

# An Encoding for Order-Preserving Matching\*

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## Abstract

Encoding data structures store enough information to answer the queries they are meant to support but not enough to recover their underlying datasets. In this paper we give the first encoding data structure for the challenging problem of order-preserving pattern matching. This problem was introduced only a few years ago but has already attracted significant attention because of its applications in data analysis. Two strings are said to be an order-preserving match if the *relative order* of their characters is the same: e.g., 4, 1, 3, 2 and 10, 3, 7, 5 are an order-preserving match. We show how, given a string  $S[1..n]$  over an arbitrary alphabet of size  $\sigma$  and a constant  $c \geq 1$ , we can build an  $O(n \log \log n)$ -bit encoding such that later, given a pattern  $P[1..m]$  with  $m \leq \log^c n$ , we can return the number of order-preserving occurrences of  $P$  in  $S$  in  $O(m)$  time. Within the same time bound we can also return the starting position of some order-preserving match for  $P$  in  $S$  (if such a match exists). We prove that our space bound is within a constant factor of optimal if  $\log \sigma = \Omega(\log \log n)$ ; our query time is optimal if  $\log \sigma = \Omega(\log n)$ . Our space bound contrasts with the  $\Omega(n \log n)$  bits needed in the worst case to store  $S$  itself, an index for order-preserving pattern matching with no restrictions on the pattern length, or an index for standard pattern matching even with restrictions on the pattern length. Moreover, we can build our encoding knowing only how each character compares to  $O(\log^c n)$  neighbouring characters.

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## 1 Introduction

As datasets have grown even faster than computer memories, researchers have designed increasingly space-efficient data structures. We can now store a sequence of  $n$  numbers from  $\{1, \dots, \sigma\}$  with  $\sigma \leq n$  in about  $n$  words, and sometimes  $n \log \sigma$  bits or even  $nH$  bits, where  $H$  is the empirical entropy of the sequence, and still support many powerful queries quickly. If we are interested only in queries of the form “what is the position of the smallest

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number between the  $i$ th and  $j$ th?”, however, we can do even better: regardless of  $\sigma$  or  $H$ , we need store only  $2n + o(n)$  bits to be able to answer in constant time [21]. Such a data structure, that stores enough information to answer the queries it is meant to support but not enough to recover the underlying dataset, is called an *encoding* [38]. As well as the variant of range-minimum queries mentioned above, there are now efficient encoding data structures for range top- $k$  [14, 24, 27], range selection [34], range majority [35], range maximum-segment-sum [23] and range nearest-larger-value [20] on sequences of numbers, and range-minimum [26] and range nearest-larger-value [30, 31] on two-dimensional arrays of numbers; all of these queries return positions but not values from the sequence or array. Perhaps Orlandi and Venturini’s [36] results about sublinear-sized data structures for substring occurrence estimation are the closest to the ones we present in this paper, in that they are more related to pattern matching than range queries: the authors showed how we can store a sequence of  $n$  numbers from  $\{1, \dots, \sigma\}$  in fewer than  $n \log \sigma$  bits but such that we can estimate quickly how often any pattern occurs in the sequence, with the additive error in our estimate proportional to our compression ratio.

Encoding data structures can offer better space bounds than traditional data structures that store the underlying dataset somehow (even in succinct or compressed form), and possibly even security guarantees: if we can build an encoding data structure using only public information, then we need not worry about it being reverse-engineered to reveal private information. From the theoretical point of view, encoding data structures pose new interesting combinatorial problems and promise to be a challenging field for future research.

In this paper we give the first encoding for *order-preserving pattern matching*, a problem which asks us to search in a text for substrings whose characters have the same relative order as those in a pattern. For example, in 6, 3, 9, 2, 7, 5, 4, 8, 1, the order-preserving matches of 2, 1, 3 are 6, 3, 9 and 5, 4, 8. Kubica et al. [33] and Kim et al. [32] formally introduced this problem and gave efficient online algorithms for it. Other researchers have continued their investigation, and we briefly survey their results in Section 2. As well as its theoretical interest, this problem has practical applications in data analysis. For example, mining for correlations in large datasets is complicated by amplification or damping — e.g., the euro fluctuating against the dollar may cause the pound to fluctuate similarly a few days later, but to a greater or lesser extent — and if we search only for sequences of values that rise or fall by exactly the same amount at each step we are likely to miss many potentially interesting leads. In such settings, searching for sequences in which only the relative order of the values is constrained to be the same is certainly more robust.

In Section 2 we discuss some previous work on order-preserving pattern matching. In Section 3 we review the algorithmic tools we use in the rest of the paper. In Section 4 we prove our first result showing how, given a string  $S[1..n]$  over an arbitrary alphabet  $[\sigma]$  and a constant  $c \geq 1$ , we can store  $\mathcal{O}(n \log \log n)$  bits — regardless of  $\sigma$  — such that later, given a pattern  $P[1..m]$  with  $m < \log^c n$ , in  $\mathcal{O}(n \log^c n)$  time we can scan our encoding and report all the order-preserving matches of  $P$  in  $S$ . Our space bound contrasts with the  $\Omega(n \log n)$  bits needed in the worst case, when  $\log \sigma = \Omega(\log n)$ , to store  $S$  itself, an index for order-preserving pattern matching with no restriction on the pattern length, or an index for standard pattern matching even with restrictions on the pattern length. (If  $S$  is a permutation then we can recover it from an index for unrestricted order-preserving pattern matching, or from an index for standard matching of patterns of length 2, even when they do not report the positions of the matches. Notice this does not contradict Orlandi and Venturini’s result, mentioned above, about estimating substring frequency, since that permits additive error.) In fact, we build our representation of  $S$  knowing only how

each character compares to  $2 \log^c n$  neighbouring characters. We show in Section 5 how to adapt and build on this representation to obtain indexed order-preserving pattern matching, instead of scan-based, allowing queries in  $\mathcal{O}(m \log^3 n)$  time but now reporting the position of only one match.

In Section 6 we give our main result showing how to speed up our index using weak prefix search and other algorithmic improvements. The final index is able to *count* the number of occurrences and *return the position* of an order-preserving match (if one exists) in  $\mathcal{O}(m)$  time. This query time is optimal if  $\log \sigma = \Omega(\log n)$ . Finally, in Section 7 we show that our space bound is optimal (up to constant factors) even for data structures that only return whether or not  $S$  contains any order-preserving matches.

