Sorting suffixes of a text via its Lyndon Factorization

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The sorting of the suffixes

Our goal

The goal is to introduce a new strategy for sorting the suffixes of a word $w$.

- The process of sorting the suffixes of a word plays a fundamental role in *Text Algorithms* with several applications in many areas of Computer Science and Bioinformatics.
- For instance, it is a fundamental step, in implicit or explicit way, for the construction of
  - the Suffix Array ($SA$): the array containing the starting positions of the suffixes of a word, sorted in lexicographic order;
  - the Burrows-Wheeler Transform ($BWT$): the array containing a permutation of the symbols of a word according to the sorting of its suffixes.
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Our idea

Our strategy uses the *Lyndon factorization* and is based on a combinatorial property that allows to sort the suffixes of $w$ ("global suffixes") by using the sorting of the suffixes inside blocks of consecutive Lyndon factors of the decomposition ("local suffixes").
Lyndon Words

- Two words \( u, v \in \Sigma^* \) are conjugate, if \( u = xy \) and \( v = yx \) for some \( x, y \in \Sigma^* \). Thus conjugate words are just cyclic shifts of one another.
- A word \( w \in \Sigma^+ \) is primitive if \( w = u^h \) implies \( w = u \) and \( h = 1 \).

**Definition**

A *Lyndon word* is a (primitive) word that is smaller in lexicographic order than all of its conjugates.

**Example**

- \( u = \text{mathematics} \) is not a Lyndon word;
- \( v = \text{athematicsm} \) is a Lyndon word.

There exist linear algorithms for the computation of the Lyndon word of a given word [Duval, 1983].
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Lyndon Factorization

**Theorem (Chen, Fox and Lyndon: 1958)**

Every word $w \in \Sigma^+$ has a **unique factorization** $w = L_1 \cdots L_k$ such that

$$L_1 \geq \cdots \geq L_k$$

is a non-increasing sequence of Lyndon words.

Let $w = abaaaaabaaaaaabaaaaabaaaaaaab$. The Lyndon factorization of $w$ is

$$ab|aaaab|aaaaaabaa|aaaaaabb$$

Note that each $L_i$ is strictly less than any of its proper conjugates/suffixes.
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The Lyndon factorization of a given word can be computed
- in linear time [Duval, 1983];
- in parallel way [Apostolico and Crochemore, 1989] and [Daykin, Iliopoulos and Smyth, 1994];
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Local and Global suffixes

For each factor $u$ of $w$, we denote by $\text{first}(u)$ and $\text{last}(u)$ the position of the first and the last symbol, respectively, of the factor $u$ in $w$.

We denote by

- $\text{suf}_u(i) = w[i, \text{last}(u)]$ and we call it local suffix at the position $i$ with respect to $u$.
- $\text{suf}(i) = w[i, n]$ and we call it global suffix of $w$ at the position $i$.
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$$w = \begin{array}{c}
1 \quad \ldots \quad i \quad u \quad \ldots \quad n \\
\text{first}(u) \quad \text{last}(u) \\
\text{suf}_u(i) \\
\text{suf}(i)
\end{array}$$
Compatible sorting

Definition

Let \( w \) be a word and let \( u \) be a factor of \( w \). We say that the sorting of the local suffixes with respect to \( u \) is \textit{compatible} with the sorting of the global suffixes of \( w \) if for all \( i, j \) with \( \text{first}(u) \leq i < j \leq \text{last}(u) \),

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\text{suf}_u(i) < \text{suf}_u(j) \iff \text{suf}(i) < \text{suf}(j).
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In general, taken an arbitrary factor of a word \( w \), the sorting of its suffixes is \textit{not compatible} with the sorting of the suffixes of \( w \), as the following example shows.

Example

Consider the word \( w = abababb \) and its factor \( u = ababa \).

Then \( \text{suf}_u(1) = ababa > a = \text{suf}_u(5) \)

whereas \( \text{suf}(1) = abababb < abb = \text{suf}(5) \).

Such sorting is not compatible.
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Our result

**Theorem**

Let \( w \in \Sigma^* \) and let \( w = L_1 L_2 \cdots L_k \) be its Lyndon factorization. For each factor \( u = L_r L_{r+1} \cdots L_s \), the sorting of the local suffixes with respect to \( u \) is compatible with the sorting of the global suffixes of \( w \).
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\[
\begin{align*}
  w &= L_1 \cdots L_r \uparrow \downarrow \cdots \downarrow L_s \cdots L_k \\
  u &= L_rL_{r+1} \cdots L_s
\end{align*}
\]
Easy case

The theorem is trivially true when the two suffixes start with two different Lyndon factors.

Suppose that
- \( i \) is the position of the first symbol of \( L_r \)
- \( j \) is the position of the first symbol of \( L_s \)
- \( u \) is the smallest factor containing both \( L_r \) and \( L_s \): \( L_r L_{r+1} \cdots L_s \).

Since \( r < s \) and \( L_1 \geq \cdots \geq L_r \geq \cdots \geq L_s \geq \cdots \geq L_k \). It is easy to verify that
- \( L_r L_{r+1} \cdots L_s > L_s \)
- \( L_r L_{r+1} \cdots L_k > L_s L_{s+1} \cdots L_k \)

We don’t need to compare any symbol.
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Suppose that
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- \( u \) is the smallest factor containing both \( L_R \) and \( L_S \): \( L_R L_{R+1} \cdots L_S \).

Since \( r < s \) and \( L_1 \geq \cdots \geq L_R \geq \cdots \geq L_S \geq \cdots \geq L_k \). It is easy to verify that
- \( L_R L_{R+1} \cdots L_S > L_S \)
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We don’t need to compare any symbol.
Other cases

The theorem is true when the two suffixes of $w$ start inside the same factor $u$ of consecutive Lyndon words.

