2 \geq -1 + s_1 & & x_2 \geq -1 - r_1 \\ x_1 - x_2 \leq 1 + 3s_1 & & x_2 \leq 1 + r_1 + 2s_1 \end{array}$$

Fix $s_1 = 0$. The left hand side of the equivalence is shown in Figure 1. The right hand side is $(-1 \leq x_1 \leq 1+2r_1) \wedge (-1-r_1 \leq x_2 \leq 1+r_1)$, with Figure 1 showing the case $r_1 = 0$. It is readily checked that for no r_1 the right hand side constraint has the same set of solutions as the left hand side constraint.

The entailment problem can be reduced to the strong entailment problem. Hence, strong entailment is co-$\mathbb{NP}$ hard.

Theorem 9 *The strong entailment problem for parameterized linear constraints is co-$\mathbb{NP}$ hard.*

Proof The entailment problem can be reduced in polynomial time to the strong entailment problem. In fact, (3) holds if and only if the following holds:

$$\forall \mathbf{s}\ \ \exists \mathbf{r}\ \ \forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \leftrightarrow \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \wedge \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} + \mathbf{P} \cdot \mathbf{s}]$$

Since proving the validity of (3) is co-$\mathbb{NP}$ hard by Theorem 4, and the formula above is of the form (15), our conclusion follows. □

9 Conclusion

In this paper, we introduced satisfiability and entailment as basic tools for reasoning over parameterized linear constraints. In order to study their computational complexities, satisfiability with respect to polyhedral sets and entailment were related to classes of quantified linear implications.

Although satisfiability of parameterized linear constraints is in $\mathbb{P}$, every instance of satisfiability with respect to polyhedral sets is equivalent to a corresponding problem of the $\langle 1, \forall, \mathbf{BR} \rangle$ class, which was shown to be co-$\mathbb{NP}$ hard. On the other hand, every entailment instance is equivalent to a corresponding problem of the $\langle 2, \forall, \mathbf{BRB} \rangle$ class, which in turn is polynomial-time reducible to a problem of the $\langle 2, \forall, \mathbf{LRB} \rangle$ class (also shown to be co-$\mathbb{NP}$ hard). Hence, both satisfiability with respect to polyhedral sets and entailment were proved to be co-$\mathbb{NP}$ hard.

A weakening and a strengthening extension of the entailment problem were also examined. The weak entailment problem was shown to be $\mathbb{NP}$ complete, using the fact that every weak entailment instance is equivalent to a problem of the $\langle 1, \exists, \mathbf{LB} \rangle$ class and the fact that the decision problem for this class in $\mathbb{NP}$ complete. The strong entailment problem was shown to be co-$\mathbb{NP}$ hard.

Further, we characterized some tractable instances of the entailment problem. More specifically, we proved that the entailment problem is in $\mathbb{P}$ if:

- no primitive constraint contains both variables and universally quantified parameters (this includes the $\langle 1, \exists, \mathbf{RB} \rangle$ class),
- the coefficients of each parameter included only in the entailed constraint are either all non-negative or all non-positive,
- the entailed constraint consists of only one equality, or
- the entailed constraint can be partitioned into two or more systems for which validity is in $\mathbb{P}$ and with no overlapping existentially quantified parameter.

Finally, it is worth noting that the proofs of the tractability results are constructive, i.e., they provide algorithms for checking entailment which rely on well-known tools (linear programs, implicit equalities, Gauss-Jordan elimination).

Future work includes establishing the exact computational complexities of the entailment and strong entailment problem. In that regard, the study of the corresponding quantified linear implications may be an important tool.

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