## 2 Previous Work

Although recently introduced, order-preserving pattern matching has received considerable attention and has been studied in different settings. For the online problem, where the pattern is given in advance, the first contributions were inspired by the classical Knuth–Morris–Pratt and Boyer–Moore algorithms [5, 12, 32, 33]. The proposed algorithms have guaranteed linear time worst-case complexity or sublinear time average complexity. However, for the online problem the best results in practice are obtained by algorithms based on the concept of filtration, in which some sort of “order-preserving” fingerprint is applied to the text and the pattern [6, 7, 8, 10, 11, 18, 15]. This approach was successfully applied also to the harder problem of matching with errors [8, 25, 28].

There has also been work on indexed order-preserving pattern matching. Crochemore et al. [13] showed how, given a string  $S[1..n]$ , in  $\mathcal{O}(n\sqrt{\log n})$  time we can build an  $\mathcal{O}(n \log n)$ -bit index such that later, given a pattern  $P[1..m]$  over an alphabet polynomially bounded in  $m$ , we can return the starting positions of all the occ order-preserving matches of  $P$  in  $S$  in optimal  $\mathcal{O}(m + \text{occ})$  time. Their index is a kind of suffix tree, and other researchers [39] are trying to reduce the space bound to  $n \log \sigma + o(n \log \sigma)$  bits, where  $\sigma$  is the size of the alphabet of  $S$ , by using a kind of Burrows–Wheeler Transform instead (similar to recent work [22] on parameterized pattern matching [1]). Even if they succeed, however, when  $\sigma = n^{\Omega(1)}$  the resulting index will still take linear space — i.e.,  $\Omega(n)$  words or  $\Omega(n \log n)$  bits. (It could be interesting to apply the techniques we develop here to parameterized pattern matching, but we leave that as future work.)

In addition to Crochemore et al.’s result, other offline solutions have been proposed combining the idea of fingerprints and indexing. Chhabra et al. [9] showed how to speed up the search by building an FM-index [19] on the binary string expressing whether in the input text each element is smaller or larger than the next one. By expanding this approach, Decaroli et al. [15] show how to build a compressed file format supporting order-preserving matching without the need of full decompression. Experiments show that this compressed file format takes roughly the same space as `gzip` and that in most cases the search is orders of magnitude faster than the sequential scan of the text. We point out that these approaches, although interesting for the applications, do not have competitive worst case bounds on the search cost as we get from Crochemore et al. and in this paper.

## 3 Background

In this section we collect a set of algorithmic tools that will be used in our solutions. In the following we report each result together with a brief description of the solved problem.

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More details can be obtained by consulting the corresponding references. All the results hold in the unit cost word-RAM model, where each memory word has size  $w = \Omega(\log n)$  bits, where  $n$  is the input size. In this model arithmetic and boolean operations on memory words require  $\mathcal{O}(1)$  time.

**Rank queries on binary vector** In the next solutions we will need to support Rank queries on a binary vector  $B[1..n]$ . Given an index  $i$ ,  $\text{Rank}(i)$  on  $B$  returns the number of 1s in the prefix  $B[1..i]$ . We report here a result in [29].

► **Theorem 1.** *Given a binary vector  $B[1..n]$ , we can support Rank queries in constant time by using  $n + o(n)$  bits of space.*

**Elias-Fano representation** In the following we will need to encode an increasing sequence of values in almost optimal space. There are several solutions to this problem, we report here the result obtained with the, so-called, Elias-Fano representation [16, 17].

► **Theorem 2.** *An increasing sequence of  $n$  values up to  $u$  can be represented by using  $\log \binom{u}{n} + \mathcal{O}(n) = n \log \frac{u}{n} + \mathcal{O}(n)$  bits, so that we can access any value of the sequence in constant time.*

**Minimal perfect hash functions** In our solution we will make use of *Monotone minimal perfect hash functions* (Mmphf) [2]. Given a set  $S = \{x_1, x_2, \dots, x_n\}$  of size  $n$ , a minimal perfect hash function (Mphf) has to injectively map keys in  $S$  to the integers in  $[n]$ .

A monotone minimal perfect hash function (Mmphf) is an Mphf  $h()$  that preserves the lexicographic ordering, i.e., for any two strings  $x$  and  $y$  in the set,  $x \leq y$  iff  $h(x) \leq h(y)$ . Results on Mmphfs have been focused on dictionaries of binary strings [2]. The results can be easily generalized to dictionaries with strings over larger alphabets. The following theorem reports the obvious generalization of Theorem 3.1 in [2] and Theorem 2 in [4].

► **Theorem 3.** *Given a dictionary of  $n$  strings drawn from the alphabet  $[\sigma]$ , there is a monotone minimal perfect hash function  $h()$  that occupies  $\mathcal{O}(n \log(\ell \log \sigma))$  bits of space, where  $\ell$  is the average length of the strings in the dictionary. Given a string  $P[1..m]$ ,  $h(P)$  is computed in  $\mathcal{O}(1 + m \log \sigma/w)$  time.*

**Weak prefix search** The *Prefix Search Problem* is a well-known problem in data-structure design for strings. It asks for the preprocessing of a given set of  $n$  strings in such a way that, given a query-pattern  $P$ , (the lexicographic range of) all the strings in the dictionary which have  $P$  as a prefix can be returned efficiently in time and space.

Belazzougui et al. [4] introduced the weak variant of the problem that allows for a one-sided error in the answer. Indeed, in the *Weak Prefix Search Problem* the answer to a query is required to be correct only in the case that  $P$  is a prefix of at least one string in dictionary; otherwise, the algorithm returns an arbitrary answer.

Due to these relaxed requirements, the data structures solving the problem are allowed to use space sublinear in the total length of the indexed strings. Belazzougui et al. [4] focus their attention on dictionaries of binary strings, but their results can be easily generalized to dictionaries with strings over larger alphabets. The following theorem states the obvious generalization of Theorem 5 in [4].

► **Theorem 4.** *Given a dictionary of  $n$  strings drawn from the alphabet  $[\sigma]$ , there exists a data structure that weak prefix searches for a pattern  $P[1..m]$  in  $\mathcal{O}(m \log \sigma/w + \log(m \log \sigma))$*

time. The data structure uses  $\mathcal{O}(n \log(\ell \log \sigma))$  bits of space, where  $\ell$  is the average length of the strings in the dictionary.

We remark that the space bound in [4] is better than the one reported above as it is stated in terms of the size the hollow trie, a conceptual tool introduced in [3], associated to the indexed dictionary. This measure is always within  $\mathcal{O}(n \log(\ell \log \sigma))$  bits but it may be much better depending on the dictionary. However, the weaker space bound suffices for the aims of this paper.