Suppose that

- $i$ is a position inside $L_r$;
- $j$ is a position inside $L_s$;
- $u$ is the smallest factor containing both $L_r$ and $L_s$: $L_r L_{r+1} \cdots L_s$.

$$suf(i) = \overbrace{L_r[i, \text{last}(L_r)]}^{suf_u(i)} L_{r+1} \cdots L_s L_{k}$$

$$suf(j) = \overbrace{L_s[j, \text{last}(L_s)]}^{suf_u(j)} L_{s+1} \cdots L_{k}$$

How many symbol comparisons we need to establish the order relation between $suf(i)$ and $suf(j)$?
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How many symbol comparisons we need to establish the order relation between \( suf(i) \) and \( suf(j) \)?
How many symbol comparisons?

Possible cases:

- There is a different symbol inside $w[i, \text{last}(L_r)]$ and $w[j, \text{last}(L_s)]$.
- There is not a different symbol inside $w[i, \text{last}(L_r)]$ and $w[j, \text{last}(L_s)]$:
  - $w[i, \text{last}(L_r)] = w[j, \text{last}(L_s)]$;
  - $w[j, \text{last}(L_s)]$ is a prefix of $w[i, \text{last}(L_r)]$;
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First case

- There is a different symbol inside $w[i, \text{last}(L_r)]$ and $w[j, \text{last}(L_s)]$.
- It is easy to verify that the order relation between the local and the global suffixes is the same!
- We need $lcp(i, j) + 1 \leq \min(|w[i, \text{last}(L_r)]|, |w[j, \text{last}(L_s)]|)$ symbol comparisons, where $lcp(i, j)$ denotes the length of the longest common prefix between the suffixes $w[i, n]$ and $w[j, n]$. 
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Second case: \( w[i, \text{last}(L_r)] = w[j, \text{last}(L_s)] \)

- Since \( r < s \) and \( L_1 \geq \cdots \geq L_r \geq \cdots \geq L_s \geq \cdots \geq L_k \). It is easy to verify that the order relation between the local and the global suffixes is the same! So we don’t need to compare further symbols.
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- In order to get the mutual order between $\text{suf}(i)$ and $\text{suf}(j)$, we need to compare at most $l(j) = |\text{suf}_u(j)|$ symbol comparisons.
- Consider $w[i, i + l(j) - 1]$ and $w[j, j + l(j) - 1] = \text{suf}_u(j)$.
  - There is a mismatch, then we need $\text{lcp}(i, j) + 1 \leq l(j)$ symbol comparisons.
  - There is not a mismatch, then we use the property of the Lyndon factorization: $L_{s+1} \cdots L_k$ is smaller than any suffix of $u$ and of $w$. 

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  - There is a mismatch, then we need $lcp(i, j) + 1 \leq l(j)$ symbol comparisons.
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- In order to get the mutual order between \( suf(i) \) and \( suf(j) \), we need to compare at most \( l(j) = |suf_u(j)| \) symbol comparisons.
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How many symbol comparisons?

In order to get the mutual order between $suf(i)$ and $suf(j)$ it is sufficient to execute at most $l(j) = |suf_u(j)|$ symbol comparisons.

Note that $l(j)$, as shown by the following example, can be smaller than $lcp(i, j) + 1$. 
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Example

Let \( w = abaaaabaaaaaabaaaaabaaaaaab \). Its Lyndon factorization is
\[
ab | aaaaab | aaaaabaaaaab | aaaaab.
\]
Let \( u = ab | aaaaab | aaaaabaaaaab | aaaaab \).

\[
\begin{align*}
i &= 2 \\
\downarrow & \\
\end{align*}
\]

\[
\begin{align*}
w &= a & b & | & a & a & a & a & b & | & a & a & a & a & b & | & a & a & a & a & b
\end{align*}
\]

\[
\begin{align*}
j &= 13 \\
\downarrow & \\
\end{align*}
\]

Consider the following suffixes:

\[
\begin{align*}
suf(2) &= b & a & a & a & b & a & a & a & a & b \\
\uparrow & \\
13
\end{align*}
\]

\[
\begin{align*}
suf(13) &= b & a & a & a & b & a & a & a & a & a & b
\end{align*}
\]

We have \( lcp(2, 13) = 11 \) and \( l(13) = 6 \).

We need only 6 symbol comparisons, indeed for Lyndon properties

\[
w[8, 25] > w[19, 25] \Rightarrow suf(2) > suf(13).
\]
Example

Let $w = abaaaabaaaaaabaaaaaabaaaaaaab$. Its Lyndon factorization is $ab | aaaaab | aaaaaabaaaaab | aaaaaaab$. Let $u = ab | aaaaab | aaaaaaabaaaaab |$.

$$
i = 2
\downarrow
1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \ 16 \ 17 \ 18 \ 19 \ 20 \ 21 \ 22 \ 23 \ 24 \ 25
w = a \ b \ | \ a \ a \ a \ a \ b \ | \ a \ a \ a \ a \ a \ b \ | \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ b \ |
\uparrow
2 + \text{lcp}(2, 13) - 1
$$

$$
j = 13
\downarrow
1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \ 16 \ 17 \ 18 \ 19 \ 20 \ 21 \ 22 \ 23 \ 24 \ 25
w = a \ b \ | \ a \ a \ a \ a \ b \ | \ a \ a \ a \ a \ a \ b \ | \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ a \ b \ |
\uparrow
13 + \text{lcp}(2, 13) - 1
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$$

$$
suf(13) = b \ a \ a \ a \ a \ b \ a \ a \ a \ a \ a \ a \ a \ b
$$

We have $\text{lcp}(2, 13) = 11$ and $l(13) = 6$.

We need only 6 symbol comparisons, indeed for Lyndon properties $w[8, 25] > w[19, 25] \Rightarrow suf(2) > suf(13)$. 

