# The Burrows-Wheeler Transform between Data Compression and Combinatorics on Words

Giovanna Rosone and Marinella Sciortino

Dipartimento di Matematica e Informatica
University of Palermo, ITALY

CiE 2013

## Preliminaries

- Let $\Sigma$ denote a non-empty finite alphabet.
- A word $w$ over an alphabet $\Sigma$ is a finite sequence of letters of $\Sigma$. We denote by $\Sigma^*$ the set of all words over $\Sigma$.
- Given a finite word $w = a_1 a_2 \cdots a_n$, $a_i \in \Sigma$, a factor of $w$ is written as $w[i,j] = a_i \cdots a_j$. A factor $w[1,j]$ is called a prefix, while a factor $w[i,n]$ is called a suffix.
- A non-empty word $w \in \Sigma^*$ is *primitive* if $w = u^h$ implies $w = u$ and $h = 1$.
- Two words $u, v \in \Sigma^*$ are conjugate, if $u = xy$ and $v = yx$ for some $x, y \in \Sigma^*$. Thus conjugate words are just cyclic shifts of one another.
- A *Lyndon word* is a primitive word which is the minimum among its conjugates, with respect to the lexicographic order relation.
  - $mathematics$ is not a Lyndon word
  - $athematicsm$ is a Lyndon word.

## Preliminaries

- Let $\Sigma$ denote a non-empty finite alphabet.
- A word $w$ over an alphabet $\Sigma$ is a finite sequence of letters of $\Sigma$. We denote by $\Sigma^*$ the set of all words over $\Sigma$.
- Given a finite word $w = a_1 a_2 \cdots a_n$, $a_i \in \Sigma$, a factor of $w$ is written as $w[i, j] = a_i \cdots a_j$. A factor $w[1, j]$ is called a prefix, while a factor $w[i, n]$ is called a suffix.
- A non-empty word $w \in \Sigma^*$ is *primitive* if $w = u^h$ implies $w = u$ and $h = 1$.
- Two words $u, v \in \Sigma^*$ are conjugate, if $u = xy$ and $v = yx$ for some $x, y \in \Sigma^*$. Thus conjugate words are just cyclic shifts of one another.
- A *Lyndon word* is a primitive word which is the minimum among its conjugates, with respect to the lexicographic order relation.
  - $mathematics$ is not a Lyndon word
  - $athematicsm$ is a Lyndon word.

## Preliminaries

- Let $\Sigma$ denote a non-empty finite alphabet.
- A word $w$ over an alphabet $\Sigma$ is a finite sequence of letters of $\Sigma$. We denote by $\Sigma^*$ the set of all words over $\Sigma$.
- Given a finite word $w = a_1 a_2 \cdots a_n$, $a_i \in \Sigma$, a factor of $w$ is written as $w[i, j] = a_i \cdots a_j$. A factor $w[1, j]$ is called a prefix, while a factor $w[i, n]$ is called a suffix.
- A non-empty word $w \in \Sigma^*$ is *primitive* if $w = u^h$ implies $w = u$ and $h = 1$.
- Two words $u, v \in \Sigma^*$ are conjugate, if $u = xy$ and $v = yx$ for some $x, y \in \Sigma^*$. Thus conjugate words are just cyclic shifts of one another.
- A *Lyndon word* is a primitive word which is the minimum among its conjugates, with respect to the lexicographic order relation.
  - $mathematics$ is not a Lyndon word
  - $athematicsm$ is a Lyndon word.

## The Burrows Wheeler Transform: the goal

The Burrows Wheeler Transform (BWT) is a reversible transformation that produces a permutation $bwt(w)$ of an input sequence $w$, defined over an ordered alphabet $\Sigma$, so that occurrences of a given symbol tend to occur in clusters in the output sequence.

## The Burrows Wheeler Transform

Given a word $w \in \Sigma^*$, $bwt(w)$ is a permutation of $w$, obtained as concatenation of the last letters of the lexicographically sorted list of its conjugates.

Example: $w = mathematics$

$m\ a\ t\ h\ e\ m\ a\ t\ i\ c\ s$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $a$ | $t$ | $h$ | $e$ | $m$ | $a$ | $t$ | $i$ | $c$ | $s$ | $m$ |
| 2 | $a$ | $t$ | $i$ | $c$ | $s$ | $m$ | $a$ | $t$ | $h$ | $e$ | $m$ |
| 3 | $c$ | $s$ | $m$ | $a$ | $t$ | $h$ | $e$ | $m$ | $a$ | $t$ | $i$ |
| 4 | $e$ | $m$ | $a$ | $t$ | $i$ | $c$ | $s$ | $m$ | $a$ | $t$ | $h$ |
| 5 | $h$ | $e$ | $m$ | $a$ | $t$ | $i$ | $c$ | $s$ | $m$ | $a$ | $t$ |
| 6 | $i$ | $c$ | $s$ | $m$ | $a$ | $t$ | $h$ | $e$ | $m$ | $a$ | $t$ |
| 7 | $m$ | $a$ | $t$ | $h$ | $e$ | $m$ | $a$ | $t$ | $i$ | $c$ | $s$ |
| 8 | $m$ | $a$ | $t$ | $i$ | $c$ | $s$ | $m$ | $a$ | $t$ | $h$ | $e$ |
| 9 | $s$ | $m$ | $a$ | $t$ | $h$ | $e$ | $m$ | $a$ | $t$ | $i$ | $c$ |
| 10 | $t$ | $h$ | $e$ | $m$ | $a$ | $t$ | $i$ | $c$ | $s$ | $m$ | $a$ |
| 11 | $t$ | $i$ | $c$ | $s$ | $m$ | $a$ | $t$ | $h$ | $e$ | $m$ | $a$ |

Each row of $M$ is a conjugate of $w$ in lexicographic order.
The index $I$ is the row of $M$ containing the original word.
$bwt(w) = L = mmihttsecaa$ and $I = 7$.

## The Burrows Wheeler Transform

Given a word $w \in \Sigma^*$, $bwt(w)$ is a permutation of $w$, obtained as concatenation of the last letters of the lexicographically sorted list of its conjugates.

Example: $w = mathematics$

$m\ a\ t\ h\ e\ m\ a\ t\ i\ c\ s$

$$\begin{array}{rccccccccccc}
1 & a & t & h & e & m & a & t & i & c & s & m \\
2 & a & t & i & c & s & m & a & t & h & e & m \\
3 & c & s & m & a & t & h & e & m & a & t & i \\
4 & e & m & a & t & i & c & s & m & a & t & h \\
5 & h & e & m & a & t & i & c & s & m & a & t \\
6 & i & c & s & m & a & t & h & e & m & a & t \\
7 & m & a & t & h & e & m & a & t & i & c & s \\
8 & m & a & t & i & c & s & m & a & t & h & e \\
9 & s & m & a & t & h & e & m & a & t & i & c \\
10 & t & h & e & m & a & t & i & c & s & m & a \\
11 & t & i & c & s & m & a & t & h & e & m & a
\end{array}$$

Each row of $M$ is a conjugate of $w$ in lexicographic order.
The index $I$ is the row of $M$ containing the original word.
$bwt(w) = L = mmihttsecaa$ and $I = 7$.

## The Burrows Wheeler Transform

Given a word $w \in \Sigma^*$, $bwt(w)$ is a permutation of $w$, obtained as concatenation of the last letters of the lexicographically sorted list of its conjugates.

Example: $w = mathematics$

$$\begin{array}{ccccccccccc}
m & a & t & h & e & m & a & t & i & c & s \\
a & t & h & e & m & a & t & i & c & s & m \\
t & h & e & m & a & t & i & c & s & m & a \\
h & e & m & a & t & i & c & s & m & a & t \\
e & m & a & t & i & c & s & m & a & t & h \\
m & a & t & i & c & s & m & a & t & h & e \\
a & t & i & c & s & m & a & t & h & e & m \\
t & i & c & s & m & a & t & h & e & m & a \\
i & c & s & m & a & t & h & e & m & a & t \\
c & s & m & a & t & h & e & m & a & t & i \\
s & m & a & t & h & e & m & a & t & i & c
\end{array}$$

$$\begin{array}{cccccccccccc}
1 & a & t & h & e & m & a & t & i & c & s & m \\
2 & a & t & i & c & s & m & a & t & h & e & m \\
3 & c & s & m & a & t & h & e & m & a & t & i \\
4 & e & m & a & t & i & c & s & m & a & t & h \\
5 & h & e & m & a & t & i & c & s & m & a & t \\
6 & i & c & s & m & a & t & h & e & m & a & t \\
7 & m & a & t & h & e & m & a & t & i & c & s \\
8 & m & a & t & i & c & s & m & a & t & h & e \\
9 & s & m & a & t & h & e & m & a & t & i & c \\
10 & t & h & e & m & a & t & i & c & s & m & a \\
11 & t & i & c & s & m & a & t & h & e & m & a
\end{array}$$

Each row of $M$ is a conjugate of $w$ in lexicographic order.
The index $I$ is the row of $M$ containing the original word.
$bwt(w) = L = mmihttsecaa$ and $I = 7$.

## The Burrows Wheeler Transform

Given a word $w \in \Sigma^*$, $bwt(w)$ is a permutation of $w$, obtained as concatenation of the last letters of the lexicographically sorted list of its conjugates.

Example: $w = mathematics$

$$
\begin{array}{ccccccccccc}
m & a & t & h & e & m & a & t & i & c & s \\
a & t & h & e & m & a & t & i & c & s & m \\
t & h & e & m & a & t & i & c & s & m & a \\
h & e & m & a & t & i & c & s & m & a & t \\
e & m & a & t & i & c & s & m & a & t & h \\
m & a & t & i & c & s & m & a & t & h & e \\
a & t & i & c & s & m & a & t & h & e & m \\
t & i & c & s & m & a & t & h & e & m & a \\
i & c & s & m & a & t & h & e & m & a & t \\
c & s & m & a & t & h & e & m & a & t & i \\
s & m & a & t & h & e & m & a & t & i & c
\end{array}
$$

$M$

$$
\begin{array}{cccccccccccc}
1 & a & t & h & e & m & a & t & i & c & s & m \\
2 & a & t & i & c & s & m & a & t & h & e & m \\
3 & c & s & m & a & t & h & e & m & a & t & i \\
4 & e & m & a & t & i & c & s & m & a & t & h \\
5 & h & e & m & a & t & i & c & s & m & a & t \\
6 & i & c & s & m & a & t & h & e & m & a & t \\
7 & m & a & t & h & e & m & a & t & i & c & s \\
8 & m & a & t & i & c & s & m & a & t & h & e \\
9 & s & m & a & t & h & e & m & a & t & i & c \\
10 & t & h & e & m & a & t & i & c & s & m & a \\
11 & t & i & c & s & m & a & t & h & e & m & a
\end{array}
$$

Each row of $M$ is a conjugate of $w$ in lexicographic order.

The index $I$ is the row of $M$ containing the original word.

$bwt(w) = L = mmihttsecaa$ and $I = 7$.

## The Burrows Wheeler Transform

Given a word $w \in \Sigma^*$, $bwt(w)$ is a permutation of $w$, obtained as concatenation of the last letters of the lexicographically sorted list of its conjugates.

Example: $w = mathematics$

$$\begin{array}{ccccccccccc}
m & a & t & h & e & m & a & t & i & c & s \\
a & t & h & e & m & a & t & i & c & s & m \\
t & h & e & m & a & t & i & c & s & m & a \\
h & e & m & a & t & i & c & s & m & a & t \\
e & m & a & t & i & c & s & m & a & t & h \\
m & a & t & i & c & s & m & a & t & h & e \\
a & t & i & c & s & m & a & t & h & e & m \\
t & i & c & s & m & a & t & h & e & m & a \\
i & c & s & m & a & t & h & e & m & a & t \\
c & s & m & a & t & h & e & m & a & t & i \\
s & m & a & t & h & e & m & a & t & i & c
\end{array}$$

$$M$$

$$\begin{array}{rcccccccccccc}
 & 1 & a & t & h & e & m & a & t & i & c & s & m \\
 & 2 & a & t & i & c & s & m & a & t & h & e & m \\
 & 3 & c & s & m & a & t & h & e & m & a & t & i \\
 & 4 & e & m & a & t & i & c & s & m & a & t & h \\
 & 5 & h & e & m & a & t & i & c & s & m & a & t \\
 & 6 & i & c & s & m & a & t & h & e & m & a & t \\
I \rightarrow & 7 & m & a & t & h & e & m & a & t & i & c & s \\
 & 8 & m & a & t & i & c & s & m & a & t & h & e \\
 & 9 & s & m & a & t & h & e & m & a & t & i & c \\
 & 10 & t & h & e & m & a & t & i & c & s & m & a \\
 & 11 & t & i & c & s & m & a & t & h & e & m & a
\end{array}$$

Each row of $M$ is a conjugate of $w$ in lexicographic order.
The index $I$ is the row of $M$ containing the original word.

$bwt(w) = L = mmihttsecaa$ and $I = 7$.

## The Burrows Wheeler Transform

Given a word $w \in \Sigma^*$, $bwt(w)$ is a permutation of $w$, obtained as concatenation of the last letters of the lexicographically sorted list of its conjugates.