#### 4 An Encoding for Scan-Based Search

As an introduction to our techniques, we show an  $\mathcal{O}(n \log \log n)$  bit encoding supporting scan-based order-preserving matching. Given a sequence  $S[1..n]$  we define the rank encoding  $E(S)[1..n]$  as

$$E(S)[i] = \begin{cases} 0.5 & \text{if } S[i] \text{ is lexicographically smaller than any} \\ & \text{character in } \{S[1], \dots, S[i-1]\}, \\ j & \text{if } S[i] \text{ is equal to the lexicographically } j\text{th} \\ & \text{character in } \{S[1], \dots, S[i-1]\}, \\ j + 0.5 & \text{if } S[i] \text{ is larger than the lexicographically } j\text{th} \\ & \text{character in } \{S[1], \dots, S[i-1]\} \text{ but smaller} \\ & \text{than the lexicographically } (j+1)\text{st}, \\ |\{S[1], \dots, S[i-1]\}| + 0.5 & \text{if } S[i] \text{ is lexicographically larger than any} \\ & \text{character in } \{S[1], \dots, S[i-1]\}. \end{cases}$$

This is similar to the representations used in previous papers on order-preserving matching. We can build  $E(S)$  in  $\mathcal{O}(n \log n)$  time. However, we would ideally need  $E(S[i..n])$  for  $i = 1, \dots, n$ , since  $P[1..m]$  has an order-preserving match in  $S[i..i+m-1]$  if and only if  $E(P) = E(S[i..i+m-1])$ . Assuming  $P$  has polylogarithmic size, we can devise a more space efficient encoding.

► **Lemma 5.** *Given  $S[1..n]$  and a constant  $c \geq 1$ , let  $\ell = \log^c n$ . We can store  $\mathcal{O}(n \log \log n)$  bits such that later, given  $i$  and  $m \leq \ell$ , we can compute  $E(S[i..i+m-1])$  in  $\mathcal{O}(m)$  time.*

**Proof.** For every position  $i$  in  $S$  which is a multiple of  $\ell = \log^c n$ , we store the ranks of the characters in the window  $S[i..i+2\ell]$ . The ranks are values at most  $2\ell + 1$ , thus they are stored in  $\mathcal{O}(\log \ell)$  bits each. We concatenate the ranks of each window in a vector  $V$ , which has length  $\mathcal{O}(n)$  and takes  $\mathcal{O}(n \log \ell)$  bits. Every range  $S[i..i+m-1]$  of length  $m \leq \ell$  is fully contained in at least one window and in constant time we can convert  $i$  into  $i'$  such that  $V[i'..i'+m-1]$  contains the ranks of  $S[i], \dots, S[i+m-1]$  in that window.

Computing  $E(S[i..i+m-1])$  naïvely from these ranks would take  $\mathcal{O}(m \log m)$  time. We can speed up this computation by exploiting the fact that  $S[i..i+m-1]$  has polylogarithmic length. Indeed, a recent result [37] introduces a data structure to represent a small dynamic set  $\mathcal{S}$  of  $\mathcal{O}(w^c)$  integers of  $w$  bits each supporting, among the others, insertions and rank queries in  $\mathcal{O}(1)$  time. Given an integer  $x$ , the rank of  $x$  is the number of integers in  $\mathcal{S}$  that are smaller than or equal to  $x$ . All operations are supported in constant time for sets of size  $\mathcal{O}(w^c)$ . This result allows us to compute  $E(S[i..i+m-1])$  in  $\mathcal{O}(m)$  time. Indeed, we can use the above data structure to insert  $S[i..i+m-1]$ 's characters one after the other and compute their ranks in constant time. ◀

It follows from Lemma 5 that given  $S$  and  $c$ , we can store an  $\mathcal{O}(n \log \log n)$ -bit encoding of  $S$  such that later, given a pattern  $P[1..m]$  with  $m \leq \log^c n$ , we can compute  $E(S[i..i+m-1])$  for each position  $i$  in turn and compare it to  $E(P)$ , and thus find all the order-preserving matches of  $P$  in  $\mathcal{O}(nm)$  time. (It is possible to speed this scan-based algorithm up by avoiding computing each  $E(S[i..i+m-1])$  from scratch but, since this is only an intermediate result, we do not pursue it further here.) We note that we can construct the encoding in Lemma 5 knowing only how each character of  $S$  compares to  $\mathcal{O}(\log^c n)$  neighbouring characters.

► **Corollary 6.** *Given  $S[1..n]$  and a constant  $c \geq 1$ , we can store an encoding of  $S$  in  $\mathcal{O}(n \log \log n)$  bits such that later, given a pattern  $P[1..m]$  with  $m \leq \log^c n$ , we can find all the order-preserving matches of  $P$  in  $S$  in  $\mathcal{O}(nm)$  time.*

We will not use Corollary 6 in the rest of this paper, but we state it as a baseline easily proven from Lemma 5.

## 5 Adding an Index to the Encoding

Suppose we are given  $S[1..n]$  and a constant  $c \geq 1$ . We build the  $\mathcal{O}(n \log \log n)$ -bit encoding of Lemma 5 for  $\ell = \log^c n + \log n$  and call it  $S_\ell$ . Using  $S_\ell$  we can compute  $E(S')$  for any substring  $S'$  of  $S$  of length  $|S'| \leq \ell$  in  $\mathcal{O}(|S'|)$  time. We now show how to complement  $S_\ell$  with a kind of “sampled suffix array” using  $\mathcal{O}(n \log \log n)$  more bits, such that we can search for a pattern  $P[1..m]$  with  $m \leq \log^c n$  and return the starting position of an order-preserving match for  $P$  in  $S$ , if there is one. Our first solution has  $\mathcal{O}(m \log^3 n)$  query time; we will improve the query time to  $\mathcal{O}(m)$  in the next section.

We define the rank-encoded suffix array  $R[1..n]$  of  $S$  such that  $R[i] = j$  if  $E(S[j..n])$  is the lexicographically  $i$ th string in  $\{E(S[1..n]), E(S[2..n]), \dots, E(S[n..n])\}$ . Note that  $E(S[i..n])$  has length  $n - i + 1$ . Figure 1 shows an example. Our algorithm consists of a *searching phase* followed by a *verification phase*. The goal of the searching phase is to identify a range  $[l, r]$  in  $R$  which contains all the encodings prefixed by  $E(P)$ , if any, or an arbitrary interval if  $P$  does not occur. The verification phase has to check if there is at least one occurrence of  $P$  in this interval, and return a position at which  $P$  occurs.