Example

Let \( w = abaaaabaaaaaabaaaaabaaaaaababaaaaab \). Its Lyndon factorization is \( ab|aaaab|aaaaabaaaaab|aaaaaab | \). Let \( u = ab|aaaab|aaaaabaaaaab|aaaaaab \).

\[
\begin{array}{cccccccccccccccc}
  j = 13 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 w = a & b & | & a & a & a & a & a & b & a & a & a & a & a & b & a & a & a & a & b & a & a & a & a & b & a & b & \\
\end{array}
\]

\[
2 + l(13) - 1 \quad j = 13 \quad \text{last}(u) = 18
\]

Consider the following suffixes:

\[
\begin{array}{cccccccccccccccc}
 s \text{uf}(2) = b & a & a & a & a & b & a & a & a & a & b & a & a & a & b & a & a & a & a & b & a & a & a & a & b & a & b \\
 s \text{uf}(13) = b & a & a & a & a & b & a & a & a & a & a & b & a & a & a & a & a & b & a & a & a & a & b \\
\end{array}
\]

\[
13 \quad 13 + l(13) - 1 \quad 13 + l(13) - 1
\]

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Let \( w = abaaabaabaaaabaaaaabaaaababaaaababaaaab \). Its Lyndon factorization is \( ab|aaaab|aaaabbaaaaab|aaaabaaab | \). Let \( u = ab|aaaab|aaaabbaaaaab|aaaabaaab | \).

\[
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\end{array}
\]

2 + \( l(13) \) - 1

13 + \( l(13) \) - 1

Consider the following suffixes:

\[
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    \text{suf}(13) &= b \ a \ a \ a \ a \ b \ | & a & a & a & a & a & b & | & a & a & a & b \ a & a & a & a & a & b & |
\end{align*}
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\( w[8, 25] > w[19, 25] \Rightarrow \text{suf}(2) > \text{suf}(13) \).
Our strategy for sorting all suffixes

Let \( w = L_1 L_2 \cdots L_l L_{l+1} \cdots L_k \). We propose an algorithm that is based on the following

**Proposition**

Let \( \text{sort}(L_1 L_2 \cdots L_l) \) and \( \text{sort}(L_{l+1} L_{l+2} \cdots L_k) \) denote the sorted lists of the suffixes of \( L_1 L_2 \cdots L_l \) and the suffixes \( L_{l+1} L_{l+2} \cdots L_k \), respectively. Then

\[
\text{sort}(L_1 L_2 \cdots L_k) = \text{merge}(\text{sort}(L_1 L_2 \cdots L_l), \text{sort}(L_{l+1} L_{l+2} \cdots L_k)).
\]

- The sorted list of the global suffixes of \( w \) can be obtained by merging the sorted lists of the local suffixes inside \( L_1 L_2 \cdots L_l \) and \( L_{l+1} L_{l+2} \cdots L_k \).
- Note that the mutual order of the local suffixes is preserved after the merge operation.
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Let $\text{sort}(L_1 L_2 \cdots L_l)$ and $\text{sort}(L_{l+1} L_{l+2} \cdots L_k)$ denote the sorted lists of the suffixes of $L_1 L_2 \cdots L_l$ and the suffixes $L_{l+1} L_{l+2} \cdots L_k$, respectively. Then

$$\text{sort}(L_1 L_2 \cdots L_k) = \text{merge}(\text{sort}(L_1 L_2 \cdots L_l), \text{sort}(L_{l+1} L_{l+2} \cdots L_k)).$$

- The sorted list of the global suffixes of $w$ can be obtained by merging the sorted lists of the local suffixes inside $L_1 L_2 \cdots L_l$ and $L_{l+1} L_{l+2} \cdots L_k$.

- Note that the mutual order of the local suffixes is preserved after the merge operation.
Our algorithm

This proposition suggests a possible strategy for sorting the list of the suffixes of some word $w$:

- find the Lyndon decomposition of $w$: $L_1 L_2 \cdots L_k$;
- find the sorted list of the suffixes of $L_1$ and, separately, the sorted list of the suffixes of $L_2$;
- merge the sorted lists in order to obtain the sorted list of the suffixes of $L_1 L_2$;
- find the sorted list of the suffixes of $L_3$ and merge it to the previous sorted list;
- repeat until all the Lyndon factors are processed;

One can use this strategy for computing the suffix array and for constructing the Burrows-Wheeler Transform.
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The Burrows-Wheeler Transform

A possible definition of BWT consists in adding an end-marker symbol $ at the end of the word. Start with word $ w \in \Sigma^* $

- Append $ symbol, which is lexicographically before all other characters in the alphabet $ \Sigma $.
- Generate all of the conjugates of $ w $ and sort them lexicographically, forming a matrix $ M $ with rows and columns equal to $ |w| = |w| + 1 $.
- Construct $ L $, the transformed text of $ w $, by taking the last column of $ M $.

Example: $ w = mathematics $

<table>
<thead>
<tr>
<th>$ mathematics$</th>
<th>1 $ mathematics$</th>
<th>2 $ mathematics$</th>
<th>3 $ mathematics$</th>
<th>4 $ mathematics$</th>
<th>5 $ mathematics$</th>
<th>6 $ mathematics$</th>
<th>7 $ mathematics$</th>
<th>8 $ mathematics$</th>
<th>9 $ mathematics$</th>
<th>10 $ mathematics$</th>
<th>11 $ mathematics$</th>
<th>12 $ mathematics$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ mathematics$</td>
<td>$ mathematics$</td>
<td>$ mathematics$</td>
<td>$ mathematics$</td>
<td>$ mathematics$</td>
<td>$ mathematics$</td>
<td>$ mathematics$</td>
<td>$ mathematics$</td>
<td>$ mathematics$</td>
<td>$ mathematics$</td>
<td>$ mathematics$</td>
<td>$ mathematics$</td>
<td>$ mathematics$</td>
</tr>
</tbody>
</table>

Output: $ bwt(w) = L = smmihtt$ecaa.$

To recover the original word, it is enough to know the position of the symbol $ in $ L $.
The Burrows-Wheeler Transform

A possible definition of BWT consists in adding an end-marker symbol $ at the end of the word. Start with word $w \in \Sigma^*$.