Example: $w = mathematics$

$$\begin{array}{ccccccccccc}
m & a & t & h & e & m & a & t & i & c & s \\
a & t & h & e & m & a & t & i & c & s & m \\
t & h & e & m & a & t & i & c & s & m & a \\
h & e & m & a & t & i & c & s & m & a & t \\
e & m & a & t & i & c & s & m & a & t & h \\
m & a & t & i & c & s & m & a & t & h & e \\
a & t & i & c & s & m & a & t & h & e & m \\
t & i & c & s & m & a & t & h & e & m & a \\
i & c & s & m & a & t & h & e & m & a & t \\
c & s & m & a & t & h & e & m & a & t & i \\
s & m & a & t & h & e & m & a & t & i & c
\end{array}$$

$$M$$

$$\begin{array}{r|ccccccccccc}
 & & & & & & & & & & & \overset{L}{\downarrow} \\
1 & a & t & h & e & m & a & t & i & c & s & m \\
2 & a & t & i & c & s & m & a & t & h & e & m \\
3 & c & s & m & a & t & h & e & m & a & t & i \\
4 & e & m & a & t & i & c & s & m & a & t & h \\
5 & h & e & m & a & t & i & c & s & m & a & t \\
6 & i & c & s & m & a & t & h & e & m & a & t \\
I \rightarrow 7 & m & a & t & h & e & m & a & t & i & c & s \\
8 & m & a & t & i & c & s & m & a & t & h & e \\
9 & s & m & a & t & h & e & m & a & t & i & c \\
10 & t & h & e & m & a & t & i & c & s & m & a \\
11 & t & i & c & s & m & a & t & h & e & m & a
\end{array}$$

Each row of $M$ is a conjugate of $w$ in lexicographic order.
The index $I$ is the row of $M$ containing the original word.
$bwt(w) = L = mmihttsecaa$ and $I = 7$.

## The Burrows Wheeler Transform

Given a word $w \in \Sigma^*$, $bwt(w)$ is a permutation of $w$, obtained as concatenation of the last letters of the lexicographically sorted list of its conjugates.

Example: $w = mathematics$

$$\begin{array}{ccccccccccc}
m & a & t & h & e & m & a & t & i & c & s \\
a & t & h & e & m & a & t & i & c & s & m \\
t & h & e & m & a & t & i & c & s & m & a \\
h & e & m & a & t & i & c & s & m & a & t \\
e & m & a & t & i & c & s & m & a & t & h \\
m & a & t & i & c & s & m & a & t & h & e \\
a & t & i & c & s & m & a & t & h & e & m \\
t & i & c & s & m & a & t & h & e & m & a \\
i & c & s & m & a & t & h & e & m & a & t \\
c & s & m & a & t & h & e & m & a & t & i \\
s & m & a & t & h & e & m & a & t & i & c
\end{array}
\qquad
\begin{array}{rcccccccccccc}
 & & & & & & M & & & & & & L \\
 & & & & & & & & & & & & \downarrow \\
1 & a & t & h & e & m & a & t & i & c & s & m \\
2 & a & t & i & c & s & m & a & t & h & e & m \\
3 & c & s & m & a & t & h & e & m & a & t & i \\
4 & e & m & a & t & i & c & s & m & a & t & h \\
5 & h & e & m & a & t & i & c & s & m & a & t \\
6 & i & c & s & m & a & t & h & e & m & a & t \\
I \rightarrow 7 & m & a & t & h & e & m & a & t & i & c & s \\
8 & m & a & t & i & c & s & m & a & t & h & e \\
9 & s & m & a & t & h & e & m & a & t & i & c \\
10 & t & h & e & m & a & t & i & c & s & m & a \\
11 & t & i & c & s & m & a & t & h & e & m & a
\end{array}$$

Each row of $M$ is a conjugate of $w$ in lexicographic order.
The index $I$ is the row of $M$ containing the original word.
$bwt(w) = L = mmihttsecaa$ and $I = 7$.

## The Burrows Wheeler Transform

Given a word $w \in \Sigma^*$, $bwt(w)$ is a permutation of $w$, obtained as concatenation of the last letters of the lexicographically sorted list of its conjugates.

Example: $w = mathematics$

$$\begin{array}{ccccccccccc}
m & a & t & h & e & m & a & t & i & c & s \\
a & t & h & e & m & a & t & i & c & s & m \\
t & h & e & m & a & t & i & c & s & m & a \\
h & e & m & a & t & i & c & s & m & a & t \\
e & m & a & t & i & c & s & m & a & t & h \\
m & a & t & i & c & s & m & a & t & h & e \\
a & t & i & c & s & m & a & t & h & e & m \\
t & i & c & s & m & a & t & h & e & m & a \\
i & c & s & m & a & t & h & e & m & a & t \\
c & s & m & a & t & h & e & m & a & t & i \\
s & m & a & t & h & e & m & a & t & i & c
\end{array}$$

$$\begin{array}{r|ccccccccccc}
 & & & & & & M & & & & & \\
 & F & & & & & & & & & & L \\
 & \downarrow & & & & & & & & & & \downarrow \\
1 & a & t & h & e & m & a & t & i & c & s & m \\
2 & a & t & i & c & s & m & a & t & h & e & m \\
3 & c & s & m & a & t & h & e & m & a & t & i \\
4 & e & m & a & t & i & c & s & m & a & t & h \\
5 & h & e & m & a & t & i & c & s & m & a & t \\
6 & i & c & s & m & a & t & h & e & m & a & t \\
I \rightarrow 7 & m & a & t & h & e & m & a & t & i & c & s \\
8 & m & a & t & i & c & s & m & a & t & h & e \\
9 & s & m & a & t & h & e & m & a & t & i & c \\
10 & t & h & e & m & a & t & i & c & s & m & a \\
11 & t & i & c & s & m & a & t & h & e & m & a
\end{array}$$

Each row of $M$ is a conjugate of $w$ in lexicographic order.
The index $I$ is the row of $M$ containing the original word.
$bwt(w) = L = mmihttsecaa$ and $I = 7$.

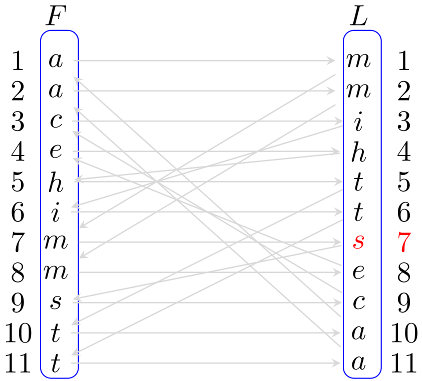
## Properties and Reversibility

Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.



$w =$ $s$

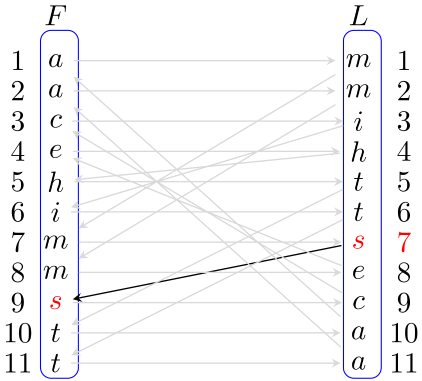
## Properties and Reversibility

Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \dots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.



$w = \qquad\qquad s$

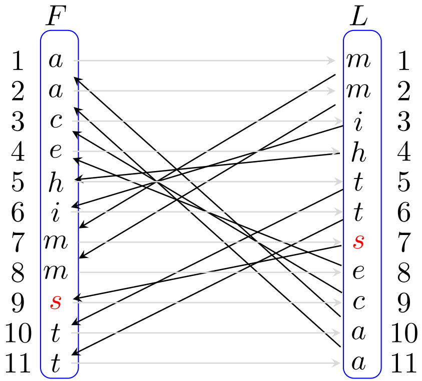
## Properties and Reversibility

Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.



$w = \qquad\qquad\qquad s$

## Properties and Reversibility

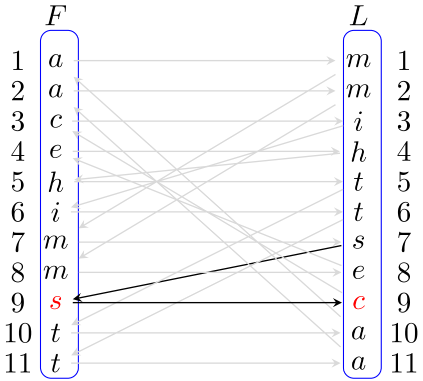
Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.

$w = \qquad cs$



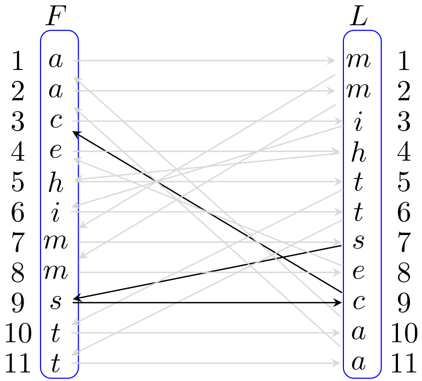
## Properties and Reversibility

Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.



$w =$ $cs$

## Properties and Reversibility

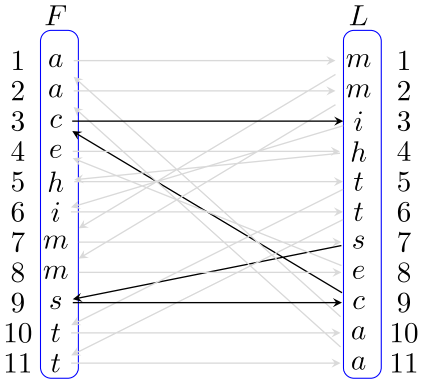
Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.

$w = \qquad ics$



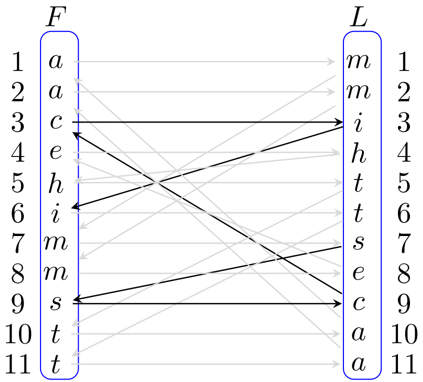
## Properties and Reversibility

Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.



$w = \qquad ics$

## Properties and Reversibility

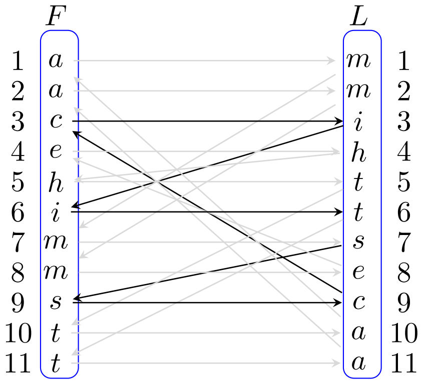
Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.

$w = \qquad tics$



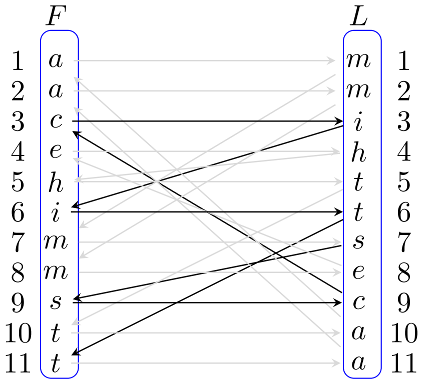
## Properties and Reversibility

Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.



$w =$ $tics$

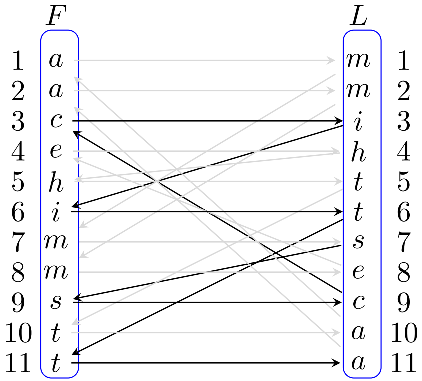
## Properties and Reversibility

Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.



$w = \qquad atics$

## Properties and Reversibility

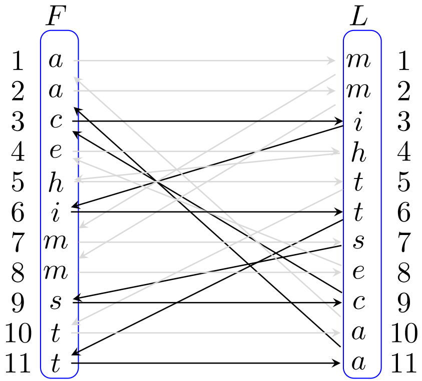
Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.

$w = \qquad atics$



## Properties and Reversibility

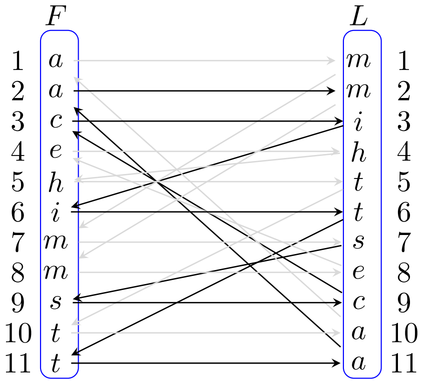
Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.

$w = \quad matics$



## Properties and Reversibility

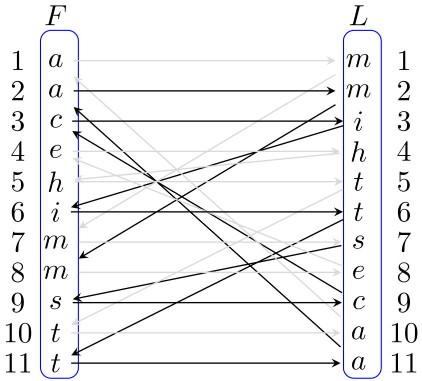
Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \dots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.