**Searching phase** Similarly to how we can use a normal suffix array and  $S$  to support normal pattern matching, we could use  $R$  and  $S$  to find all order-preserving matches for a pattern  $P[1..m]$  in  $\mathcal{O}(m \log n)$  time via binary search, i.e., at each step we choose an index  $i$ , extract  $S[R[i]..R[i] + m - 1]$ , compute its rank encoding and compare it to  $E(P)$ , all in  $\mathcal{O}(m)$  time. If  $m \leq \ell$  we can compute  $E(S[R[i]..R[i] + m - 1])$  using  $S_\ell$  instead of  $S$ , still in  $\mathcal{O}(m)$  time, but storing  $R$  still takes  $\Omega(n \log n)$  bits.

Therefore, for our searching phase we sample and store only every  $d$ -th element of  $R$ , by position, and every element of  $R$  equal 1 or  $n$  or a multiple of  $d$ , where  $d = \lfloor \log n / \log \log n \rfloor$ . This takes  $\mathcal{O}(n \log \log n)$  bits. Notice we can still find in  $\mathcal{O}(m \log n)$  time via binary search in the sampled  $R$  an order-preserving match for any pattern  $P[1..m]$  that has at least  $d$  order-preserving matches in  $S$ . If  $P$  has fewer than  $d$  order-preserving matches in  $S$  but we happen to have sampled a cell of  $R$  pointing to the starting position of one of those matches, then our binary search still finds it. Otherwise, we find an interval of length at most  $d - 1$  which contains pointers to all the order-preserving matches for  $P$  in  $S$ ; on this interval we perform the verification phase.

$i$	$R[i]$	$L[i]$	$B[i]$	$D[i]$	$E(S[R[i]..n])$
1	<b>30</b>				0.5
2	29	<b>2</b>	<b>1.5</b>	<b>4</b>	0.5 0.5
3	22	<b>2</b>	<b>0.5</b>	<b>2</b>	0.5 0.5 0.5 0.5 1.5 5 5.5 6.5 1
4	<b>13</b>				0.5 0.5 0.5 1 0.5 1.5 4 4.5 1 4 3.5 3.5 2 3 6 7 7.5 2
5	2	<b>2</b>	<b>0.5</b>	<b>1</b>	0.5 0.5 0.5 1.5 2.5 3.5 5.5 2.5 2 5 4 8 4 1 1 0.5 1.5 6 7 1 6 5 4 2 3 6 7 8 2
6	23	<b>3</b>	<b>3.5</b>	<b>3</b>	0.5 0.5 0.5 1.5 4.5 5.5 6.5 1
7	<b>8</b>				0.5 0.5 0.5 2.5 2.5 5.5 3 0.5 1 0.5 1.5 6 7 1 6 5 4 2 3 6 7 7.5 2
8	<b>14</b>				0.5 0.5 1 0.5 1.5 4 4.5 1 4 3.5 3.5 2 3 6 7 7.5 2
9	<b>20</b>				0.5 0.5 1.5 1.5 1.5 1.5 2.5 6 7 7.5 2
10	3	<b>3</b>	<b>3.5</b>	<b>1</b>	0.5 0.5 1.5 2.5 3.5 5.5 2.5 2 5 4 7.5 4 1 1 0.5 1.5 6 7 1 6 5 4 2 3 6 7 8 2
11	<b>16</b>				0.5 0.5 1.5 3.5 4.5 1 4 3.5 3.5 2 3 6 7 7.5 2
12	<b>24</b>				0.5 0.5 1.5 3.5 4.5 5.5 1
13	11	<b>2</b>	<b>0.5</b>	<b>3</b>	0.5 0.5 2.5 1 0.5 1 0.5 1.5 4 5 1 4 3.5 3.5 2 3 6 7 7.5 2
14	9	<b>3</b>	<b>3.5</b>	<b>3</b>	0.5 0.5 2.5 2.5 4.5 3 0.5 1 0.5 1.5 6 7 1 6 5 4 2 3 6 7 7.5 2
15	15	<b>2</b>	<b>1.5</b>	<b>1</b>	0.5 1 0.5 1.5 3.5 4.5 1 4 3.5 3.5 2 3 6 7 7.5 2
16	<b>28</b>				0.5 1.5 0.5
17	7	<b>3</b>	<b>1.5</b>	<b>4</b>	0.5 1.5 0.5 0.5 3 2.5 5.5 3 0.5 1 0.5 1.5 6 7 1 6 5 4 2 3 6 7 7.5 2
18	19	<b>3</b>	<b>1.5</b>	<b>5</b>	0.5 1.5 0.5 2 1.5 1.5 1.5 2.5 6 7 7.5 2
19	<b>12</b>				0.5 1.5 1 0.5 1 0.5 1.5 4 4.5 1 4 3.5 3.5 2 3 6 7 7.5 2
20	<b>1</b>				0.5 1.5 1.5 0.5 2 2.5 3.5 5.5 2.5 2 5 4 8 4 1 1 0.5 1.5 6 7 1 6 5 4 2 3 6 7 8 2
21	21	<b>2</b>	<b>2.5</b>	<b>1</b>	0.5 1.5 1.5 1.5 1.5 2.5 6 6.5 7.5 2
22	10	<b>2</b>	<b>1.5</b>	<b>2</b>	0.5 1.5 1.5 3.5 2 0.5 1 0.5 1.5 5 6 1 5 4.5 4 2 3 6 7 7.5 2
23	27	<b>4</b>	<b>1.5</b>	<b>2</b>	0.5 1.5 2.5 0.5
24	<b>6</b>				0.5 1.5 2.5 0.5 0.5 4 3 5.5 3 0.5 1 0.5 1.5 6 7 1 6 5 4 2 3 6 7 7.5 2
25	18	<b>4</b>	<b>1</b>	<b>3</b>	0.5 1.5 2.5 0.5 3 2.5 2.5 2 2.5 6 7 7.5 2
26	26	<b>4</b>	<b>0.5</b>	<b>1</b>	0.5 1.5 2.5 3.5 0.5
27	17	<b>2</b>	<b>2.5</b>	<b>3</b>	0.5 1.5 2.5 3.5 1 3 2.5 2.5 2 2.5 6 7 7.5 2
28	<b>5</b>				0.5 1.5 2.5 3.5 1.5 1 4 3 5.5 3 0.5 1 0.5 1.5 6 7 1 6 5 4 2 3 6 7 7.5 2
29	25	<b>2</b>	<b>2.5</b>	<b>4</b>	0.5 1.5 2.5 3.5 4.5 1
30	<b>4</b>				0.5 1.5 2.5 3.5 4.5 2.5 2 5 4 6.5 4 1 1 0.5 1.5 6 7 1 6 5 4 2 3 6 7 7.5 2

■ **Figure 1** The rank-encoded suffix array  $R[1..30]$  for  $S[1..30] = 397235684365952201560543125671$ , with  $L[i]$ ,  $B[i]$  and  $D[i]$  computed for  $d = 4$ . Stored values are shown in boldface.