- Append $ symbol, which is lexicographically before all other characters in the alphabet $\Sigma$.
- Generate all of the conjugates of $w$ and sort them lexicographically, forming a matrix $M$ with rows and columns equal to $|w| = |w| + 1$.
- Construct $L$, the transformed text of $w$, by taking the last column of $M$.

Example: $w = mathematics$

```
matheemathics $
```

```
  1 $mathemathics
  2 atheemathics $m
  3 atics $mathem
  4 cs $mathemati
  5 eemathics $math
  6 hemathics $mat
  7 ics $mathemat
  8 mathemathics $
  9 maties $mathem
 10 s $mathematic
11 theemathics $ma
12 tics $mahem
```

Output: $bwt(w) = L = smmihtt$ecaa.

To recover the original word, it is enough to know the position of the symbol $ in $L$. 

The Burrows-Wheeler Transform

A possible definition of BWT consists in adding an end-marker symbol $ at the end of the word. Start with word $ w \in \Sigma^* $.

- Append $ symbol, which is lexicographically before all other characters in the alphabet $ \Sigma $.
- Generate all of the conjugates of $ w $ and sort them lexicographically, forming a matrix $ M $ with rows and columns equal to $ |w| + 1 $.
- Construct $ L $, the transformed text of $ w $, by taking the last column of $ M $.

Example: $ w = mathematics $

```
mathematics $ \\
mathematics m \\
mathematics a \\
mathematics t \\
mathematics h \\
mathematics e \\
mathematics m \\
mathematics a \\
mathematics t \\
mathematics i \\
mathematics c \\
mathematics i \\
mathematics c \\
mathematics s \\

m  1 2 3 4 5 6 7 8 9 10 11 12
a  $ m a t h e m a t i c s \\
the  $ m a t h e m a t i c s \\
math e $ m a t h e m a t i c s \\
em a t h e m a t i c s $ m a t h e m a t i c s \\
c s $ m a t h e m a t i c s \\
mat $ m a t h e m a t i c s \\
the $ m a t h e m a t i c s \\
s $ m a t h e m a t i c s \\
mat $ m a t h e m a t i c s \\
the $ m a t h e m a t i c s \\
```

Output: $ bwt(w) = L = smmihtt$ecaa. 

To recover the original word, it is enough to know the position of the symbol $ in L. $
The Burrows-Wheeler Transform

A possible definition of BWT consists in adding an end-marker symbol $ at the end of the word. Start with word $ w \in \Sigma^* $.

- Append $ symbol, which is lexicographically before all other characters in the alphabet $ \Sigma $.
- Generate all of the conjugates of $ w $ and sort them lexicographically, forming a matrix $ M $ with rows and columns equal to $ |w| + 1 $.
- Construct $ L $, the transformed text of $ w $, by taking the last column of $ M $.

Example: $ w = mathematics $

\[
\begin{align*}
  m & at h e m a t i c s $ \\
  a t h e m a t i c s $ m \\
  t h e m a t i c s $ m a t \\
  h e m a t i c s $ m a t h \\
  e m a t i c s $ m a t h e \\
  m a t i c s $ m a t h e m \\
  a t i c s $ m a t h e m a \\
  i c s $ m a t h e m a t \\
  c s $ m a t h e m a t i \\
  s $ m a t h e m a t i c \\
  $ m a t h e m a t i c s \\
  \end{align*}
\]

\[
\begin{align*}
  1 & $ m a t h e m a t i c s \\
  2 & a t h e m a t i c s $ m \\
  3 & a t i c s $ m a t h e m \\
  4 & c s $ m a t h e m a t i \\
  5 & e m a t i c s $ m a t h \\
  6 & h e m a t i c s $ m a t h e \\
  7 & i c s $ m a t h e m a t \\
  8 & m a t h e m a t i c s $ \\
  9 & m a t i c s $ m a t h e \\
  10 & s $ m a t h e m a t i c \\
  11 & t h e m a t i c s $ m a t h e \\
  12 & t i c s $ m a t h e m a \\
\end{align*}
\]

\[ M \]

⇒

Output: $ bwt(w) = L = smmihtt$ecaa.$

To recover the original word, it is enough to know the position of the symbol $ in $ L $. 

The Burrows-Wheeler Transform

A possible definition of BWT consists in adding an end-marker symbol $ at the end of the word. Start with word $ w \in \Sigma^* $. 

- Append $ symbol, which is lexicographically before all other characters in the alphabet $ \Sigma $. 
- Generate all of the conjugates of $ w^\$ $ and sort them lexicographically, forming a matrix $ M $ with rows and columns equal to $ |w^\$| = |w| + 1 $. 
- Construct $ L $, the transformed text of $ w^\$ $, by taking the last column of $ M $.