$w = \quad matics$



## Properties and Reversibility

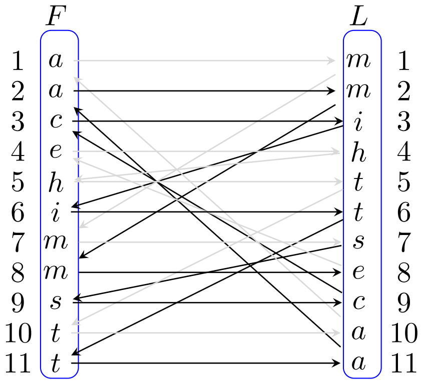
Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.

$w = \qquad ematics$



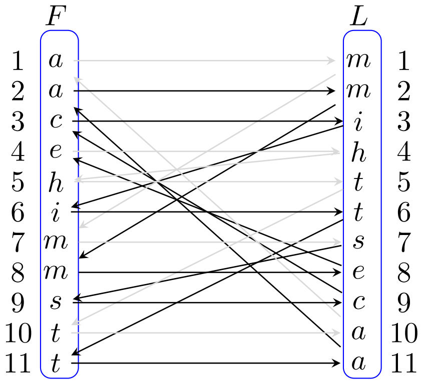
## Properties and Reversibility

Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.



$w = \quad ematics$

## Properties and Reversibility

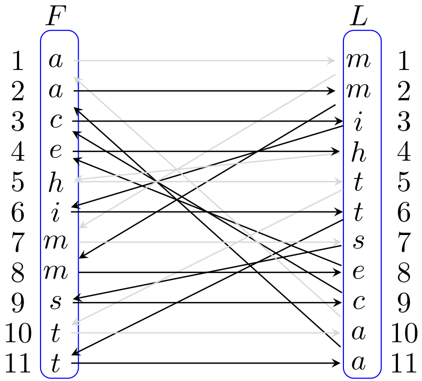
Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.

$w = \quad hematics$



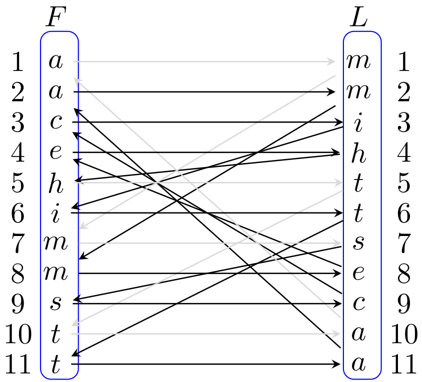
## Properties and Reversibility

Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.



$w = \quad hematics$

## Properties and Reversibility

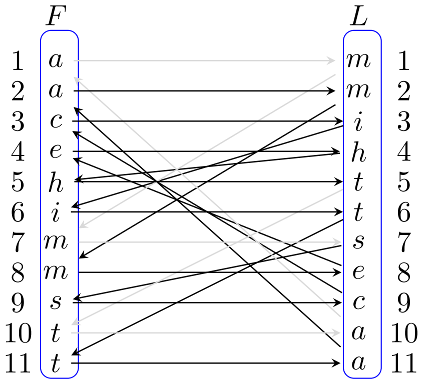
Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.

$w = \quad thematics$



## Properties and Reversibility

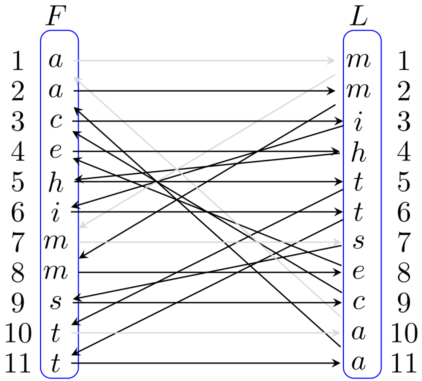
Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \dots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.

$w = \quad thematics$



## Properties and Reversibility

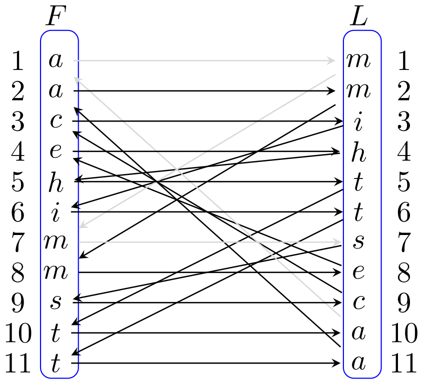
Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.

$w = \quad athematics$



## Properties and Reversibility

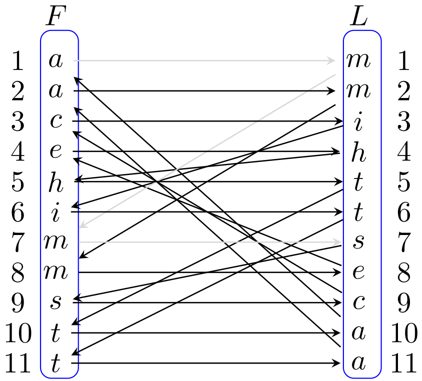
Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.

$w = \quad athematics$



## Properties and Reversibility

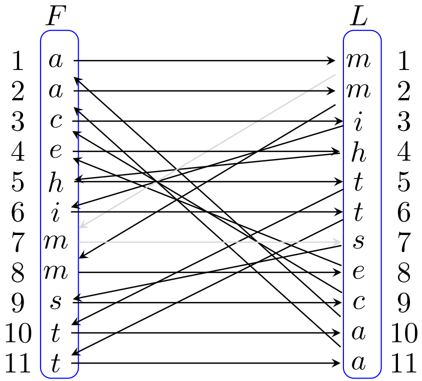
Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.

$w = mathematics$



## Properties and Reversibility

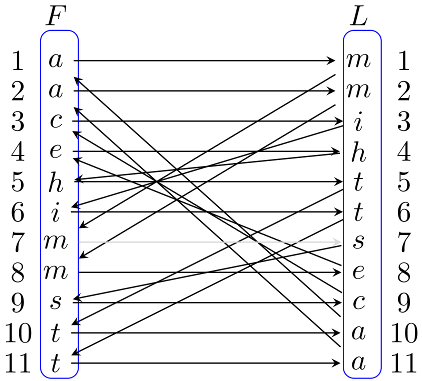
Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.

$w = mathematics$



## Properties and Reversibility

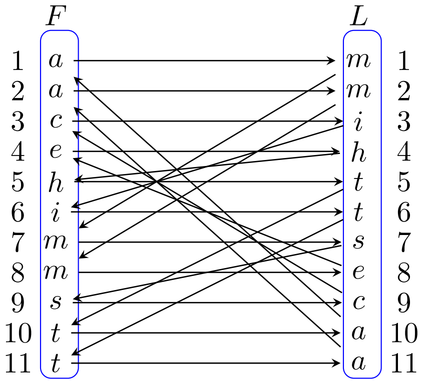
Example: $bwt(w) = L = mmihttsecaa$ and $I = 7$

- The last letter of $w$ is $L[I]$.
- For each letter $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 6 & 5 & 10 & 11 & 9 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- For all $i = 1, \ldots, n$ and $i \neq I$, the letter $L[i]$ precedes $F[i]$ in the original word.

$w = mathematics$



## Sorting of the conjugates

$$
\begin{array}{ccccccccccc}
F & \multicolumn{9}{c}{\text{Sorted Conjugates}} & L \\
\downarrow & & & & & & & & & & \downarrow \\
a & t & h & e & m & a & t & i & c & s & m \\
a & t & i & c & s & m & a & t & h & e & m \\
c & s & m & a & t & h & e & m & a & t & i \\
e & m & a & t & i & c & s & m & a & t & h \\
h & e & m & a & t & i & c & s & m & a & t \\
i & c & s & m & a & t & h & e & m & a & t \\
m & a & t & h & e & m & a & t & i & c & s \\
m & a & t & i & c & s & m & a & t & h & e \\
s & m & a & t & h & e & m & a & t & i & c \\
t & h & e & m & a & t & i & c & s & m & a \\
t & i & c & s & m & a & t & h & e & m & a
\end{array}
$$

In general, the computation of the sorting of the conjugates of a word is slow!

Sorting the suffixes of a word is a simpler problem. So, in practical applications the sorting of the suffixes is used!

## Sorting of the conjugates

$$\begin{array}{ccccccccccc}
F & & & & & \text{Sorted Conjugates} & & & & & L \\
\downarrow & & & & & & & & & & \downarrow \\
a & t & h & e & m & a & t & i & c & s & m \\
a & t & i & c & s & m & a & t & h & e & m \\
c & s & m & a & t & h & e & m & a & t & i \\
e & m & a & t & i & c & s & m & a & t & h \\
h & e & m & a & t & i & c & s & m & a & t \\
i & c & s & m & a & t & h & e & m & a & t \\
m & a & t & h & e & m & a & t & i & c & s \\
m & a & t & i & c & s & m & a & t & h & e \\
s & m & a & t & h & e & m & a & t & i & c \\
t & h & e & m & a & t & i & c & s & m & a \\
t & i & c & s & m & a & t & h & e & m & a
\end{array}$$

In general, the computation of the sorting of the conjugates of a word is slow!

Sorting the suffixes of a word is a simpler problem. So, in practical applications the sorting of the suffixes is used!

## Sorting of the conjugates

$$\begin{array}{ccccccccccc}
F & \multicolumn{9}{c}{\text{Sorted Conjugates}} & L \\
\downarrow & & & & & & & & & & \downarrow \\
a & t & h & e & m & a & t & i & c & s & m \\
a & t & i & c & s & m & a & t & h & e & m \\
c & s & m & a & t & h & e & m & a & t & i \\
e & m & a & t & i & c & s & m & a & t & h \\
h & e & m & a & t & i & c & s & m & a & t \\
i & c & s & m & a & t & h & e & m & a & t \\
m & a & t & h & e & m & a & t & i & c & s \\
m & a & t & i & c & s & m & a & t & h & e \\
s & m & a & t & h & e & m & a & t & i & c \\
t & h & e & m & a & t & i & c & s & m & a \\
t & i & c & s & m & a & t & h & e & m & a
\end{array}$$

In general, the computation of the sorting of the conjugates of a word is slow!

Sorting the suffixes of a word is a simpler problem. So, in practical applications the sorting of the suffixes is used!

## Sorting of the suffixes

To ensure the reversibility of the transform, one needs to append the symbol $\$$ at the end of the input string $w \in \Sigma^*$, where $\$ \notin \Sigma = \{a_1, a_2, \ldots, a_k\}$ and $\$ < a_1 < a_2 < \ldots < a_k$.

$bwt(w\$)$ is a permutation of $w\$$, obtained as concatenation of the letters that (circularly) precede the first symbol of the suffix in the list of its lexicographically sorted suffixes.

| $BWT$ | Sorted Suffixes |
|---|---|
| $s$ | $\$$ |
| $m$ | $a\ t\ h\ e\ m\ a\ t\ i\ c\ s\ \$$ |
| $m$ | $a\ t\ i\ c\ s\ \$$ |
| $i$ | $c\ s\ \$$ |
| $h$ | $e\ m\ a\ t\ i\ c\ s\ \$$ |
| $t$ | $h\ e\ m\ a\ t\ i\ c\ s\ \$$ |
| $t$ | $i\ c\ s\ \$$ |
| $\$$ | $m\ a\ t\ h\ e\ m\ a\ t\ i\ c\ s\ \$$ |
| $e$ | $m\ a\ t\ i\ c\ s\ \$$ |
| $c$ | $s\ \$$ |
| $a$ | $t\ h\ e\ m\ a\ t\ i\ c\ s\ \$$ |
| $a$ | $t\ i\ c\ s\ \$$ |

## The suffix array and the BWT

Given a word $w \in \Sigma^*$, with $|w| = n$:

- $SA[i]$: The starting position of the $i$th smallest suffix of $w\$$.
- $BWT[i]$: The symbol that (circularly) precedes the first symbol of the $i$th smallest suffix.

$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ v = m & a & t & h & e & m & a & t & i & c & s & \$ \end{array}$$

| $SA$ | $BWT$ | Sorted Suffixes |
|---|---|---|
| 12 | $s$ | $\$$ |
| 2 | $m$ | $a\ t\ h\ e\ m\ a\ t\ i\ c\ s\ \$$ |
| 7 | $m$ | $a\ t\ i\ c\ s\ \$$ |
| 9 | $i$ | $c\ s\ \$$ |
| 5 | $h$ | $e\ m\ a\ t\ i\ c\ s\ \$$ |
| 4 | $t$ | $h\ e\ m\ a\ t\ i\ c\ s\ \$$ |
| 9 | $t$ | $i\ c\ s\ \$$ |
| 1 | $\$$ | $m\ a\ t\ h\ e\ m\ a\ t\ i\ c\ s\ \$$ |
| 6 | $e$ | $m\ a\ t\ i\ c\ s\ \$$ |
| 11 | $c$ | $s\ \$$ |
| 3 | $a$ | $t\ h\ e\ m\ a\ t\ i\ c\ s\ \$$ |
| 8 | $a$ | $t\ i\ c\ s\ \$$ |

More efficient!

## By sorting of the suffixes

- There exist several algorithms in time linear for the construction of the SA (see survey of [*Puglisi and Smyth*, 2007]).
- There exist several algorithms in external memory for the construction either of the BWT or of the SA (for instance [*Ferragina, Gagie and Manzini*, 2012]).
- Recently, an in-place computation of the BWT has been proposed in [*Crochemore, Grossi, Kärkkäinen and Landau*, 2013], in which the space occupied by word $w$ is used to store the $bwt(w)$.