**Verification phase** The verification phase receives a range  $R[l, r]$  (although  $R$  is not stored completely) and has to check if that range contains the starting position of an order preserving match for  $P$  and, if so, return its position. This is done by adding auxiliary data structures to the sampled entries of  $R$ .

Suppose that for each unsampled element  $R[i] = j$  we store the following data.

- the smallest number  $L[i]$  (if one exists) such that  $S[j - 1..j + L[i] - 1]$  has at most  $d$  order-preserving matches in  $S$ ;
- the rank  $B[i] = E(S[j - 1..j + L[i] - 1]^{rev})[L[i] + 1] \leq L[i] + 1/2$  of  $S[j - 1]$  in  $S[j..j + L[i] - 1]$ , where the superscript  $rev$  indicates that the string is reversed;
- the distance  $D[i]$  to the cell of  $R$  containing  $j - 1$  from the last sampled element  $x$  such that  $E(S[x..x + L[i]])$  is lexicographically smaller than  $E(S[j - 1..j + L[i] - 1])$ .

Figure 1 shows the values in  $L$ ,  $B$  and  $D$  for our example.

Assume we are given  $P[1..m]$  and  $i$  and told that  $S[R[i]..R[i] + m - 1]$  is an order-preserving match for  $P$ , but we are not told the value  $R[i] = j$ . If  $R[i]$  is sampled, of course, then we can return  $j$  immediately. If  $L[i]$  does not exist or is greater than  $m$  then  $P$  has at least  $d$  order-preserving matches in  $S$ , so we can find one in  $\mathcal{O}(m)$  time: we consider



the sampled values from  $R$  that precede and follow  $R[i]$  and check with Lemma 5 whether there are order-preserving matches starting at those sampled values. Otherwise, from  $L[i]$ ,  $B[i]$  and  $P$ , we can compute  $E(S[j - 1..j + L[i] - 1])$  in  $\mathcal{O}(m \log m)$  time: we take the length- $L[i]$  prefix of  $P$ ; if  $B[i]$  is an integer, we prepend to  $P[1..L[i]]$  a character equal to the lexicographically  $B[i]$ th character in that prefix; if  $B[i]$  is  $r + 0.5$  for some integer  $r$  with  $1 \leq r < L[i]$ , we prepend a character lexicographically between the lexicographically  $r$ th and  $(r + 1)$ st characters in the prefix; if  $B[i] = 0.5$  or  $B[i] = L[i] + 0.5$ , we prepend a character lexicographically smaller or larger than any in the prefix, respectively. We can then find in  $\mathcal{O}(m \log n)$  time the position in  $R$  of  $x$ , the last sampled element such that  $E(S[x..x + L[i]])$  is lexicographically smaller than  $E(S[j - 1..j + L[i] - 1])$ . Adding  $D[i]$  to this position gives us the position  $i'$  of  $j - 1$  in  $R$ . Repeating this procedure until we reach a sampled cell of  $R$  takes  $\mathcal{O}(m \log^2 n / \log \log n) = \mathcal{O}(m \log^2 n)$  time, and we can then compute and return  $j$ . As the reader may have noticed, the procedure is very similar to how we use backward stepping to locate occurrences of a pattern with an FM-index [19], so we refer to it as a backward step at position  $i$ .

Even if we do not really know whether  $S[R[i]..R[i] + m - 1]$  is an order-preserving match for  $P$ , we can still start at the cell  $R[i]$  and repeatedly apply this procedure: if we do not find a sampled cell after  $d - 1$  repetitions, then  $S[R[i]..R[i] + m - 1]$  is not an order-preserving match for  $P$ ; if we do, then we add the number of times we have repeated the procedure to the contents of the sampled cell to obtain the contents of  $R[i] = j$ . Then, using  $S_\ell$  we compute  $E(S[j..j + m - 1])$  in  $\mathcal{O}(m)$  time, compare it to  $E(P)$  and, if they are the same, return  $j$ . This still takes  $\mathcal{O}(m \log^2 n)$  time. Therefore, after our searching phase, if we find an interval  $[l, r]$  of length at most  $d - 1$  which contains pointers to all the order-preserving matches for  $P$  in  $S$  (instead of an order-preserving match directly), then we can check each cell in that interval with this procedure, in a total of  $\mathcal{O}(m \log^3 n)$  time.

If  $R[i] = j$  is the starting position of an order-preserving match for a pattern  $P[1..m]$  with  $m \leq \log^c n$  that has at most  $d$  order-preserving matches in  $S$ , then  $L[i] \leq \log^c n$ . Moreover, if  $R[i'] = j - 1$  then  $L[i'] \leq \log^c n + 1$  and, more generally, if  $R[i''] = j - t$  then  $L[i''] \leq \log^c n + t$ . Therefore, we can repeat the stepping procedure described above and find  $j$  without ever reading a value in  $L$  larger than  $\log^c n + \log n$  and, since each value in  $B$  is bounded in terms of the corresponding value in  $L$ , without ever reading a value in  $B$  larger than  $\log^c n + \log n + 1/2$ . It follows that we can replace any values in  $L$  and  $B$  greater than  $\log^c n + \log n + 1/2$  by the flag  $-1$ , indicating that we can stop the procedure when we read it. With this modification, each value in  $L$  and  $B$  takes  $\mathcal{O}(\log \log n)$  bits, so  $L$ ,  $B$  and  $D$  take a total of  $\mathcal{O}(n \log \log n)$  bits. Since also the encoding  $S_\ell$  from Lemma 5 with  $\ell = \log^c n + \log n$  takes  $\mathcal{O}(n \log \log n)$  bits, the following intermediate theorem summarizes our results so far.