Example: $ w = \text{mathematics} $

<table>
<thead>
<tr>
<th>$ M $</th>
<th>$ L $</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ m a t h e m a t i c s $</td>
<td>$ m a t h e m a t i c s $</td>
</tr>
<tr>
<td>$ a t h e m a t i c s $</td>
<td>$ a t h e m a t i c s $</td>
</tr>
<tr>
<td>$ t h e m a t i c s $</td>
<td>$ t h e m a t i c s $</td>
</tr>
<tr>
<td>$ h e m a t i c s $</td>
<td>$ h e m a t i c s $</td>
</tr>
<tr>
<td>$ e m a t i c s $</td>
<td>$ e m a t i c s $</td>
</tr>
<tr>
<td>$ m a t i c s $</td>
<td>$ m a t i c s $</td>
</tr>
<tr>
<td>$ a t i c s $</td>
<td>$ a t i c s $</td>
</tr>
<tr>
<td>$ t i c s $</td>
<td>$ t i c s $</td>
</tr>
<tr>
<td>$ i c s $</td>
<td>$ i c s $</td>
</tr>
<tr>
<td>$ c s $</td>
<td>$ c s $</td>
</tr>
<tr>
<td>$ s $</td>
<td>$ s $</td>
</tr>
<tr>
<td>$ ^$ $</td>
<td>$ ^$ $</td>
</tr>
</tbody>
</table>

Output: $ \text{bwt}(w^\$) = L = \text{smmihtt$ecaa}$. 

To recover the original word, it is enough to know the position of the symbol $ in $ $ L $. 

Example:

<table>
<thead>
<tr>
<th>$ F $</th>
<th>$ M $</th>
<th>$ L $</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ \downarrow $</td>
<td>$ \downarrow $</td>
<td>$ \Rightarrow $</td>
</tr>
<tr>
<td>1</td>
<td>$ m a t h e m a t i c s $</td>
<td>$ m a t h e m a t i c s $</td>
</tr>
<tr>
<td>2</td>
<td>$ a t h e m a t i c s $</td>
<td>$ a t h e m a t i c s $</td>
</tr>
<tr>
<td>3</td>
<td>$ t h e m a t i c s $</td>
<td>$ t h e m a t i c s $</td>
</tr>
<tr>
<td>4</td>
<td>$ h e m a t i c s $</td>
<td>$ h e m a t i c s $</td>
</tr>
<tr>
<td>5</td>
<td>$ e m a t i c s $</td>
<td>$ e m a t i c s $</td>
</tr>
<tr>
<td>6</td>
<td>$ m a t i c s $</td>
<td>$ m a t i c s $</td>
</tr>
<tr>
<td>7</td>
<td>$ a t i c s $</td>
<td>$ a t i c s $</td>
</tr>
<tr>
<td>8</td>
<td>$ t i c s $</td>
<td>$ t i c s $</td>
</tr>
<tr>
<td>9</td>
<td>$ i c s $</td>
<td>$ i c s $</td>
</tr>
<tr>
<td>10</td>
<td>$ c s $</td>
<td>$ c s $</td>
</tr>
<tr>
<td>11</td>
<td>$ s $</td>
<td>$ s $</td>
</tr>
</tbody>
</table>
The Burrows-Wheeler Transform

A possible definition of BWT consists in adding an end-marker symbol \( \$ \) at the end of the word. Start with word \( w \in \Sigma^* \).

- Append \( \$ \) symbol, which is lexicographically before all other characters in the alphabet \( \Sigma \).
- Generate all of the conjugates of \( w\$ \) and sort them lexicographically, forming a matrix \( M \) with rows and columns equal to \(|w\$| = |w| + 1 \).
- Construct \( L \), the transformed text of \( w\$ \), by taking the last column of \( M \).

Example: \( w = \text{mathematics} \)

\[
\begin{array}{cccccccccccccccc}
\mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} & \mathtt{\$} \\
\mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} & \mathtt{\$} & \mathtt{m} \\
\mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} & \mathtt{\$} & \mathtt{m} & \mathtt{a} \\
\mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} & \mathtt{\$} & \mathtt{m} & \mathtt{a} & \mathtt{t} \\
\mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} & \mathtt{\$} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} \\
\mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} & \mathtt{\$} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} \\
\mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} & \mathtt{\$} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} \\
\mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} & \mathtt{\$} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} \\
\mathtt{i} & \mathtt{c} & \mathtt{s} & \mathtt{\$} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} \\
\mathtt{c} & \mathtt{s} & \mathtt{\$} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} \\
\mathtt{s} & \mathtt{\$} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} \\
\$ & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} \\
\end{array}
\]

\[
\begin{array}{cccccccccccccccc}
F &&&&&&&&&&&&& \\
\downarrow &&&&&&&&&&&&& \\
1 & \$ & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} \\
2 & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} & \$ & \mathtt{m} \\
3 & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} & \$ & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} \\
4 & \mathtt{c} & \mathtt{s} & \$ & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} \\
5 & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} & \$ & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} \\
6 & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} & \$ & \mathtt{m} & \mathtt{a} & \mathtt{t} \\
7 & \mathtt{i} & \mathtt{c} & \mathtt{s} & \$ & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} \\
\Rightarrow & 8 & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} & \$ \\
9 & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} & \$ & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} \\
10 & \mathtt{s} & \$ & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} \\
11 & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} & \$ & \mathtt{m} & \mathtt{a} \\
12 & \mathtt{t} & \mathtt{i} & \mathtt{c} & \mathtt{s} & \$ & \mathtt{m} & \mathtt{a} & \mathtt{t} & \mathtt{h} & \mathtt{e} & \mathtt{m} & \mathtt{a} \\
\end{array}
\]

Output: \( \text{bwt}(w\$) = L = \text{smmihtt}\$\text{ecaa} \).

To recover the original word, it is enough to know the position of the symbol \( \$ \) in \( L \).
The Burrows-Wheeler Transform

A possible definition of BWT consists in adding an end-marker symbol $ at the end of the word. Start with word $w \in \Sigma^*$.

- Append $ symbol, which is lexicographically before all other characters in the alphabet $\Sigma$.
- Generate all of the conjugates of $w$ and sort them lexicographically, forming a matrix $M$ with rows and columns equal to $|w| = |w| + 1$.
- Construct $L$, the transformed text of $w$, by taking the last column of $M$.