## By sorting of the suffixes

- There exist several algorithms in time linear for the construction of the SA (see survey of [*Puglisi and Smyth*, 2007]).
- There exist several algorithms in external memory for the construction either of the BWT or of the SA (for instance [*Ferragina, Gagie and Manzini*, 2012]).
- Recently, an in-place computation of the BWT has been proposed in [*Crochemore, Grossi, Kärkkäinen and Landau*, 2013], in which the space occupied by word $w$ is used to store the $bwt(w)$.

## By sorting of the suffixes

- There exist several algorithms in time linear for the construction of the SA (see survey of [*Puglisi and Smyth*, 2007]).
- There exist several algorithms in external memory for the construction either of the BWT or of the SA (for instance [*Ferragina, Gagie and Manzini*, 2012]).
- Recently, an in-place computation of the BWT has been proposed in [*Crochemore, Grossi, Kärkkäinen and Landau*, 2013], in which the space occupied by word $w$ is used to store the $bwt(w)$.

## Conjugates and Suffixes

In spite of the closeness of these variants, the sorting processes involve different sorting relations on different objects:

- lexicographic order among suffixes of a single word;
- lexicographic order among conjugates of a single word;

Note that,

- in general, the sorting of the conjugates of a word $w$ and the sorting of the suffixes of a word $w\$$ is different.
- for a Lyndon word lexicographic sorting of the suffixes and lexicographic sorting of the conjugates are equivalent. So, one can obtain in linear time the sorting of conjugates of a word by using Lyndon word (cf. *Giancarlo, Restivo and S.*, 2007).

## Conjugates and Suffixes

In spite of the closeness of these variants, the sorting processes involve different sorting relations on different objects:

- lexicographic order among suffixes of a single word;
- lexicographic order among conjugates of a single word;

Note that,

- in general, the sorting of the conjugates of a word $w$ and the sorting of the suffixes of a word $w\$$ is different.
- for a Lyndon word lexicographic sorting of the suffixes and lexicographic sorting of the conjugates are equivalent. So, one can obtain in linear time the sorting of conjugates of a word by using Lyndon word (cf. *Giancarlo, Restivo and S.*, 2007).

## Conjugates and Suffixes

In spite of the closeness of these variants, the sorting processes involve different sorting relations on different objects:

- lexicographic order among suffixes of a single word;
- lexicographic order among conjugates of a single word;

Note that,

- in general, the sorting of the conjugates of a word $w$ and the sorting of the suffixes of a word $w\$$ is different.
- for a Lyndon word lexicographic sorting of the suffixes and lexicographic sorting of the conjugates are equivalent. So, one can obtain in linear time the sorting of conjugates of a word by using Lyndon word (cf. *Giancarlo, Restivo and S.*, 2007).

## Conjugates and Suffixes

In spite of the closeness of these variants, the sorting processes involve different sorting relations on different objects:

- lexicographic order among suffixes of a single word;
- lexicographic order among conjugates of a single word;

Note that,

- in general, the sorting of the conjugates of a word $w$ and the sorting of the suffixes of a word $w\$$ is different.
- for a Lyndon word lexicographic sorting of the suffixes and lexicographic sorting of the conjugates are equivalent. So, one can obtain in linear time the sorting of conjugates of a word by using Lyndon word (cf. *Giancarlo, Restivo and S.*, 2007).

## Conjugates and Suffixes

In spite of the closeness of these variants, the sorting processes involve different sorting relations on different objects:

- lexicographic order among suffixes of a single word;
- lexicographic order among conjugates of a single word;

Note that,

- in general, the sorting of the conjugates of a word $w$ and the sorting of the suffixes of a word $w\$$ is different.
- for a Lyndon word lexicographic sorting of the suffixes and lexicographic sorting of the conjugates are equivalent. So, one can obtain in linear time the sorting of conjugates of a word by using Lyndon word (cf. *Giancarlo, Restivo and S.*, 2007).

## BWT of a word by its Lyndon Factorization

Theorem (*Chen, Fox and Lyndon*, 1958)

*Every word $w \in \Sigma^+$ has a unique factorization $w = w_1 \cdots w_s$ such that $w_1 \geq_{lex} \cdots \geq_{lex} w_s$ is a non-increasing sequence of Lyndon words.*

Let $w = abaaaabaaaaabaaaabaaaaaab$. The Lyndon factorization of $w$ is

$$ab|aaaab|aaaaabaaaab|aaaaaab$$

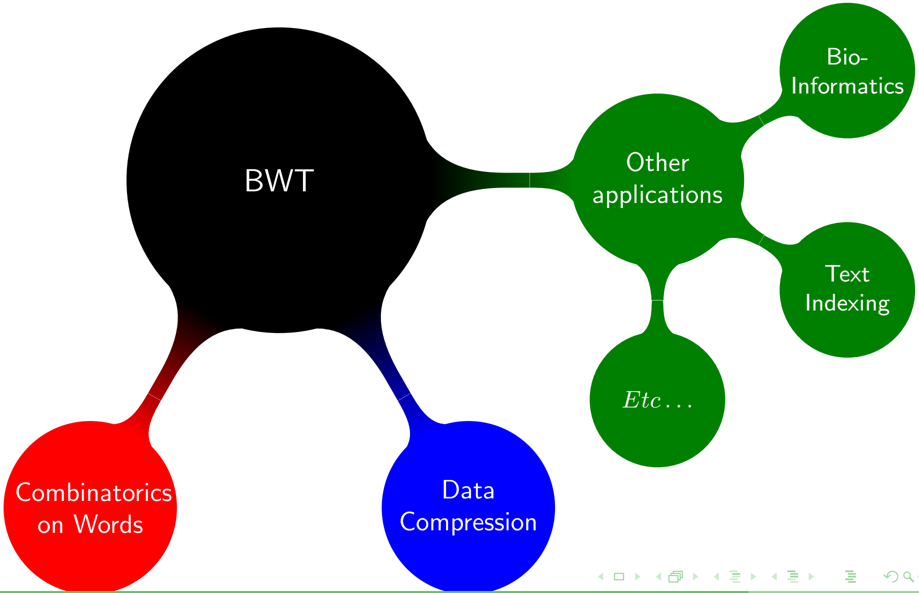
The Lyndon factorization of a given word can be computed in linear time [*Duval* 1983].

Theorem (*Mantaci, Restivo, Rosone and S.*, 2013)

*BWT of $w$ can be computed by sorting the suffixes of the Lyndon factors of $w$.*

Property: the local suffixes inside factors keep their mutual order when extended to the suffixes of the whole word.

## BWT as tool


BWT
Other applications
Bio-Informatics
Text Indexing
*Etc . . .*
Combinatorics on Words
Data Compression

## Characterization of $bwt$ images

- The word $v = caraab$ is a $bwt$ image, because $bwt(abraca) = caraab$.
- The word $u = bccaaab$ is not a $bwt$ image.
- All words $a^p b^q$ are not $bwt$ images.

**Problem**

Characterizing all the words in $\Sigma^*$ that are images by $bwt$ of some word in $\Sigma^*$.

**Theorem** (*Likhomanov and Shur*, 2011)

*A characterization of the words that are $bwt$ images is given in terms of combinatorial properties of the permutation $\pi$.*

## Characterization of $bwt$ images

- The word $v = caraab$ is a $bwt$ image, because $bwt(abraca) = caraab$.
- The word $u = bccaaab$ is not a $bwt$ image.
- All words $a^p b^q$ are not $bwt$ images.

**Problem**

Characterizing all the words in $\Sigma^*$ that are images by $bwt$ of some word in $\Sigma^*$.

**Theorem** (*Likhomanov and Shur*, 2011)

*A characterization of the words that are $bwt$ images is given in terms of combinatorial properties of the permutation $\pi$.*

## Characterization of $bwt$ images

- The word $v = caraab$ is a $bwt$ image, because $bwt(abraca) = caraab$.
- The word $u = bccaaab$ is not a $bwt$ image.
- All words $a^p b^q$ are not $bwt$ images.

**Problem**

Characterizing all the words in $\Sigma^*$ that are images by $bwt$ of some word in $\Sigma^*$.

**Theorem** (*Likhomanov and Shur*, 2011)

*A characterization of the words that are $bwt$ images is given in terms of combinatorial properties of the permutation $\pi$.*

## Characterization of $bwt$ images

- The word $v = caraab$ is a $bwt$ image, because $bwt(abraca) = caraab$.
- The word $u = bccaaab$ is not a $bwt$ image.
- All words $a^p b^q$ are not $bwt$ images.

**Problem**

Characterizing all the words in $\Sigma^*$ that are images by $bwt$ of some word in $\Sigma^*$.

**Theorem** (*Likhomanov and Shur*, 2011)

*A characterization of the words that are $bwt$ images is given in terms of combinatorial properties of the permutation $\pi$.*

## Characterization of $bwt$ images

- The word $v = caraab$ is a $bwt$ image, because $bwt(abraca) = caraab$.
- The word $u = bccaaab$ is not a $bwt$ image.
- All words $a^p b^q$ are not $bwt$ images.

**Problem**

Characterizing all the words in $\Sigma^*$ that are images by $bwt$ of some word in $\Sigma^*$.

**Theorem** (*Likhomanov and Shur*, 2011)

*A characterization of the words that are $bwt$ images is given in terms of combinatorial properties of the permutation $\pi$.*

## Perfectly clustering words

**Problem**

Characterizing the perfectly clustering words by $bwt$, that are the words that are transformed by $bwt$ into expressions in which all the occurrences of the same characters are consecutive, such as $c^i b^j a^h$ or $d^i b^j c^h a^k$.

**Theorem** (*Mantaci, Restivo and S.*, 2003)

*Given a word $u$ over the alphabet $\{a, b\}$, $bwt(u) = b^p a^q$ (with $\gcd(p, q) = 1$) if and only if $u$ is a conjugate of a standard sturmian word.*

**Standard sturmian words**

Let $d_1, d_2, \ldots, d_n, \ldots$ be, with $d_1 \geq 0$ and $d_i > 0$ for $i = 2, \ldots, n, \ldots$, the directive sequence, each finite word $s_n$, where $s_0 = b$, $s_1 = a$, and $s_{n+1} = s_n^{d_n} s_{n-1}$, for $n \geq 1$, is a standard sturmian word.

## Simple $bwt$ words

A special attention has been given to the words with *simple* $bwt$.

**Definition**

A word $w$ over an ordered alphabet $\Sigma = \{a_1, a_2, \ldots, a_k\}$ with $a_1 < a_2 < \ldots < a_k$, has a simple $bwt$, if $bwt(w)$ is of the form $a_k^{n_k} a_{k-1}^{n_{k-1}} \cdots a_1^{n_1}$, for some positive integers $n_1, n_2, \ldots, n_k$.

**Example**

The word $v = acbcbcadad$ is a simple $bwt$ word, in fact $bwt(v) = ddcccbbaaa$.

## Simple $bwt$ words: three letters alphabets

Simpson and Puglisi get a constructive characterization of the set of simple $bwt$ words in the case of three letters alphabet.

**Theorem** (*Simpson and Puglisi*, 2008, *Pak and Redlich*, 2008)

*The word $u$ is a primitive word having a simple $bwt$ on the alphabet $\Sigma = \{a_1, a_2, a_3\}$, i.e. $bwt(u) = a_3^{n_3} a_2^{n_2} a_1^{n_1}$, if and only if $(n_1, n_2, n_3)$ is a triple of integers satisfying both the conditions $\gcd(n_1, n_2, n_3) = 1$ and $\gcd(n_1 + n_2, n_2 + n_3) = 1$.*

**Open problem**

This result that involves the vector of the occurrences of the characters cannot be naturally extended for greater alphabets.
The question is still open.

## Simple $bwt$ words

**Theorem** (*Restivo and Rosone*, 2009)

*If the word $w \in \Sigma^*$ of length $n$ has a simple $bwt$ then $ww$ has $2n+1$ distinct palindromic factors.*

**Example**

The word $v = acbcbcadad$ is a simple $bwt$, $|v| = 10$, in fact $bwt(acbcbcadad) = ddcccbbaaa$. The word $vv$ contains $21$ distinct palindromic factors.

We note that the converse of this result is false, for instance $bwt(ccaaccb) = cacccba$ and $ccaaccbccaaccb$ has $15$ distinct palindromic factors.

## Simple $bwt$ words

**Theorem (*Restivo and Rosone*, 2009)**

*If the word $w \in \Sigma^*$ of length $n$ has a simple $bwt$ then $ww$ has $2n + 1$ distinct palindromic factors.*

**Example**

The word $v = acbcbcadad$ is a simple $bwt$, $|v| = 10$, in fact $bwt(acbcbcadad) = ddcccbbaaa$. The word $vv$ contains $21$ distinct palindromic factors.

We note that the converse of this result is false, for instance $bwt(ccaaccb) = cacccba$ and $ccaaccbccaaccb$ has $15$ distinct palindromic factors.

## Perfectly clustering words

In [*Ferenczi and Zamboni*, 2013] it is proved that perfectly clustering words are intrinsically related to $k$-discrete interval exchange transformations.

**Theorem**

*Perfectly clustering words are exactly those words $w \in \Sigma^*$ such that $ww$ occurs in a trajectory of a $k$-discrete interval exchange transformation, where $k$ is the size of $\Sigma$.*

## BWT, Clustering effect and Compression

$$\begin{tikzcd} v \arrow[r] & \boxed{BWT} \arrow[r] & bwt(v) \arrow[r] & \boxed{\text{Compressor}} \arrow[r] & \text{Output} \end{tikzcd}$$

- The application of the BWT produces a clustering effect.
- BWT-based compressors, in general, take advantage of such clustering effect.
- Perfect clustering corresponds to optimal performances of some BWT-based compression algorithms.