► **Theorem 7.** *Given  $S[1..n]$  and a constant  $c \geq 1$ , we can store an encoding of  $S$  in  $\mathcal{O}(n \log \log n)$  bits such that later, given a pattern  $P[1..m]$  with  $m \leq \log^c n$ , in  $\mathcal{O}(m \log^3 n)$  time we can return the position of an order-preserving match of  $P$  in  $S$  (if one exists).*

**A complete search example** Suppose we are searching for order-preserving matches for  $P = 2312$  in the string  $S[1..30]$  shown in Figure 1. Binary search on  $R$  tells us that pointers to all the matches are located in  $R$  strictly between  $R[16] = 28$  and  $R[19] = 12$ , because

$$\begin{aligned} E(S[28..30]) = E(671) = 0.5 \ 1.5 \ 0.5 &< E(P) = E(2312) = 0.5 \ 1.5 \ 0.5 \ 2 \\ &< E(S[12..14]) = E(595) = 0.5 \ 1.5 \ 1; \end{aligned}$$

notice  $R[16] = 28$  and  $R[19] = 12$  are stored because 16, 28 and 12 are multiples of  $d = 4$ .



We first check whether  $R[17]$  points to an order-preserving match for  $P$ . That is, we assume (incorrectly) that it does; we take the first  $L[17] = 3$  characters of  $P$ ; and, because  $B[17] = 1.5$ , we prepend a character between the lexicographically first and second, say 1.5. This gives us 1.5 2 3 1, whose encoding is 0.5 1.5 2.5 0.5. Another binary search on  $R$  shows that  $R[20] = 1$  is the last sampled element  $x$  such that  $E(S[x..x + 3])$ , in this case 0.5 1.5 1.5 0.5, is lexicographically smaller than 0.5 1.5 2.5 0.5. Adding  $D[17] = 4$  to 20, we would conclude that  $R[24] = R[17] - 1$  (which happens to be true in this case) and that 0.5 1.5 2.5 0.5 is a prefix of  $E(S[R[24]..n])$  (which also happens to be true). Since  $R[24] = 6$  is sampled, however, we compute  $E(S[7..10]) = 0.5 1.5 0.5 0.5$  and, since it is not the same as  $P$ 's encoding, we reject our initial assumption that  $R[17]$  points to an order-preserving match for  $P$ .

We now check whether  $R[18]$  points to an order preserving match for  $P$ . That is, we assume (correctly this time) that it does; we take the first  $L[18] = 3$  characters of  $P$ ; and, because  $B[18] = 1.5$ , we prepend a character between the lexicographically first and second, say 1.5. This again gives us 1.5 3 2 1, whose encoding is 0.5 1.5 2.5 0.5. As before, a binary search on  $R$  shows that  $R[20] = 1$  is the last sampled element  $x$  such that  $E(S[x..x + 3])$  is lexicographically smaller than 0.5 1.5 2.5 0.5. Adding  $D[18] = 5$  to 20, we conclude (correctly) that  $R[25] = R[18] - 1$  and that 0.5 1.5 2.5 0.5 is a prefix of  $E(S[R[25]..n])$ .

Repeating this procedure with  $L[25] = 4$ ,  $B[25] = 1$  and  $D[25] = 3$ , we build a string with encoding 0.5 1.5 2.5 0.5, say 2 3 4 1, and prepend a character equal to the lexicographically first, 1. This gives us 1 2 3 4 1, whose encoding is 0.5 1.5 2.5 3.5 1. Another binary search shows that  $R[24] = 6$  is the last sampled element  $x$  such that  $E(S[x..x + 4])$  is lexicographically smaller than 0.5 1.5 2.5 3.5 1. We conclude (again correctly) that  $R[27] = R[18] - 2$  and that 0.5 1.5 2.5 3.5 1 is a prefix of  $E(S[R[27]..n])$ .

Finally, repeating this procedure with  $L[27] = 2$ ,  $B[27] = 2.5$  and  $D[27] = 3$ , we build a string with encoding 0.5 1.5, say 1 2, and prepend a character lexicographically greater than any currently in the string, say 3. This gives us 3 1 2, whose encoding is 0.5 0.5 1.5. A final binary search show that  $R[8] = 14$  is the last sampled element  $x$  such that  $E(S[x..x + 2])$  is lexicographically smaller than 0.5 0.5 1.5. We conclude (again correctly) that  $R[11] = R[18] - 3$  and that 0.5 0.5 1.5 is a prefix of  $E(S[R[11]..n])$ . Since  $R[11] = 16$  is sampled, we compute  $E(S[19..22]) = 0.5 1.5 0.5 2$  and, since it matches  $P$ 's encoding, we indeed report  $S[19..22]$  as an order-preserving match for  $P$ .

## 6 Achieving $\mathcal{O}(m)$ query time

In this section we prove our main result:

► **Theorem 8.** *Given  $S[1..n]$  and a constant  $c \geq 1$ , we can store an encoding of  $S$  in  $\mathcal{O}(n \log \log n)$  bits such that later, given a pattern  $P[1..m]$  with  $m \leq \log^c n$ , in  $\mathcal{O}(m)$  time we can return the position of an order-preserving match of  $P$  in  $S$  (if one exists). In  $\mathcal{O}(m)$  time we can also report the total number of order-preserving occurrences of  $P$  in  $S$ .*

Compared to Theorem 7, we improve the query time from  $\mathcal{O}(m \log^3 n)$  to  $\mathcal{O}(m)$ . This is achieved by speeding up several steps of the algorithm described in the previous section.

**Speeding up pattern's encoding** Given a pattern  $P[1..m]$ , the algorithm has to compute its encoding  $E(P[1..m])$ . Doing this naïvely as in the previous section would cost  $\mathcal{O}(m \log m)$  time, which is, by itself, larger than our target time complexity. However, since  $m$  is polylogarithmic in  $n$ , we can speed this up as we sped up the computation of the rank-encoding of  $S[i..i + m - 1]$  in the proof of Lemma 5, and obtain  $E(P)$  in  $\mathcal{O}(m)$  time. Indeed,

we can insert  $P$ 's characters one after the other in the data structures of [37] and compute their ranks in constant time.

**Dealing with short patterns** The approach used by our solution cannot achieve a  $o(d)$  query time. This is because we answer a query by performing  $\Theta(d)$  backward steps regardless of the pattern's length. This means that for very short patterns, namely  $m = o(d) = o(\log n / \log \log n)$ , the solution cannot achieve  $\mathcal{O}(m)$  query time. However, we can precompute and store the answers of all these short patterns in  $o(n)$  bits. Indeed, we can find a constant  $c$  such that the encoding of a pattern of length at most  $c \log n / \log \log n$  is a binary string of length at most  $\frac{1}{2} \log n$  bits. Thus, there are  $\mathcal{O}(\sqrt{n})$  possible encodings. For each of these encodings we explicitly store the number of its occurrences and the position of one of them in  $o(n)$  bits. From now on, thus, we can safely assume that  $m = \Omega(\log n / \log \log n)$ .