Example: $w = mathematics$

```
matheematics $  
athematics $ m  
theematics $ m a  
theematics $ m a t  
ematheematics $ m at  
matices $ m at h  
matices $ m at h e  
atices $ m at he m  
tices $ m at he ma  
ic s $ m at he ma t  
c s $ m at he ma t i  
s $ m at he ma tic  
$ m at he ma tic s
```

\[ \Rightarrow \]

```
F  
\downarrow  
1 $ m a t h e m a t i c s  
2 a t h e m a t i c s $ m  
3 a t i c s $ m a t h e m  
4 c s $ m a t h e m a t i  
5 e m a t i c s $ m a t h  
6 h e m a t i c s $ m a t  
7 i c s $ m a t h e m a t  
8 \rightarrow m a t h e m a t i c s $  
9 m a t i c s $ m a t h e  
10 s $ m a t h e m a t i c  
11 t h e m a t i c s $ m a  
12 t i c s $ m a t h e m a
```

Output: $bwt(w) = L = smmihtt$ecaa.

To recover the original word, it is enough to know the position of the symbol $ in $L$. 

Sorting suffixes of a text via its Lyndon Factorization

Incontro di Combinatoria delle Parole
Progetto PRIN 2010/2011 "Automi e Linguaggi Formali: aspetti matematici e applicativi"
Palermo, 10-11 ottobre 2013

20 / 29
This is equivalent to sort the suffixes of $w$. 

$$w = m a t h e m a t i c s$$

<table>
<thead>
<tr>
<th>$M$</th>
<th>$L$</th>
<th>$BWT$</th>
<th>Sorted Suffixes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td></td>
<td>$s$</td>
<td>$s$</td>
</tr>
<tr>
<td>$m$</td>
<td>$a$</td>
<td>$m$</td>
<td>$m$</td>
</tr>
<tr>
<td>$m$</td>
<td>$i$</td>
<td>$m$</td>
<td>$m$</td>
</tr>
<tr>
<td>$c$</td>
<td>$t$</td>
<td>$c$</td>
<td>$c$</td>
</tr>
<tr>
<td>$e$</td>
<td>$h$</td>
<td>$e$</td>
<td>$e$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$h$</td>
<td>$h$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t$</td>
<td>$t$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$i$</td>
<td>$i$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$c$</td>
<td>$c$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$s$</td>
<td>$s$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$m$</td>
<td>$m$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a$</td>
<td>$a$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t$</td>
<td>$t$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$i$</td>
<td>$i$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$c$</td>
<td>$c$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$s$</td>
<td>$s$</td>
</tr>
</tbody>
</table>

Note that one can build the BWT of a string without needing to compute its suffix array.
### BWT and SA

This is equivalent to sort the suffixes of $w$.

In order to obtain it, one can compute the **suffix array**.

- $SA[i]$: The starting position of the $i$th smallest suffix of $w$.
- $BWT[i]$: The symbol that (circularly) precedes the first symbol of the $i$th smallest suffix.