## Balanced words and compression

What kind of regularity of the input text produces a good compression ratio?

The (experimental) answer:

Balanced input text!

*It seems that the output of BWT is more compressible if the input is very close to be balanced.*

Is there a statistic that allows to decide whether a text is more compressible by using the BWT?

The (experimental) answer:

Local Entropy of the input text!

*The notion of local entropy seems to be a measure of the degree of balance of a text.*

**Conjecture**

The more balanced the input word is, the more local similarity one has after BWT, and the better the compression is.

Intuitively a word is *circularly balanced* if for any pair of different factors of the same length, the frequency of the characters of the alphabet is almost the same.

## Balanced words and compression

What kind of regularity of the input text produces a good compression ratio?
The (experimental) answer:
Balanced input text!
*It seems that the output of BWT is more compressible if the input is very close to be balanced.*

Is there a statistic that allows to decide whether a text is more compressible by using the BWT?
The (experimental) answer:
Local Entropy of the input text!
*The notion of local entropy seems to be a measure of the degree of balance of a text.*

**Conjecture**

The more balanced the input word is, the more local similarity one has after BWT, and the better the compression is.

Intuitively a word is *circularly balanced* if for any pair of different factors of the same length, the frequency of the characters of the alphabet is almost the same.

## Balanced words and compression

What kind of regularity of the input text produces a good compression ratio?
The (experimental) answer:
Balanced input text!
*It seems that the output of BWT is more compressible if the input is very close to be balanced.*

Is there a statistic that allows to decide whether a text is more compressible by using the BWT?
The (experimental) answer:
Local Entropy of the input text!
*The notion of local entropy seems to be a measure of the degree of balance of a text.*

**Conjecture**

The more balanced the input word is, the more local similarity one has after BWT, and the better the compression is.

Intuitively a word is *circularly balanced* if for any pair of different factors of the same length, the frequency of the characters of the alphabet is almost the same.

## Balanced words and compression

What kind of regularity of the input text produces a good compression ratio?
The (experimental) answer:
Balanced input text!
*It seems that the output of BWT is more compressible if the input is very close to be balanced.*

Is there a statistic that allows to decide whether a text is more compressible by using the BWT?
The (experimental) answer:
Local Entropy of the input text!
*The notion of local entropy seems to be a measure of the degree of balance of a text.*

**Conjecture**

The more balanced the input word is, the more local similarity one has after BWT, and the better the compression is.

Intuitively a word is *circularly balanced* if for any pair of different factors of the same length, the frequency of the characters of the alphabet is almost the same.

## Balanced words and compression

What kind of regularity of the input text produces a good compression ratio?
The (experimental) answer:
Balanced input text!
*It seems that the output of BWT is more compressible if the input is very close to be balanced.*

Is there a statistic that allows to decide whether a text is more compressible by using the BWT?
The (experimental) answer:
Local Entropy of the input text!
*The notion of local entropy seems to be a measure of the degree of balance of a text.*

**Conjecture**

The more balanced the input word is, the more local similarity one has after BWT, and the better the compression is.

Intuitively a word is *circularly balanced* if for any pair of different factors of the same length, the frequency of the characters of the alphabet is almost the same.

## Statistic: Local Entropy based on Distance Coding

Distance coding: for each symbol of the input word, the DC algorithm outputs the distance to the previous occurrence of the same symbol (in circular way).

**Example**

$$\begin{array}{rccccccc} v = & a & c & b & c & a & a & b \\ dc(v) = & & & & & & & \end{array}$$

Let $v = b_1 b_2 \cdots b_n$, $b_i \in A$ and $dc(v) = d_1 d_2 \cdots d_n$, where $0 \leq d_i < n$. Define the Local Entropy of $v$:

$$LE(v) = \frac{1}{n} \sum_{i=1}^{n} \log(d_i + 1)$$

Local entropy (LE) has been considered by

- *Bentley, Sleator, Tarjan and Wei*, 1986
- *Manzini*, 2001
- *Kaplan, Landau and Verbin*, 2007

## Statistic: Local Entropy based on Distance Coding

Distance coding: for each symbol of the input word, the DC algorithm outputs the distance to the previous occurrence of the same symbol (in circular way).

**Example**

$$\begin{array}{rccccccc} v = & a & c & b & c & a & a & b \\ dc(v) = & 1 & & & & & & \end{array}$$

Let $v = b_1 b_2 \cdots b_n$, $b_i \in A$ and $dc(v) = d_1 d_2 \cdots d_n$, where $0 \leq d_i < n$.
Define the Local Entropy of $v$:

$$LE(v) = \frac{1}{n} \sum_{i=1}^{n} \log(d_i + 1)$$

Local entropy (LE) has been considered by

- *Bentley, Sleator, Tarjan and Wei*, 1986
- *Manzini*, 2001
- *Kaplan, Landau and Verbin*, 2007

## Statistic: Local Entropy based on Distance Coding

Distance coding: for each symbol of the input word, the DC algorithm outputs the distance to the previous occurrence of the same symbol (in circular way).

**Example**

$$\begin{array}{rccccccc} v = & a & c & b & c & a & a & b \\ dc(v) = & 1 & 4 & & & & & \end{array}$$

Let $v = b_1 b_2 \cdots b_n$, $b_i \in A$ and $dc(v) = d_1 d_2 \cdots d_n$, where $0 \leq d_i < n$.
Define the Local Entropy of $v$:

$$LE(v) = \frac{1}{n} \sum_{i=1}^{n} \log(d_i + 1)$$

Local entropy (LE) has been considered by

- *Bentley, Sleator, Tarjan and Wei*, 1986
- *Manzini*, 2001
- *Kaplan, Landau and Verbin*, 2007

## Statistic: Local Entropy based on Distance Coding

Distance coding: for each symbol of the input word, the DC algorithm outputs the distance to the previous occurrence of the same symbol (in circular way).

**Example**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $v =$ | $a$ | $c$ | $b$ | $c$ | $a$ | $a$ | $b$ |
| $dc(v) =$ | $1$ | $4$ | $2$ | | | | |

Let $v = b_1 b_2 \cdots b_n$, $b_i \in A$ and $dc(v) = d_1 d_2 \cdots d_n$, where $0 \leq d_i < n$.
Define the Local Entropy of $v$:

$$LE(v) = \frac{1}{n} \sum_{i=1}^{n} \log(d_i + 1)$$

Local entropy (LE) has been considered by

- *Bentley, Sleator, Tarjan and Wei*, 1986
- *Manzini*, 2001
- *Kaplan, Landau and Verbin*, 2007

## Statistic: Local Entropy based on Distance Coding

Distance coding: for each symbol of the input word, the DC algorithm outputs the distance to the previous occurrence of the same symbol (in circular way).

**Example**

$$\begin{array}{rccccccc} v = & a & c & b & c & a & a & b \\ dc(v) = & 1 & 4 & 2 & 1 & & & \end{array}$$

Let $v = b_1 b_2 \cdots b_n$, $b_i \in A$ and $dc(v) = d_1 d_2 \cdots d_n$, where $0 \leq d_i < n$.
Define the Local Entropy of $v$:

$$LE(v) = \frac{1}{n} \sum_{i=1}^{n} \log(d_i + 1)$$

Local entropy (LE) has been considered by

- *Bentley, Sleator, Tarjan and Wei*, 1986
- *Manzini*, 2001
- *Kaplan, Landau and Verbin*, 2007

## Statistic: Local Entropy based on Distance Coding

Distance coding: for each symbol of the input word, the DC algorithm outputs the distance to the previous occurrence of the same symbol (in circular way).

**Example**

$$\begin{array}{rccccccc} v = & a & c & b & c & a & a & b \\ dc(v) = & 1 & 4 & 2 & 1 & 3 & & \end{array}$$

Let $v = b_1 b_2 \cdots b_n$, $b_i \in A$ and $dc(v) = d_1 d_2 \cdots d_n$, where $0 \leq d_i < n$.
Define the Local Entropy of $v$:

$$LE(v) = \frac{1}{n} \sum_{i=1}^{n} \log(d_i + 1)$$

Local entropy (LE) has been considered by

- *Bentley, Sleator, Tarjan and Wei*, 1986
- *Manzini*, 2001
- *Kaplan, Landau and Verbin*, 2007

## Statistic: Local Entropy based on Distance Coding

Distance coding: for each symbol of the input word, the DC algorithm outputs the distance to the previous occurrence of the same symbol (in circular way).

**Example**

$$\begin{array}{rccccccc} v = & a & c & b & c & a & a & b \\ dc(v) = & 1 & 4 & 2 & 1 & 3 & 0 & \end{array}$$

Let $v = b_1 b_2 \cdots b_n$, $b_i \in A$ and $dc(v) = d_1 d_2 \cdots d_n$, where $0 \leq d_i < n$.
Define the Local Entropy of $v$:

$$LE(v) = \frac{1}{n} \sum_{i=1}^{n} \log(d_i + 1)$$

Local entropy (LE) has been considered by

- *Bentley, Sleator, Tarjan and Wei*, 1986
- *Manzini*, 2001
- *Kaplan, Landau and Verbin*, 2007

## Statistic: Local Entropy based on Distance Coding

Distance coding: for each symbol of the input word, the DC algorithm outputs the distance to the previous occurrence of the same symbol (in circular way).

**Example**

$$\begin{array}{rccccccc} v = & a & c & b & c & a & a & b \\ dc(v) = & 1 & 4 & 2 & 1 & 3 & 0 & 3 \end{array}$$

Let $v = b_1 b_2 \cdots b_n$, $b_i \in A$ and $dc(v) = d_1 d_2 \cdots d_n$, where $0 \leq d_i < n$. Define the Local Entropy of $v$:

$$LE(v) = \frac{1}{n} \sum_{i=1}^{n} \log(d_i + 1)$$

Local entropy (LE) has been considered by

- *Bentley, Sleator, Tarjan and Wei*, 1986
- *Manzini*, 2001
- *Kaplan, Landau and Verbin*, 2007

## Statistic: Local Entropy based on Distance Coding

Distance coding: for each symbol of the input word, the DC algorithm outputs the distance to the previous occurrence of the same symbol (in circular way).

**Example**

$$\begin{array}{rccccccc} v = & a & c & b & c & a & a & b \\ dc(v) = & 1 & 4 & 2 & 1 & 3 & 0 & 3 \end{array}$$

Let $v = b_1 b_2 \cdots b_n$, $b_i \in A$ and $dc(v) = d_1 d_2 \cdots d_n$, where $0 \leq d_i < n$.
Define the Local Entropy of $v$:

$$LE(v) = \frac{1}{n} \sum_{i=1}^{n} \log(d_i + 1)$$

Local entropy (LE) has been considered by

- *Bentley, Sleator, Tarjan and Wei*, 1986
- *Manzini*, 2001
- *Kaplan, Landau and Verbin*, 2007

## Bounds

**Theorem (*Restivo and Rosone*, 2011)**

*For any word $v$ one has:*

- $G(v) \leq LE(v) \leq H_0(v)$
- $LE(v) = H_0(v)$ *if and only if* $v$ *is a constant gap word.*
- $LE(v) = G(v)$ *if and only if* $v$ *is a clustered word.*

*where*
$H_0(v) = \sum_{a \in A} \frac{|v|_a}{|v|} \log \frac{|v|}{|v|_a}$, *and* $G(v) = \sum_{a \in A} \frac{1}{|v|}[\log(|v| - |v|_a + 1)]$.

The notion of local entropy can be used in order to define a measure of the degree of balance of a text.

**Constant gap words**

A finite word $v$ is *constant gap* if, for each letter $a$, the distance (the number of letters) between two consecutive occurrences of $a$ is constant (in circular way).
$|v|_a$ denotes the number of occurrences of the letter $a$ in the word $v$.

## Preliminary experiments

| File name | Size | $H_0$ | Bst | Gzip | Diff % | $\delta(v)$ | $\tau(bwt(v))$ |
|---|---|---|---|---|---|---|---|
| bible | 4,047,392 | 4.343 | 796,231 | 1,191,071 | 9.755 | 0.117 | 0.233 |
| english | 52,428,800 | 4.529 | 11,533,171 | 19,672,355 | 15.524 | 0.136 | 0.238 |
| etext99 | 105,277,340 | 4.596 | 24,949,871 | 39,493,346 | 13.814 | 0.141 | 0.264 |
| english | 104,857,600 | 4.556 | 23,993,810 | 39,437,704 | 14.728 | 0.143 | 0.250 |
| dblp.xml | 52,428,800 | 5.230 | 4,871,450 | 9,034,902 | 7.941 | 0.152 | 0.093 |
| dblp.xml | 296,135,874 | 5.262 | 25,597,003 | 50,481,103 | 8.403 | 0.164 | 0.086 |
| world192 | 2,473,400 | 4.998 | 430,225 | 724,606 | 11.902 | 0.174 | 0.183 |
| rctail96 | 114,711,151 | 5.154 | 11,429,406 | 24,007,508 | 10.965 | 0.178 | 0.097 |
| sprot34.dat | 109,617,186 | 4.762 | 18,850,472 | 26,712,981 | 7.173 | 0.215 | 0.206 |
| jdk13c | 69,728,899 | 5.531 | 3,187,900 | 7,525,172 | 6.220 | 0.224 | 0.041 |
| howto | 39,886,973 | 4.857 | 8,713,851 | 12,638,334 | 9.839 | 0.231 | 0.229 |
| rfc | 116,421,901 | 4.623 | 17,565,908 | 26,712,981 | 7.857 | 0.239 | 0.163 |
| w3c2 | 104,201,579 | 5.954 | 7,021,478 | 15,159,804 | 7.810 | 0.246 | 0.058 |
| chr22.dna | 34,553,758 | 2.137 | 8,015,707 | 8,870,068 | 2.473 | 0.341 | 0.575 |
| pitches | 52,428,800 | 5.633 | 18,651,999 | 16,884,651 | -3.371 | 0.530 | 0.344 |
| pitches | 55,832,855 | 5.628 | 19,475,065 | 16,040,370 | -6.152 | 0.533 | 0.337 |

- $\delta(v)$ measures the degree of balancing of the input text $v$;
- $\tau(bwt(v))$ measures the degree of clustering of the $bwt(v)$.