**Speeding up searching phase** The searching phase of the previous algorithm has two important drawbacks. First, it costs  $\mathcal{O}(m \log n)$  time and, thus, it is obviously too expensive for our target time complexity. Second, binary searching on the sampled entries in  $R$  gives too imprecise results. Indeed, it finds a range  $[l, r]$  of positions in  $R$  which may be potential matches for  $P$ . However, if the entire range is within two consecutive sampled positions, we are only guaranteed that all the occurrences of  $P$  are in the range but there may exist positions in the range which do not match  $P$ . This uncertainty forces us to explicitly check *every* single position in the range until a match for  $P$  is found, if any. This implies that we have to check  $r - l + 1 = \mathcal{O}(d)$  positions in the worst case. Since every check has a cost proportional to  $m$ , this gives  $\omega(m)$  query time.

We use the data structure for weak prefix search of Theorem 4 to index the encodings of all suffixes of the text truncated at length  $\ell = \log^c n + \log n$ . This way, we can find the range  $[l, r]$  of suffixes prefixed by  $E(P[1..m])$  in  $\mathcal{O}(m \log \log n / w + \log(m \log \log n)) = \mathcal{O}(m \log \log n / w + \log \log n)$  time with a data structure of size  $\mathcal{O}(n \log \log n)$  bits. This is because  $E(P[1..m])$  is drawn from an alphabet of size  $\mathcal{O}(\log^c n)$ , and both  $m$  and  $\ell$  are in  $\mathcal{O}(\log^c n)$ . Apart from its faster query time, this solution has stronger guarantees. Indeed, if the pattern  $P$  has at least one occurrence, the range  $[l, r]$  contains all and only the occurrences of  $P$ . Instead, if the pattern  $P$  does not occur,  $[l, r]$  is an arbitrary and meaningless range. In both cases, just a single check of any position in the range is enough to answer the order-preserving query. This property gives an  $\mathcal{O}(\log n / \log \log n)$  factor improvement over the previous solution.

**Speeding up verification phase** It is clear by the discussion above that the verification phase has to check only one position in the range  $[l, r]$ . If the range contains at least one sampled entry of  $R$ , we are done. Otherwise, we have to perform at most  $d$  backward steps.

We now improve the computation of every single backward step. Assume we have to perform a backward step at  $i$ , where  $R[i] = j$ . Before performing the backward step, we have to compute the encoding  $E(S[j - 1..j + L[i] - 1])$  given  $E(S[j..j + u])$ , for some value of  $L[i]$  and  $u$  with  $u \geq L[i]$ . Our goal is to do this in  $\mathcal{O}(1 + m \log \log n / w)$  time. Notice that removing symbols at the end of  $S[j..j + u]$  does not change the encoding of the remaining symbols. However, after the insertion of  $S[j - 1]$  the encoding of  $S[k]$  in  $S[j - 1..j + L[i] - 1]$  either does not change, if  $S[j - 1] \geq S[k]$ , or has to be increased by one, if  $S[j - 1] > S[k]$ . The main issue is that we have to process  $\mathcal{O}(w / \log \log n)$  symbols in parallel. To this end, apart from  $E(S[j - 1..j + L[i] - 1])$ , we also keep a different encoding  $R(S[j..j + u])$  for  $S[j..j + u]$ . The encoding  $R$  stores  $\mathcal{O}(\log \log n)$ -bit ranks which represent the relative

order among symbols in  $S[j..j+u]$ . More precisely, for any two symbols  $S[k]$  and  $S[k']$ ,  $R(S[j..j+u])[k] < R(S[j..j+u])[k']$  iff  $S[k] < S[k']$ . Notice that we are not constraining these ranking values to form a consecutive interval, i.e., there may be missing values.

Our goal is to compute  $R(S[j-1..j+L[i]-1])$  from  $R(S[j..j+u])$  as we perform backward steps. For this reason, we no longer store the value  $B[i]$  as in the previous solution. Instead, we store the values  $P[i]$  and  $O[i]$  which are the positions of the predecessor and an occurrence of  $S[j-1]$  in  $S[j..j+L[i]-1]$ , if any. This way, we can compute  $R(S[j-1..j+L[i]-1])$  by prepending an appropriate rank  $r$  for symbol  $S[j-1]$ . It is  $r = R(S[j..L[i]-1])[O[i]]$ , if there already exists an occurrence of  $S[j-1]$  in  $S[j..L[i]-1]$ , or  $r = R(S[j..L[i]-1])[P[i]] + 1$ , otherwise. In the latter case, we increase any value in  $R(S[j..L[i]-1])$  which is larger than or equal to  $r$  to guarantee that there is no collision with the assigned rank. This can be done in  $\mathcal{O}(1 + L[i] \log \log n/w) = \mathcal{O}(1 + m \log \log n/w)$  time by exploiting word parallelism. We observe that the positions with a rank larger than  $r$  are exactly the positions that we need to increase by one in order to compute  $E(S[j-1..j+L[i]-1])$ . The backward step at  $i$  is  $i' = k + D[i]$ , where  $k$  is the sampled entry in  $R$  whose encoding has the prefix of length  $L[i]$  which is the largest prefix which is (lexicographically) smaller than or equal to  $E(S[j-1..j+L[i]-1])$ . Notice that equality may occur only for at most one prefix as otherwise  $S[j-1..j+L[i]-1]$  would occur more than  $d$  times.

To compute  $k$ , given  $i$  and  $E(S[j-1..j+L[i]-1])$ , we observe that  $E(S[j-1..j+L[i]-1])$  depends only on  $S$  and  $L[i]$  and not on the pattern  $P$  we are searching for. Thus, there exists just one valid  $E(S[j-1..j+L[i]-1])$  that could be used at query time for a backward step at  $i$ . Notice that, if the pattern  $P$  does not occur, the encoding that will be used at  $i$  may be different, but in this case it is not necessary to compute a correct backward step. Consider the set  $\mathcal{E}$  of all these, at most  $n$ , encodings. The goal is to map each encoding in  $\mathcal{E}$  to its corresponding sampled entry in  $R$ . This can be done as follows. We build a monotone minimal perfect hash function  $h()$  on  $\mathcal{E}$  to map each encoding to its lexicographic rank. Obviously, the encodings to be mapped to a certain sampled entry  $i$  in  $R$  form a consecutive range in the lexicographic ordering. Moreover, none of these ranges overlap. Thus, we can use a binary vector  $B$  to mark each of these ranges, so that, given the lexicographic rank of an encoding, we can infer its closest sampled entry. The binary vector is obtained by processing the sampled entries in  $R$  in lexicographic order and by writing the size of its range in unary. It is easy to see that the sampled entry prefixed by  $x = E(S[j-1..j+L[i]-1])$  can be computed as  $\text{Rank}_1(h(x))$  in constant time. The data structure that stores  $B$  and supports  $\text{Rank}$  requires  $\mathcal{O}(n)$  bits (see Theorem 1).