$$w = mathematics$$

```latex
tabular
<table>
<thead>
<tr>
<th>M</th>
<th>L</th>
<th>$SA$</th>
<th>$BWT$</th>
<th>Sorted Suffixes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>a</td>
<td>12</td>
<td>s</td>
<td>$mathematics$</td>
</tr>
<tr>
<td>a</td>
<td>t</td>
<td>2</td>
<td>$m$</td>
<td>$mathematics$</td>
</tr>
<tr>
<td>t</td>
<td>h</td>
<td>7</td>
<td>$m$</td>
<td>$mathematics$</td>
</tr>
<tr>
<td>h</td>
<td>e</td>
<td>9</td>
<td>$i$</td>
<td>$mathematics$</td>
</tr>
<tr>
<td>e</td>
<td>m</td>
<td>5</td>
<td>$h$</td>
<td>$mathematics$</td>
</tr>
<tr>
<td>m</td>
<td>a</td>
<td>4</td>
<td>$t$</td>
<td>$mathematics$</td>
</tr>
<tr>
<td>a</td>
<td>t</td>
<td>9</td>
<td>$i$</td>
<td>$mathematics$</td>
</tr>
<tr>
<td>t</td>
<td>h</td>
<td>1</td>
<td>$s$</td>
<td>$mathematics$</td>
</tr>
<tr>
<td>h</td>
<td>e</td>
<td>6</td>
<td>$e$</td>
<td>$mathematics$</td>
</tr>
<tr>
<td>e</td>
<td>m</td>
<td>11</td>
<td>$c$</td>
<td>$mathematics$</td>
</tr>
<tr>
<td>m</td>
<td>a</td>
<td>3</td>
<td>$a$</td>
<td>$mathematics$</td>
</tr>
<tr>
<td>a</td>
<td>t</td>
<td>8</td>
<td>$i$</td>
<td>$mathematics$</td>
</tr>
<tr>
<td>t</td>
<td>i</td>
<td>10</td>
<td>$s$</td>
<td>$mathematics$</td>
</tr>
<tr>
<td>i</td>
<td>c</td>
<td>11</td>
<td>$c$</td>
<td>$mathematics$</td>
</tr>
<tr>
<td>c</td>
<td>s</td>
<td>10</td>
<td>$a$</td>
<td>$mathematics$</td>
</tr>
<tr>
<td>s</td>
<td>$m$</td>
<td>1</td>
<td>$s$</td>
<td>$mathematics$</td>
</tr>
<tr>
<td>$m$</td>
<td>a</td>
<td>12</td>
<td>s</td>
<td>$mathematics$</td>
</tr>
</tbody>
</table>

Note that one can build the BWT of a string without needing to compute its suffix array.
Compute the BWT and the SA

- the **Suffix Array** \((SA)\): the array containing the starting positions of the suffixes of a word, sorted in lexicographic order;
- the **Burrows-Wheeler Transform** \((BWT)\): the array containing a permutation of the symbols of a word according to the sorting of its suffixes.

Let \(w = aabcabbaabaabdabbaaabbbdc\). Its Lyndon factorization is \(aabcabb|aabaabdabb|aaabbdc\).

\[
\begin{align*}
\text{w$} &= L_1 = aabcabb \quad L_2 = aabaabdabb \quad L_3 = aaabbdc \quad L_4 = \$
\end{align*}
\]
Compute the BWT and the SA

- the **Suffix Array** ($SA$): the array containing the starting positions of the suffixes of a word, sorted in lexicographic order;
- the **Burrows-Wheeler Transform** ($BWT$): the array containing a permutation of the symbols of a word according to the sorting of its suffixes.

Let $w = aabcabbaabaabdabbaaabbdabbaabbdc$. Its Lyndon factorization is $aabcabb|aabaabdabb|aaabbdc$.

$$w^\$ = L_1 = aabcabb \quad L_2 = aabaabdabb \quad L_3 = aaabbdc \quad L_4 = $$
Compute the BWT and the SA

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Let $w = aabcabbaabaabdbbaaabbdc$. Its Lyndon factorization is $aabcabb|aabaabdbb|aaabbdc$.

$w$ = $L_1 = aabcabb$  $L_2 = aabaabdbb$  $L_3 = aaabbdc$  $L_4 =$

Consider: $L_1 = aabcabb$

$w$ = $L_1 = aabcabb$  $L_2 = aabaabdbb$  $L_3 = aaabbdc$  $L_4 =$

Consider: $L_1 = aabcabb$
Compute the BWT and the SA

- the **Suffix Array** (SA): the array containing the starting positions of the suffixes of a word, sorted in lexicographic order;
- the **Burrows-Wheeler Transform** (BWT): the array containing a permutation of the symbols of a word according to the sorting of its suffixes.

Let $w = aabcabbaabaabdabbaaabbdc$. Its Lyndon factorization is $aabcabb|aabaabdabb|aaabbdc$.

Consider: $L_1$ = $aabcabb$

Compute the $BWT(L_1)$ and $SA(L_1)$:

<table>
<thead>
<tr>
<th>$L_1$</th>
<th>SA</th>
<th>BWT</th>
<th>Sorted Suffixes</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>$b$</td>
<td></td>
<td>$$</td>
</tr>
<tr>
<td>1</td>
<td>$$</td>
<td>$aabcabb$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$c$</td>
<td>$abb$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$a$</td>
<td>$abcabb$</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$b$</td>
<td>$b$</td>
<td>$b$</td>
</tr>
<tr>
<td>6</td>
<td>$a$</td>
<td>$bb$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$a$</td>
<td>$bcabb$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$b$</td>
<td>$cab$</td>
<td></td>
</tr>
</tbody>
</table>
Compute the BWT and the SA

\[ w = L_1 = aabcabb \quad L_2 = aabaabdabb \quad L_3 = aaabbdcc \quad L_4 = \$ \]

\[ L_1$ = aabcabb$ \]

\[
\begin{array}{|c|c|c|}
\hline
SA & BWT & Sorted Suffixes \\
\hline
8 & b & \$ \\
1 & $ & aabcabb$ \\
5 & c & abb$ \\
2 & a & abcabb$ \\
7 & b & b$ \\
6 & a & bb$ \\
3 & a & bcabb$ \\
4 & b & cabb$ \\
\hline
\end{array}
\]
Compute the BWT and the SA

\[ w = L_1 = aabcabb \quad L_2 = aabaabdabb \quad L_3 = aaabbdac \quad L_4 = $ \]

\[ L_1$ = aabcabb$ \]

Consider: \[ L_2$ = aabaabdabb$ \]

Note that \(|L_1| = j_1 = 7\). Compute the \( BWT(L_2$) and \( SA(L_2$).
Compute the BWT and the SA

**L₁ $**

<table>
<thead>
<tr>
<th>SA</th>
<th>BWT</th>
<th>Sorted Suffixes</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>b</td>
<td>$</td>
</tr>
<tr>
<td>1</td>
<td>$</td>
<td>aabcabb$</td>
</tr>
<tr>
<td>5</td>
<td>c</td>
<td>abb$</td>
</tr>
<tr>
<td>2</td>
<td>a</td>
<td>abcabb$</td>
</tr>
<tr>
<td>7</td>
<td>b</td>
<td>b$</td>
</tr>
<tr>
<td>6</td>
<td>a</td>
<td>bb$</td>
</tr>
<tr>
<td>3</td>
<td>a</td>
<td>bcabb$</td>
</tr>
<tr>
<td>4</td>
<td>b</td>
<td>cabb$</td>
</tr>
</tbody>
</table>

**L₂ $**

<table>
<thead>
<tr>
<th>G</th>
<th>SA</th>
<th>BWT</th>
<th>Sorted Suffixes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>11 + 7 = 18</td>