The experiments show that when $\delta(v)$ is less than $0.23$, then $\tau(bwt(v))$ is less than $0.3$ and the BWT-based compressor (bst) has good performances.

Practical application: the computation of $\delta(v)$ is a fast test for the choice between bst and gzip.

## Preliminary experiments

| File name | Size | $H_0$ | Bst | Gzip | Diff % | $\delta(v)$ | $\tau(bwt(v))$ |
|---|---|---|---|---|---|---|---|
| bible | 4,047,392 | 4.343 | 796,231 | 1,191,071 | 9.755 | 0.117 | 0.233 |
| english | 52,428,800 | 4.529 | 11,533,171 | 19,672,355 | 15.524 | 0.136 | 0.238 |
| etext99 | 105,277,340 | 4.596 | 24,949,871 | 39,493,346 | 13.814 | 0.141 | 0.264 |
| english | 104,857,600 | 4.556 | 23,993,810 | 39,437,704 | 14.728 | 0.143 | 0.250 |
| dblp.xml | 52,428,800 | 5.230 | 4,871,450 | 9,034,902 | 7.941 | 0.152 | 0.093 |
| dblp.xml | 296,135,874 | 5.262 | 25,597,003 | 50,481,103 | 8.403 | 0.164 | 0.086 |
| world192 | 2,473,400 | 4.998 | 430,225 | 724,606 | 11.902 | 0.174 | 0.183 |
| rctail96 | 114,711,151 | 5.154 | 11,429,406 | 24,007,508 | 10.965 | 0.178 | 0.097 |
| sprot34.dat | 109,617,186 | 4.762 | 18,850,472 | 26,712,981 | 7.173 | 0.215 | 0.206 |
| jdk13c | 69,728,899 | 5.531 | 3,187,900 | 7,525,172 | 6.220 | 0.224 | 0.041 |
| howto | 39,886,973 | 4.857 | 8,713,851 | 12,638,334 | 9.839 | 0.231 | 0.229 |
| rfc | 116,421,901 | 4.623 | 17,565,908 | 26,712,981 | 7.857 | 0.239 | 0.163 |
| w3c2 | 104,201,579 | 5.954 | 7,021,478 | 15,159,804 | 7.810 | 0.246 | 0.058 |
| chr22.dna | 34,553,758 | 2.137 | 8,015,707 | 8,870,068 | 2.473 | 0.341 | 0.575 |
| pitches | 52,428,800 | 5.633 | 18,651,999 | 16,884,651 | -3.371 | 0.530 | 0.344 |
| pitches | 55,832,855 | 5.628 | 19,475,065 | 16,040,370 | -6.152 | 0.533 | 0.337 |

- $\delta(v)$ measures the degree of balancing of the input text $v$;
- $\tau(bwt(v))$ measures the degree of clustering of the $bwt(v)$.

The experiments show that when $\delta(v)$ is less than $0.23$, then $\tau(bwt(v))$ is less than $0.3$ and the BWT-based compressor (bst) has good performances.

Practical application: the computation of $\delta(v)$ is a fast test for the choice between bst and gzip.

## Preliminary experiments

| File name | Size | $H_0$ | Bst | Gzip | Diff % | $\delta(v)$ | $\tau(bwt(v))$ |
|---|---|---|---|---|---|---|---|
| bible | 4,047,392 | 4.343 | 796,231 | 1,191,071 | 9.755 | 0.117 | 0.233 |
| english | 52,428,800 | 4.529 | 11,533,171 | 19,672,355 | 15.524 | 0.136 | 0.238 |
| etext99 | 105,277,340 | 4.596 | 24,949,871 | 39,493,346 | 13.814 | 0.141 | 0.264 |
| english | 104,857,600 | 4.556 | 23,993,810 | 39,437,704 | 14.728 | 0.143 | 0.250 |
| dblp.xml | 52,428,800 | 5.230 | 4,871,450 | 9,034,902 | 7.941 | 0.152 | 0.093 |
| dblp.xml | 296,135,874 | 5.262 | 25,597,003 | 50,481,103 | 8.403 | 0.164 | 0.086 |
| world192 | 2,473,400 | 4.998 | 430,225 | 724,606 | 11.902 | 0.174 | 0.183 |
| rctail96 | 114,711,151 | 5.154 | 11,429,406 | 24,007,508 | 10.965 | 0.178 | 0.097 |
| sprot34.dat | 109,617,186 | 4.762 | 18,850,472 | 26,712,981 | 7.173 | 0.215 | 0.206 |
| jdk13c | 69,728,899 | 5.531 | 3,187,900 | 7,525,172 | 6.220 | 0.224 | 0.041 |
| howto | 39,886,973 | 4.857 | 8,713,851 | 12,638,334 | 9.839 | 0.231 | 0.229 |
| rfc | 116,421,901 | 4.623 | 17,565,908 | 26,712,981 | 7.857 | 0.239 | 0.163 |
| w3c2 | 104,201,579 | 5.954 | 7,021,478 | 15,159,804 | 7.810 | 0.246 | 0.058 |
| chr22.dna | 34,553,758 | 2.137 | 8,015,707 | 8,870,068 | 2.473 | 0.341 | 0.575 |
| pitches | 52,428,800 | 5.633 | 18,651,999 | 16,884,651 | -3.371 | 0.530 | 0.344 |
| pitches | 55,832,855 | 5.628 | 19,475,065 | 16,040,370 | -6.152 | 0.533 | 0.337 |

- $\delta(v)$ measures the degree of balancing of the input text $v$;
- $\tau(bwt(v))$ measures the degree of clustering of the $bwt(v)$.

The experiments show that when $\delta(v)$ is less than $0.23$, then $\tau(bwt(v))$ is less than $0.3$ and the BWT-based compressor (bst) has good performances.

Practical application: the computation of $\delta(v)$ is a fast test for the choice between bst and gzip.

## Multiset of words

**Problem**

Is it possible to extend the notion of BWT to a multiset of words?

## The Extended Burrows-Wheeler Transform [*Mantaci, Restivo, Rosone and S.*, 2005]

Given two words $u, v \in \Sigma^*$ we define the following order relation:

$$u \preceq_\omega v \Longleftrightarrow u^\omega <_{lex} v^\omega$$

where $u^\omega = uuuuu\cdots$ and $v^\omega = vvvvv\cdots$.

Given a set of words , $S = \{w_1, w_2, \ldots w_m\}$ with $w_1, w_2, \ldots w_m \in \Sigma^*$ the EBWT is a transformation that produces a word obtained by sorting according to the $\preceq_\omega$ order the conjugates of the words in $S$ and by taking the concatenation of the last letters of the sorted list.

## The Extended Burrows-Wheeler Transform

Consider the set $S = \{abac, bca, cbab, cba\}$.

- Sort all the conjugates of the words in $S$ by the $\preceq_\omega$ order relation;
- Consider the list of the sorted words and take the word $L$ obtained by concatenating the last letter of each word;
- Take the set $\mathcal{I}$ containing the positions of the words corresponding to the ones in $S$;

| | | | |
|---|---|---|---|
| $a\ b\ a\ c\ a\ b \cdots$ | | 1 | $a\ b\ a\ \mathbf{c}$ |
| $a\ b\ c\ a\ b\ c \cdots$ | | 2 | $a\ b\ \mathbf{c}$ |
| $a\ b\ c\ b\ a\ b \cdots$ | | 3 | $a\ b\ c\ \mathbf{b}$ |
| $a\ c\ a\ b\ a\ c \cdots$ | | 4 | $a\ c\ a\ \mathbf{b}$ |
| $a\ c\ b\ a\ c\ b \cdots$ | | 5 | $a\ c\ \mathbf{b}$ |
| $b\ a\ b\ c\ b\ a \cdots$ | | 6 | $b\ a\ b\ \mathbf{c}$ |
| $b\ a\ c\ a\ b\ a \cdots$ | | 7 | $b\ a\ c\ \mathbf{a}$ |
| $b\ a\ c\ b\ a\ c \cdots$ | $\Longrightarrow$ | 8 | $b\ a\ \mathbf{c}$ |
| $b\ c\ a\ b\ c\ a \cdots$ | | 9 | $b\ c\ \mathbf{a}$ |
| $b\ c\ b\ a\ b\ c \cdots$ | | 10 | $b\ c\ b\ \mathbf{a}$ |
| $c\ a\ b\ a\ c\ a \cdots$ | | 11 | $c\ a\ b\ \mathbf{a}$ |
| $c\ a\ b\ c\ a\ b \cdots$ | | 12 | $c\ a\ \mathbf{b}$ |
| $c\ b\ a\ b\ c\ b \cdots$ | | 13 | $c\ b\ a\ \mathbf{b}$ |
| $c\ b\ a\ c\ b\ a \cdots$ | | 14 | $c\ b\ \mathbf{a}$ |

The output of $ebwt(S)$ is the couple $(L, \mathcal{I})$ where $L = ccbbbcacaaabba$ and $\mathcal{I} = \{1, 9, 13, 14\}$

## The Extended Burrows-Wheeler Transform

Consider the set $S = \{abac, bca, cbab, cba\}$.

- Sort all the conjugates of the words in $S$ by the $\preceq_\omega$ order relation;
- Consider the list of the sorted words and take the word $L$ obtained by concatenating the last letter of each word;
- Take the set $\mathcal{I}$ containing the positions of the words corresponding to the ones in $S$;

| | | | |
|---|---|---|---|
| $a\ b\ a\ c\ a\ b \cdots$ | | 1 | $a\ b\ a\ \mathbf{c}$ |
| $a\ b\ c\ a\ b\ c \cdots$ | | 2 | $a\ b\ \mathbf{c}$ |
| $a\ b\ c\ b\ a\ b \cdots$ | | 3 | $a\ b\ c\ \mathbf{b}$ |
| $a\ c\ a\ b\ a\ c \cdots$ | | 4 | $a\ c\ a\ \mathbf{b}$ |
| $a\ c\ b\ a\ c\ b \cdots$ | | 5 | $a\ c\ \mathbf{b}$ |
| $b\ a\ b\ c\ b\ a \cdots$ | | 6 | $b\ a\ b\ \mathbf{c}$ |
| $b\ a\ c\ a\ b\ a \cdots$ | | 7 | $b\ a\ c\ \mathbf{a}$ |
| $b\ a\ c\ b\ a\ c \cdots$ | $\Longrightarrow$ | 8 | $b\ a\ \mathbf{c}$ |
| $b\ c\ a\ b\ c\ a \cdots$ | | 9 | $b\ c\ \mathbf{a}$ |
| $b\ c\ b\ a\ b\ c \cdots$ | | 10 | $b\ c\ b\ \mathbf{a}$ |
| $c\ a\ b\ a\ c\ a \cdots$ | | 11 | $c\ a\ b\ \mathbf{a}$ |
| $c\ a\ b\ c\ a\ b \cdots$ | | 12 | $c\ a\ \mathbf{b}$ |
| $c\ b\ a\ b\ c\ b \cdots$ | | 13 | $c\ b\ a\ \mathbf{b}$ |
| $c\ b\ a\ c\ b\ a \cdots$ | | 14 | $c\ b\ \mathbf{a}$ |

The output of $ebwt(S)$ is the couple $(L, \mathcal{I})$ where $L = ccbbbcacaaabba$ and $\mathcal{I} = \{1, 9, 13, 14\}$

## The Extended Burrows-Wheeler Transform

Consider the set $S = \{abac, bca, cbab, cba\}$.

- Sort all the conjugates of the words in $S$ by the $\preceq_\omega$ order relation;
- Consider the list of the sorted words and take the word $L$ obtained by concatenating the last letter of each word;
- Take the set $\mathcal{I}$ containing the positions of the words corresponding to the ones in $S$;

$$
\begin{array}{l}
a\ b\ a\ c\ a\ b \cdots \\
a\ b\ c\ a\ b\ c \cdots \\
a\ b\ c\ b\ a\ b \cdots \\
a\ c\ a\ b\ a\ c \cdots \\
a\ c\ b\ a\ c\ b \cdots \\
b\ a\ b\ c\ b\ a \cdots \\
b\ a\ c\ a\ b\ a \cdots \\
b\ a\ c\ b\ a\ c \cdots \\
b\ c\ a\ b\ c\ a \cdots \\
b\ c\ b\ a\ b\ c \cdots \\
c\ a\ b\ a\ c\ a \cdots \\
c\ a\ b\ c\ a\ b \cdots \\
c\ b\ a\ b\ c\ b \cdots \\
c\ b\ a\ c\ b\ a \cdots
\end{array}
\quad \Longrightarrow \quad
\begin{array}{rl}
1 & a\ b\ a\ \mathbf{c} \\
2 & a\ b\ \mathbf{c} \\
3 & a\ b\ c\ \mathbf{b} \\
4 & a\ c\ a\ \mathbf{b} \\
5 & a\ c\ \mathbf{b} \\
6 & b\ a\ b\ \mathbf{c} \\
7 & b\ a\ c\ \mathbf{a} \\
8 & b\ a\ \mathbf{c} \\
9 & b\ c\ \mathbf{a} \\
10 & b\ c\ b\ \mathbf{a} \\
11 & c\ a\ b\ \mathbf{a} \\
12 & c\ a\ \mathbf{b} \\
13 & c\ b\ a\ \mathbf{b} \\
14 & c\ b\ \mathbf{a}
\end{array}
$$

The output of $ebwt(S)$ is the couple $(L, \mathcal{I})$ where $L = ccbbbcacaaabba$ and $\mathcal{I} = \{1, 9, 13, 14\}$

## The Extended Burrows-Wheeler Transform

Consider the set $S = \{abac, bca, cbab, cba\}$.