Since the evaluation of  $h()$  is the dominant cost, a backward step takes  $\mathcal{O}(1 + m \log \log n/w)$  time. The overall space usage of this solution is  $\mathcal{O}(n \log \log n)$  bits, because  $B$  has at most  $2n$  bits and  $h()$  requires  $\mathcal{O}(n \log \log n)$  bits by Theorem 3. Since we perform at most  $d$  backward steps, the overall query time is  $\mathcal{O}(d \times (1 + m \log \log n/w) = \mathcal{O}(m)$ . The equality follows by observing that  $d = \mathcal{O}(\log n / \log \log n)$ ,  $m = \Omega(\log n / \log \log n)$  and  $w = \Omega(\log n)$ .

**Query algorithm** We report here the query algorithm for a pattern  $P[1..m]$ , with  $m = \Omega(\log n / \log \log n)$ . Recall that for shorter patterns we store all possible answers.

We first compute  $E(P[1..m])$  in  $\mathcal{O}(1 + m \log \log n/w)$  time. Then, we perform a weak prefix search to identify the range  $[l, r]$  of encodings that are prefixed by  $E(P[1..m])$  in  $\mathcal{O}(m \log \log n/w + \log \log n)$  time. If  $P$  has at least one occurrence, the search is guaranteed to find the correct range; otherwise, the range may be arbitrary but the subsequent check will identify the mistake and report zero occurrences.

In the checking phase there are two possible cases. If  $[l, r]$  contains a sampled entry,

say  $i$ , in  $R$  we use the encoding from Lemma 5 to compare  $E(S[R[i]..R[i] + m - 1])$  and  $E(P[1..m])$  in  $\mathcal{O}(m)$  time. If they are equal, we report  $R[i]$ ; otherwise, we are guaranteed that there is no occurrence of  $P$  in  $S$ . If  $[l, r]$  contains no sampled entry we arbitrarily select an index  $i \in [l, r]$  and we perform a sequence of backward steps starting from  $i$ . If  $P$  has at least one occurrence, we are guaranteed to find a sampled entry  $e$  in at most  $d$  backward steps. The overall time of these backward steps is  $\mathcal{O}(d \times m \log \log n/w) = \mathcal{O}(m)$ . If  $e$  is not found, we conclude that  $P$  has no occurrence. Otherwise, we explicitly compare  $E(S[R[e] + b..R[e] + m + b - 1])$  and  $E(P[1..m])$  in  $\mathcal{O}(m)$  time, where  $b$  is the number of performed backward steps. We report  $R[e] + b$  only in case of a successful comparison. Note that if  $P$  occurs, then the number of its occurrences is  $r - l + 1$ .

## 7 Space Lower Bound

In this section we prove that our solution is space optimal. This is done by showing a lower bound on the space that any data structure must use to solve the easier problem of just establishing if a given pattern  $P$  has at least one order-preserving occurrence in  $S$ .

► **Theorem 9.** *For any  $n$ , for any  $\sigma$  such that  $\log \sigma = \Omega(\log \log n)$ , and for any encoding data structure that, given a pattern  $P[1..m]$  with  $m = \log n$ , establishes if  $P$  has any order-preserving occurrence in a given string, there exists a string  $S[1..n]$  over the alphabet  $[\sigma]$  such that the encoding must use  $\Omega(n \log \log n)$  bits of space.*

By contradiction assume there exists a data structure  $D$  that uses  $o(n \log \log n)$  bits. We prove this implies we can store any string  $S[1..n]$  in less than  $n \log \sigma$  bits, which is impossible. We start by splitting  $S$  into  $n/m$  blocks of size  $m = \log n$ . Let  $B_i$  denote the  $i$ th block. Observe that if we know the set  $L(B_i)$  of characters that occur in  $B_i$ , we can recover  $B_i$ . This is because  $E(B_i)$  implicitly tells us how to permute the characters in  $L(B_i)$  to obtain  $B_i$ . Obviously, if we are able to reconstruct each  $B_i$ , we can reconstruct  $S$ . Thus, our goal is to use  $D$  together with additional data structures to obtain  $E(B_i)$  and  $L(B_i)$ , for any  $B_i$ .

We encode  $L(B_i)$  for each  $i$  by encoding the sorted sequence of characters with the Elias-Fano representation. By Theorem 2, we know that this requires  $\ell \log \frac{\sigma}{\ell} + \mathcal{O}(\ell)$  bits, where  $\ell = |L(B_i)| \leq m$ . If  $\sigma \geq m$ , this is at most  $m \log \frac{\sigma}{m} + \mathcal{O}(m)$  bits. Summing over all the blocks, the overall space is at most  $n \log \frac{\sigma}{m} + \mathcal{O}(n)$  bits. If  $\sigma < m$ , the representation uses  $\mathcal{O}(m)$  bits per block and, thus,  $\mathcal{O}(n)$  bits overall.

To represent the encodings of all the blocks, consider the set  $\mathcal{E}$  of the encodings of all the substrings of  $S$  of length  $m$ . We do not store  $\mathcal{E}$  because it would require too much space. Instead, for each block  $B_i$ , we store the lexicographic rank of  $B_i$  in  $\mathcal{E}$ . This way, we are keeping track of those elements in  $\mathcal{E}$  that are blocks and their positions in  $S$ . This requires  $\mathcal{O}(n)$  bits, because there are  $n/\log n$  blocks and storing each rank needs  $\mathcal{O}(\log n)$  bits.

We are now ready to retrieve the encoding of all the blocks. This is done by searching in  $D$  for every possible encoding of exactly  $m$  characters. The data structure will tell us the ones that occur in  $S$ , i.e., we are retrieving the entire set  $\mathcal{E}$ . Thus, we sort  $\mathcal{E}$  and replace each the stored rank of each block with its original encoding. Thus, we are able to reconstruct  $S$  by using  $D$  and additional data structures which uses at most  $\max(0, n \log \sigma - n \log \log n) + \mathcal{O}(n)$  bits of space. This implies that  $D$  cannot use  $o(n \log \log n)$  bits.

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