<td>b</td>
<td>$</td>
</tr>
<tr>
<td>0</td>
<td>1 + 7 = 8</td>
<td>$</td>
<td>aabaabdabb$</td>
</tr>
<tr>
<td>2</td>
<td>4 + 7 = 11</td>
<td>b</td>
<td>aabdabb$</td>
</tr>
<tr>
<td>2</td>
<td>2 + 7 = 9</td>
<td>a</td>
<td>abaadabb$</td>
</tr>
<tr>
<td>2</td>
<td>8 + 7 = 15</td>
<td>d</td>
<td>abb$</td>
</tr>
<tr>
<td>4</td>
<td>5 + 7 = 12</td>
<td>a</td>
<td>abdabb$</td>
</tr>
<tr>
<td>4</td>
<td>10 + 7 = 17</td>
<td>b</td>
<td>b$</td>
</tr>
<tr>
<td>5</td>
<td>3 + 7 = 10</td>
<td>a</td>
<td>baabdbabb$</td>
</tr>
<tr>
<td>5</td>
<td>9 + 7 = 16</td>
<td>a</td>
<td>bb$</td>
</tr>
<tr>
<td>7</td>
<td>6 + 7 = 13</td>
<td>a</td>
<td>bdabb$</td>
</tr>
<tr>
<td>8</td>
<td>7 + 7 = 14</td>
<td>b</td>
<td>dabb$</td>
</tr>
</tbody>
</table>

merge ⇒

<table>
<thead>
<tr>
<th>L₁L₂ $</th>
<th>SA</th>
<th>BWT</th>
<th>Sorted Suffixes</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>b</td>
<td>$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>b</td>
<td>aabaabdabb$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$</td>
<td>aabcabbaabaabdabb$</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>b</td>
<td>aabdabb$</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>a</td>
<td>abaadabb$</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>d</td>
<td>abb$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>c</td>
<td>abbaabaabdabb$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>a</td>
<td>abcabaabaabdabb$</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>a</td>
<td>abdabb$</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>b</td>
<td>b$</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>b</td>
<td>baabaabdabb$</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>a</td>
<td>baabdabb$</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>a</td>
<td>bb$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>a</td>
<td>bbaabaabdabb$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>a</td>
<td>bcabbaabaabdabb$</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>a</td>
<td>baddb$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>b</td>
<td>cabaabaabdabb$</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>b</td>
<td>dabb$</td>
<td></td>
</tr>
</tbody>
</table>
Compute the BWT and the SA

\[ W = \begin{array}{c}
L_1 = aabcabb \\
L_2 = aabaabdabb \\
L_3 = aaabbdc \\
L_4 = $
\end{array} \]

\[ L_1 L_2$ = aabcabbaabaabdabb$ \]

By merging the sorted list of the suffixes of $L_1 L_2$ and of $L_3$, we obtain the SA/BWT of $W = L_1 L_2 L_3$. 
Compute the BWT and the SA

\[ w = \begin{array}{c}
   L_1 = aabcabb \\
   L_2 = aabaabdabb \\
   L_3 = aaabbdc \\
   L_4 = \\
\end{array} \]

\[ L_1 L_2$ = aabcabbaabaabdabb$ \]

Consider:\n
\[ L_3$ = aaabbdc$ \]

Compute the $BWT(L_3$) and $SA(L_3$).

\[
\begin{array}{|c|c|c|}
\hline
SA & L_3$ & Sorted Suffixes \\
\hline
17 + 8 = 25 & c & $ \\
17 + 1 = 18 & $ & aaabbdc$ \\
17 + 2 = 19 & a & aaabbdc$ \\
17 + 3 = 20 & a & abbd$ \\
17 + 4 = 21 & a & bbdc$ \\
17 + 5 = 22 & b & bdc$ \\
17 + 7 = 24 & d & c$ \\
17 + 6 = 23 & b & dc$ \\
\hline
\end{array}
\]

By merging the sorted list of the suffixes of $L_1L_2$ and of $L_3$, we obtain the SA/BWT of \[ w = L_1L_2L_3$. \]
Compute the BWT and the SA

\[ w = L_1 = aabcabb \quad L_2 = aabaabdabb \quad L_3 = aaabbdcc \quad L_4 = \$

\[ L_1 L_2$ = aabcabbaabaabdabb$

Consider: \[ L_3$ = aaabbdcc$

Compute the \( BWT(L_3$)\) and \( SA(L_3$).

<table>
<thead>
<tr>
<th>( SA )</th>
<th>( BWT )</th>
<th>Sorted Suffixes</th>
</tr>
</thead>
<tbody>
<tr>
<td>17 + 8 = 25</td>
<td>c</td>
<td>$</td>
</tr>
<tr>
<td>17 + 1 = 18</td>
<td>$</td>
<td>aaabbdcc$</td>
</tr>
<tr>
<td>17 + 2 = 19</td>
<td>a</td>
<td>aabbdcc$</td>
</tr>
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<td>a</td>
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</tr>
<tr>
<td>17 + 4 = 21</td>
<td>a</td>
<td>bbdc$</td>
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<tr>
<td>17 + 5 = 22</td>
<td>b</td>
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<tr>
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<td>c$</td>
</tr>
<tr>
<td>17 + 6 = 23</td>
<td>b</td>
<td>dc$</td>
</tr>
</tbody>
</table>

By merging the sorted list of the suffixes of $L_1 L_2$ and of $L_3$, we obtain the SA/BWT of $w$ = $L_1 L_2 L_3$.
Further work: Parallel sorting

- The word could be partitioned into several sequences of consecutive blocks of Lyndon words, and the sorting algorithm can be applied \textit{in parallel} to each of those sequences. Then one should merge the sorted lists.

- Furthermore, also the Lyndon factorization can be performed in parallel, as shown in [Apostolico and Crochemore, 1989] and [Daykin, Iliopoulos and Smyth, 1994].
Further work

One can compute the BWT without the SA by using our strategy and the strategies already used in the following papers:

- Hon, Lam, Sadakane, Sung and Yiu, 2007;
- Ferragina, Gagie and Manzini, 2010 and 2012;
- Bauer, Cox and R., 2011 and 2013;

In this way, one could obtain algorithms that work:

- in external memory;
- in place.

One could use efficient dynamic data structures for the rank and insert operations, for instance by using Navarro and Nekrich’s recent results on optimal representations of dynamic sequences.
Further work: linear algorithm

Does there exist a linear algorithm that uses the Lyndon Factorization in order to sort (implicitly or explicitly) the suffixes?

Open problem!
Thank you for your attention!