- Sort all the conjugates of the words in $S$ by the $\preceq_\omega$ order relation;
- Consider the list of the sorted words and take the word $L$ obtained by concatenating the last letter of each word;
- Take the set $\mathcal{I}$ containing the positions of the words corresponding to the ones in $S$;

$$\begin{array}{l} a\ b\ a\ c\ a\ b \cdots \\ a\ b\ c\ a\ b\ c \cdots \\ a\ b\ c\ b\ a\ b \cdots \\ a\ c\ a\ b\ a\ c \cdots \\ a\ c\ b\ a\ c\ b \cdots \\ b\ a\ b\ c\ b\ a \cdots \\ b\ a\ c\ a\ b\ a \cdots \\ b\ a\ c\ b\ a\ c \cdots \\ b\ c\ a\ b\ c\ a \cdots \\ b\ c\ b\ a\ b\ c \cdots \\ c\ a\ b\ a\ c\ a \cdots \\ c\ a\ b\ c\ a\ b \cdots \\ c\ b\ a\ b\ c\ b \cdots \\ c\ b\ a\ c\ b\ a \cdots \end{array} \Longrightarrow \begin{array}{rl} 1 & a\ b\ a\ \mathbf{c} \\ 2 & a\ b\ \mathbf{c} \\ 3 & a\ b\ c\ \mathbf{b} \\ 4 & a\ c\ a\ \mathbf{b} \\ 5 & a\ c\ \mathbf{b} \\ 6 & b\ a\ b\ \mathbf{c} \\ 7 & b\ a\ c\ \mathbf{a} \\ 8 & b\ a\ \mathbf{c} \\ 9 & b\ c\ \mathbf{a} \\ 10 & b\ c\ b\ \mathbf{a} \\ 11 & c\ a\ b\ \mathbf{a} \\ 12 & c\ a\ \mathbf{b} \\ 13 & c\ b\ a\ \mathbf{b} \\ 14 & c\ b\ \mathbf{a} \end{array}$$

The output of $ebwt(S)$ is the couple $(L, \mathcal{I})$ where $L = ccbbbcacaaabba$ and $\mathcal{I} = \{1, 9, 13, 14\}$

## The Extended Burrows-Wheeler Transform

Consider the set $S = \{abac, bca, cbab, cba\}$.

- Sort all the conjugates of the words in $S$ by the $\preceq_\omega$ order relation;
- Consider the list of the sorted words and take the word $L$ obtained by concatenating the last letter of each word;
- Take the set $\mathcal{I}$ containing the positions of the words corresponding to the ones in $S$;

| | | | |
|---|---|---|---|
| $abacab\cdots$ | | 1 | $aba\mathbf{c}$ |
| $abcabc\cdots$ | | 2 | $ab\mathbf{c}$ |
| $abcbab\cdots$ | | 3 | $abc\mathbf{b}$ |
| $acabac\cdots$ | | 4 | $aca\mathbf{b}$ |
| $acbacb\cdots$ | | 5 | $ac\mathbf{b}$ |
| $babcba\cdots$ | | 6 | $bab\mathbf{c}$ |
| $bacaba\cdots$ | $\Longrightarrow$ | 7 | $bac\mathbf{a}$ |
| $bacbac\cdots$ | | 8 | $ba\mathbf{c}$ |
| $bcabca\cdots$ | | 9 | $bc\mathbf{a}$ |
| $bcbabc\cdots$ | | 10 | $bcb\mathbf{a}$ |
| $cabaca\cdots$ | | 11 | $cab\mathbf{a}$ |
| $cabcab\cdots$ | | 12 | $ca\mathbf{b}$ |
| $cbabcb\cdots$ | | 13 | $cba\mathbf{b}$ |
| $cbacba\cdots$ | | 14 | $cb\mathbf{a}$ |

The output of $ebwt(S)$ is the couple $(L, \mathcal{I})$ where $L = ccbbbcacaaabba$ and $\mathcal{I} = \{1, 9, 13, 14\}$.

## Properties and Reversibility

Example: $L = ccbbbcacaaabba$ and $\mathcal{I} = \{1, 9, 13, 14\}$.

- The last character of each word $w_j$ is $L[I_j]$;
- For each character $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;
- In any row $i \neq \mathcal{I}$, the character $F[i]$ follows $L[i]$ in a word in $S$.

| | | | | |
|---|---|---|---|---|
| 1 | $a$ | $b$ | $a$ | **c** |
| 2 | $a$ | $b$ | **c** | |
| 3 | $a$ | $b$ | $c$ | **b** |
| 4 | $a$ | $c$ | $a$ | **b** |
| 5 | $a$ | $c$ | **b** | |
| 6 | $b$ | $a$ | $b$ | **c** |
| 7 | $b$ | $a$ | $c$ | **a** |
| 8 | $b$ | $a$ | **c** | |
| 9 | $b$ | $c$ | **a** | |
| 10 | $b$ | $c$ | $b$ | **a** |
| 11 | $c$ | $a$ | $b$ | **a** |
| 12 | $c$ | $a$ | **b** | |
| 13 | $c$ | $b$ | $a$ | **b** |
| 14 | $c$ | $b$ | **a** | |

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 11 & 12 & 6 & 7 & 8 & 13 & 1 & 14 & 2 & 3 & 4 & 9 & 10 & 5 \end{pmatrix} = (\,11\ 4\ 7\ 1\,)(\,9\ 2\ 12\,)(\,13\ 10\ 3\ 6\,)(\,14\ 5\ 8\,)$$

So, we can recover each word of the multiset

$$S = \{abac, bca, cbab, cba\}.$$

## Properties and Reversibility

Example: $L = ccbbbcacaaabba$ and $\mathcal{I} = \{1, 9, 13, 14\}$.

| | | | | |
|---|---|---|---|---|
| 1 | $a$ | $b$ | $a$ | **c** |
| 2 | $a$ | $b$ | **c** | |
| 3 | $a$ | $b$ | $c$ | **b** |
| 4 | $a$ | $c$ | $a$ | **b** |
| 5 | $a$ | $c$ | **b** | |
| 6 | $b$ | $a$ | $b$ | **c** |
| 7 | $b$ | $a$ | $c$ | **a** |
| 8 | $b$ | $a$ | **c** | |
| 9 | $b$ | $c$ | **a** | |
| 10 | $b$ | $c$ | $b$ | **a** |
| 11 | $c$ | $a$ | $b$ | **a** |
| 12 | $c$ | $a$ | **b** | |
| 13 | $c$ | $b$ | $a$ | **b** |
| 14 | $c$ | $b$ | **a** | |

- The last character of each word $w_j$ is $L[I_j]$;
- For each character $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;
- In any row $i \neq \mathcal{I}$, the character $F[i]$ follows $L[i]$ in a word in $S$.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 11 & 12 & 6 & 7 & 8 & 13 & 1 & 14 & 2 & 3 & 4 & 9 & 10 & 5 \end{pmatrix} = (\,11\ 4\ 7\ 1\,)(\,9\ 2\ 12\,)(\,13\ 10\ 3\ 6\,)(\,14\ 5\ 8\,)$$

So, we can recover each word of the multiset

$$S = \{abac, bca, cbab, cba\}.$$

## Properties and Reversibility

Example: $L = ccbbbcacaaabba$ and $\mathcal{I} = \{1, 9, 13, 14\}$.

- The last character of each word $w_j$ is $L[I_j]$;
- For each character $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;
- In any row $i \neq \mathcal{I}$, the character $F[i]$ follows $L[i]$ in a word in $S$.

| | |
|---|---|
| 1 | $a\ b\ a$ **c** |
| 2 | $a\ b$ **c** |
| 3 | $a\ b\ c$ **b** |
| 4 | $a\ c\ a$ **b** |
| 5 | $a\ c$ **b** |
| 6 | $b\ a\ b$ **c** |
| 7 | $b\ a\ c$ **a** |
| 8 | $b\ a$ **c** |
| 9 | $b\ c$ **a** |
| 10 | $b\ c\ b$ **a** |
| 11 | $c\ a\ b$ **a** |
| 12 | $c\ a$ **b** |
| 13 | $c\ b\ a$ **b** |
| 14 | $c\ b$ **a** |

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 11 & 12 & 6 & 7 & 8 & 13 & 1 & 14 & 2 & 3 & 4 & 9 & 10 & 5 \end{pmatrix} = (\,11\ 4\ 7\ 1\,)(\,9\ 2\ 12\,)(\,13\ 10\ 3\ 6\,)(\,14\ 5\ 8\,)$$

So, we can recover each word of the multiset

$$S = \{abac, bca, cbab, cba\}.$$

## Properties and Reversibility

Example: $L = ccbbbcacaaabba$ and $\mathcal{I} = \{1, 9, 13, 14\}$.

- The last character of each word $w_j$ is $L[I_j]$;
- For each character $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;
- In any row $i \neq \mathcal{I}$, the character $F[i]$ follows $L[i]$ in a word in $S$.

| | |
|---|---|
| 1 | $a\ b\ a\ \mathbf{c}$ |
| 2 | $a\ b\ \mathbf{c}$ |
| 3 | $a\ b\ c\ \mathbf{b}$ |
| 4 | $a\ c\ a\ \mathbf{b}$ |
| 5 | $a\ c\ \mathbf{b}$ |
| 6 | $b\ a\ b\ \mathbf{c}$ |
| 7 | $b\ a\ c\ \mathbf{a}$ |
| 8 | $b\ a\ \mathbf{c}$ |
| 9 | $b\ c\ \mathbf{a}$ |
| 10 | $b\ c\ b\ \mathbf{a}$ |
| 11 | $c\ a\ b\ \mathbf{a}$ |
| 12 | $c\ a\ \mathbf{b}$ |
| 13 | $c\ b\ a\ \mathbf{b}$ |
| 14 | $c\ b\ \mathbf{a}$ |

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 11 & 12 & 6 & 7 & 8 & 13 & 1 & 14 & 2 & 3 & 4 & 9 & 10 & 5 \end{pmatrix} = (\ 11\ 4\ 7\ 1\ )(\ 9\ 2\ 12\ )(\ 13\ 10\ 3\ 6\ )(\ 14\ 5\ 8\ )$$

So, we can recover each word of the multiset

$$S = \{abac, bca, cbab, cba\}.$$

## Properties and Reversibility

Example: $L = ccbbbcacaaabba$ and $\mathcal{I} = \{1, 9, 13, 14\}$.

- The last character of each word $w_j$ is $L[I_j]$;
- For each character $z$, the $i$-th occurrence of $z$ in $L$ corresponds to the $i$-th occurrence of $z$ in $F$;
- In any row $i \neq \mathcal{I}$, the character $F[i]$ follows $L[i]$ in a word in $S$.

| | | | | |
|---|---|---|---|---|
| 1 | $a$ | $b$ | $a$ | **c** |
| 2 | $a$ | $b$ | **c** | |
| 3 | $a$ | $b$ | $c$ | **b** |
| 4 | $a$ | $c$ | $a$ | **b** |
| 5 | $a$ | $c$ | **b** | |
| 6 | $b$ | $a$ | $b$ | **c** |
| 7 | $b$ | $a$ | $c$ | **a** |
| 8 | $b$ | $a$ | **c** | |
| 9 | $b$ | $c$ | **a** | |
| 10 | $b$ | $c$ | $b$ | **a** |
| 11 | $c$ | $a$ | $b$ | **a** |
| 12 | $c$ | $a$ | **b** | |
| 13 | $c$ | $b$ | $a$ | **b** |
| 14 | $c$ | $b$ | **a** | |

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 11 & 12 & 6 & 7 & 8 & 13 & 1 & 14 & 2 & 3 & 4 & 9 & 10 & 5 \end{pmatrix} = (\ 11\ 4\ 7\ 1\ )(\ 9\ 2\ 12\ )(\ 13\ 10\ 3\ 6\ )(\ 14\ 5\ 8\ )$$

So, we can recover each word of the multiset

$$S = \{abac, bca, cbab, cba\}.$$

## EBWT as bijection

Let $M$ be the family of multisets of conjugacy classes of primitive words of $\Sigma^*$. Then, if we don't care about the indices $EBWT : M \longrightarrow \Sigma^*$

- The transformation $EBWT$ is injective.
- The $EBWT$ is surjective. For each word $u \in \Sigma^*$, there exists a multiset $S \in M$ such that $EBWT(S) = u$. For instance, $EBWT(ab, abcac) = (bccaaab)$.

**Theorem** (*Gessel and Reutenauer*, 1993. *Mantaci, Restivo, Rosone and S.*, 2007)

*There exists a bijection between $\Sigma^*$ and the family of multisets of conjugacy classes of primitive words in $\Sigma^*$.*

## EBWT as bijection

Let $M$ be the family of multisets of conjugacy classes of primitive words of $\Sigma^*$. Then, if we don't care about the indices $EBWT : M \longrightarrow \Sigma^*$

- The transformation $EBWT$ is injective.
- The $EBWT$ is surjective. For each word $u \in \Sigma^*$, there exists a multiset $S \in M$ such that $EBWT(S) = u$. For instance, $EBWT(ab, abcac) = (bccaaab)$.

Theorem (*Gessel and Reutenauer*, 1993. *Mantaci, Restivo, Rosone and S.*, 2007)

*There exists a bijection between $\Sigma^*$ and the family of multisets of conjugacy classes of primitive words in $\Sigma^*$.*

## EBWT as bijection

Let $M$ be the family of multisets of conjugacy classes of primitive words of $\Sigma^*$. Then, if we don't care about the indices $EBWT : M \longrightarrow \Sigma^*$

- The transformation $EBWT$ is injective.
- The $EBWT$ is surjective. For each word $u \in \Sigma^*$, there exists a multiset $S \in M$ such that $EBWT(S) = u$. For instance, $EBWT(ab, abcac) = (bccaaab)$.

Theorem (*Gessel and Reutenauer*, 1993. *Mantaci, Restivo, Rosone and S.*, 2007)

*There exists a bijection between $\Sigma^*$ and the family of multisets of conjugacy classes of primitive words in $\Sigma^*$.*

## Sorting of the conjugates

| | |
|---|---|
| 1 | $a\ b\ a\ \mathbf{c}$ |
| 2 | $a\ b\ \mathbf{c}$ |
| 3 | $a\ b\ c\ \mathbf{b}$ |
| 4 | $a\ c\ a\ \mathbf{b}$ |
| 5 | $a\ c\ \mathbf{b}$ |
| 6 | $b\ a\ b\ \mathbf{c}$ |
| 7 | $b\ a\ c\ \mathbf{a}$ |
| 8 | $b\ a\ \mathbf{c}$ |
| 9 | $b\ c\ \mathbf{a}$ |
| 10 | $b\ c\ b\ \mathbf{a}$ |
| 11 | $c\ a\ b\ \mathbf{a}$ |
| 12 | $c\ a\ \mathbf{b}$ |
| 13 | $c\ b\ a\ \mathbf{b}$ |
| 14 | $c\ b\ \mathbf{a}$ |

Sorting the conjugates of each word of the multiset in according to $\preceq_\omega$ order is the bottleneck of the algorithm.

- *Mantaci, Restivo, Rosone and S.*: An extension of the Burrows-Wheeler Transform, 2007. Use a periodicity theorem to reduce the number of comparisons.
- *Hon, Ku, Lu, Shah and Thankachan*: Efficient Algorithm for Circular Burrows-Wheeler Transform, 2011. A $O(n \log n)$ algorithm is provided, where $n$ denotes the total length of the words in $S$.

## Efficient strategy by sorting the suffixes

To ensure the reversibility of the transform, one needs to append a different end-marker at the end of each input string of the multiset.

Let $S'$ be the set of the strings of $S$ included the end-markers. $ebwt(S')$ is a permutation of $S'$, obtained as concatenation of the letters that (circularly) precede the first symbol of the suffix in the list of its lexicographically sorted suffixes of $S'$.

- *Bauer, Cox and Rosone*: Lightweight algorithms for constructing and inverting the BWT of string collections. 2013.

## Efficient strategy by sorting the suffixes

To ensure the reversibility of the transform, one needs to append a different end-marker at the end of each input string of the multiset.

Let $S'$ be the set of the strings of $S$ included the end-markers. $ebwt(S')$ is a permutation of $S'$, obtained as concatenation of the letters that (circularly) precede the first symbol of the suffix in the list of its lexicographically sorted suffixes of $S'$.

- *Bauer, Cox and Rosone*: Lightweight algorithms for constructing and inverting the BWT of string collections. 2013.

## Different sorting relations

In order to compute EBWT of a multiset of words, different sorting processes can be involved.

- Lexicographic order among suffixes of a multiset of words;
- $\preceq_\omega$ order among conjugates of a multiset of words.

A study of the combinatorial aspects that connect these two sorting can be found in *Bonomo, Mantaci, Restivo, Rosone and S.* 2013.
An important role is played by the notion of Lyndon word.

## Different sorting relations

In order to compute EBWT of a multiset of words, different sorting processes can be involved.

- Lexicographic order among suffixes of a multiset of words;
- $\preceq_\omega$ order among conjugates of a multiset of words.

A study of the combinatorial aspects that connect these two sorting can be found in *Bonomo, Mantaci, Restivo, Rosone and S.* 2013.
An important role is played by the notion of Lyndon word.

## Different sorting relations

In order to compute EBWT of a multiset of words, different sorting processes can be involved.

- Lexicographic order among suffixes of a multiset of words;
- $\preceq_\omega$ order among conjugates of a multiset of words.

A study of the combinatorial aspects that connect these two sorting can be found in *Bonomo, Mantaci, Restivo, Rosone and S.* 2013.
An important role is played by the notion of Lyndon word.

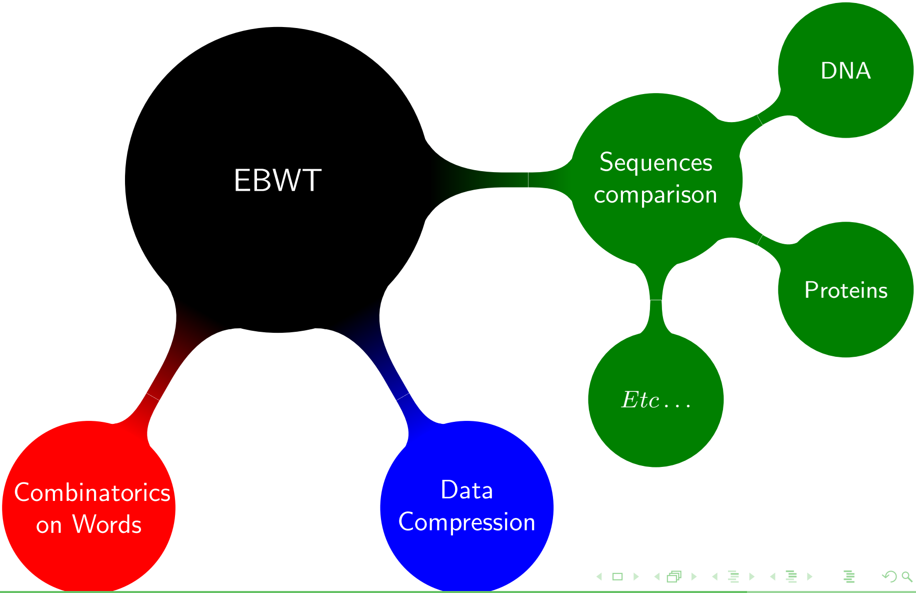
## Different sorting relations

In order to compute EBWT of a multiset of words, different sorting processes can be involved.

- Lexicographic order among suffixes of a multiset of words;
- $\preceq_\omega$ order among conjugates of a multiset of words.

A study of the combinatorial aspects that connect these two sorting can be found in *Bonomo, Mantaci, Restivo, Rosone and S.* 2013.
An important role is played by the notion of Lyndon word.

## EBWT as tool


EBWT
Sequences comparison
DNA
Proteins
*Etc...*
Combinatorics on Words
Data Compression

## Sequences comparison

The transformation EBWT is used in order to define an alignment-free method for comparing sequences.

The comparison method based on transformation EBWT measures how similar $u$ and $v$ are, by taking into account how much their conjugates are mixed.

## Different possible formalizations of distance measures

For instance,

- by computing the number of the alternations in the sequence of colors [*Mantaci, Restivo, Rosone and S.*, 2007].
- by using different partitioning of the colored output of the $EBWT$ and by finally counting the difference of frequencies of colors into each block of the partition [*Mantaci, Restivo, Rosone and S.*, 2008].

For instance, let $S = \{u = ababccb, v = ababccc\}$, the output colored is $bcbbcaaaacccbb$.

| Sorted conjugates | EBWT |
|---|---|
| $ababccb$ | $b$ |
| $ababccc$ | $c$ |
| $abccbab$ | $b$ |
| $abcccab$ | $b$ |
| $bababcc$ | $c$ |
| $babccba$ | $a$ |
| $babccca$ | $a$ |
| $bccbaba$ | $a$ |
| $bcccaba$ | $a$ |
| $cababcc$ | $c$ |
| $cbababc$ | $c$ |
| $ccababc$ | $c$ |
| $ccbabab$ | $b$ |
| $cccabab$ | $b$ |

## Different possible formalizations of distance measures

For instance,

- by computing the number of the alternations in the sequence of colors [*Mantaci, Restivo, Rosone and S.*, 2007].
- by using different partitioning of the colored output of the $EBWT$ and by finally counting the difference of frequencies of colors into each block of the partition [*Mantaci, Restivo, Rosone and S.*, 2008].

For instance, let $S = \{u = ababccb, v = ababccc\}$, the output colored is $bcbbcaaaacccbb$.

| Sorted conjugates | EBWT | $\delta(u,v)$ |
|---|---|---|
| $ababccb$ | $b$ | $0$ |
| $ababccc$ | $c$ | $0$ |
| $abccbab$ | $b$ | $0$ |
| $abcccab$ | $b$ | $0$ |
| $bababcc$ | $c$ | |
| $babccba$ | $a$ | $1$ |
| $babccca$ | $a$ | $0$ |
| $bccbaba$ | $a$ | $0$ |
| $bcccaba$ | $a$ | |
| $cababcc$ | $c$ | $1$ |
| $cbababc$ | $c$ | $0$ |
| $ccababc$ | $c$ | $0$ |
| $ccbabab$ | $b$ | $0$ |
| $cccabab$ | $b$ | $0$ |

For instance, the number of the alternations in the sequence of colors, computed as:

$$\delta(u,v) = \sum_{\substack{i=1,\\ n_i \neq 0}}^{k} (n_i - 1),$$

is equal to $2$.

## Different possible formalizations of distance measures

For instance,

- by computing the number of the alternations in the sequence of colors [*Mantaci, Restivo, Rosone and S.*, 2007].
- by using different partitioning of the colored output of the $EBWT$ and by finally counting the difference of frequencies of colors into each block of the partition [*Mantaci, Restivo, Rosone and S.*, 2008].

For instance, let $S = \{u = ababccb, v = ababccc\}$, the output colored is $bcbbcaaaacccbb$.

| Sorted conjugates | EBWT |
|---|---|
| $ababccb$ | **b** |
| $ababccc$ | $c$ |
| $abccbab$ | **b** |
| $abcccab$ | **b** |
| $bababcc$ | $c$ |
| $babccba$ | **a** |
| $babccca$ | **a** |
| $bccbaba$ | **a** |
| $bcccaba$ | **a** |
| $cababcc$ | $c$ |
| $cbababc$ | $c$ |
| $ccababc$ | $c$ |
| $ccbabab$ | **b** |
| $cccabab$ | **b** |

## Different possible formalizations of distance measures

For instance,

- by computing the number of the alternations in the sequence of colors [*Mantaci, Restivo, Rosone and S.*, 2007].
- by using different partitioning of the colored output of the $EBWT$ and by finally counting the difference of frequencies of colors into each block of the partition [*Mantaci, Restivo, Rosone and S.*, 2008].

For instance, let $S = \{u = ababccb, v = ababccc\}$, the output colored is $bcbbcaaaacccbb$.

| Sorted conjugates | EBWT | $\varrho(u,v)$ |
|---|---|---|
| $ababccb$ | b | 1 |
| $ababccc$ | $c$ | 1 |
| $abccbab$ | b | 0 |
| $abcccab$ | b | |
| $bababcc$ | $c$ | 1 |
| $babccba$ | a | 0 |
| $babccca$ | a | |
| $bccbaba$ | a | |
| $bcccaba$ | a | |
| $cababcc$ | $c$ | 1 |
| $cbababc$ | $c$ | |
| $ccababc$ | $c$ | |
| $ccbabab$ | b | 0 |
| $cccabab$ | b | |

$$\varrho(u,v) = \sum_{i=1}^{k} |c_i(u) - c_i(v)| = 4$$

## Applications to biological sequences

Such distances have been successfully used in several biological datasets, as for instance mitochondrial DNA genomes, expressed sequence tags and proteins

- *Mantaci, Restivo, Rosone and S.*: A New Combinatorial Approach to Sequence Comparison, 2008.
- *Yang, Chang, Zhang and Wang*: Use of the Burrows-Wheeler similarity distribution to the comparison of the proteins, 2010.
- *Yang, Zhang and Wang*: The Burrows-Wheeler similarity distribution between biological sequences based on Burrows-Wheeler transform, 2010.
- *Cox, Jakobi, Rosone and Schulz-Trieglaff*: Comparing DNA Sequence Collections by Direct Comparison of Compressed Text Indexes, 2012.
- *Ng, Ho, and Phon-Amnuaisuk*: A hybrid distance measure for clustering expressed sequence tags originating from the same gene family, 2012.

## Massive Datasets

The EBWT has been used as a preprocessing for compression of big sets of $m$ texts.

- *Cox, Bauer, Jakobi and Rosone*: Large-scale compression of genomic sequence databases with the Burrows-Wheeler transform, 2012.
- *Janin, Rosone and Cox*: Adaptive reference-free compression of sequence quality scores, 2013.

The method is also used for the computation of the LCP of very large collections of sequences.

- *Cox, Bauer, Rosone and S.*: Lightweight LCP Construction for Next-Generation Sequencing Datasets, 2012. The code is available as part of the BEETL Library - http://beetl.github.com/BEETL/

## Further works and open problems

- Use EBWT to define lightweight data structures for indexing big datasets of sequences;
- Study of the clustering effect of the EBWT from Combinatorics on Words viewpoint.

Thanks for your attention!