The data structure at the core of large-scale search engines is the inverted index, which is essentially a collection of sorted integer sequences called inverted lists. Because of the many documents indexed by such engines and stringent performance requirements imposed by the heavy load of queries, the inverted index stores billions of integers that must be searched efficiently. In this scenario, index compression is essential because it leads to a better exploitation of the computer memory hierarchy for faster query processing and, at the same time, allows reducing the number of storage machines.

The aim of this article is twofold: first, surveying the encoding algorithms suitable for inverted index compression and, second, characterizing the performance of the inverted index through experimentation.
representation: feed faster memory levels with more data and, hence, speed up the query processing algorithms. As a result, the design of techniques that compress the index effectively while maintaining a noticeable decoding speed is a well-studied problem, that dates back to more than 50 years ago, and still a very active field of research. In fact, many representation for inverted lists are known, each exposing a different space/time trade-off: refer to the timeline shown in Table 1 and references therein.

**Organization.** We classify the techniques in a hierarchical manner by identifying three main classes. The first class comprehends algorithms that compress a single integer (Section 2). The second class covers algorithms that compress many integers together, namely an inverted list (Section 3). The third class describes a family of algorithms that represent many lists together, i.e., the whole inverted index (Section 4). In our intention, this first part of the survey is devoted to readers who are new to the field of integer compression.

This hierarchical division is natural and intuitive. Firstly, it reflects the flexibility of the algorithms, given that algorithms in a higher class can be used to represent the unit of compression of the algorithms in a lower class, but not the other way round. For example, an algorithm that compresses a single integer at a time (first class) can obviously be used to represent a list of integers (unit of compression of the second class) by just concatenating the encoding of each single integer in

---

**Table 1. Timeline of techniques.**

<table>
<thead>
<tr>
<th>Year</th>
<th>Techniques</th>
</tr>
</thead>
<tbody>
<tr>
<td>1949</td>
<td>Shannon-Fano [29, 79]</td>
</tr>
<tr>
<td>1952</td>
<td>Huffman [38]</td>
</tr>
<tr>
<td>1963</td>
<td>Arithmetic [1] ¹</td>
</tr>
<tr>
<td>1966</td>
<td>Golomb [35]</td>
</tr>
<tr>
<td>1971</td>
<td>Elias-Fano [27, 30]; Rice [74]</td>
</tr>
<tr>
<td>1972</td>
<td>Variable-Byte and Nibble [84]</td>
</tr>
<tr>
<td>1975</td>
<td>Gamma and Delta [28]</td>
</tr>
<tr>
<td>1987</td>
<td>Fibonacci-based [4]</td>
</tr>
<tr>
<td>1996</td>
<td>Interpolative [55, 56]</td>
</tr>
<tr>
<td>1998</td>
<td>Frame-of-Reference (For) [34]</td>
</tr>
<tr>
<td>2003</td>
<td>SC-dense [9]</td>
</tr>
<tr>
<td>2004</td>
<td>Zeta [6, 7]</td>
</tr>
<tr>
<td>2005</td>
<td>Simple-9, Relative-10 and Carryover-12 [2]</td>
</tr>
<tr>
<td>2006</td>
<td>PForDelta [96]</td>
</tr>
<tr>
<td>2008</td>
<td>Simple-16 [94]</td>
</tr>
<tr>
<td>2009</td>
<td>ANS [24]; Varint-GB [20]; Opt-PFor [93]</td>
</tr>
<tr>
<td>2010</td>
<td>Simple8b [3]; VSE [81]; SIMD-Gamma [78]</td>
</tr>
<tr>
<td>2011</td>
<td>Varint-G8IU [82]</td>
</tr>
<tr>
<td>2013</td>
<td>DAC [10]; Quasi-Succinct [90]</td>
</tr>
<tr>
<td>2014</td>
<td>partitioned Elias-Fano [61]; QMX [86]; Roaring [14, 43, 45]</td>
</tr>
<tr>
<td>2015</td>
<td>BP32, SIMD-BP128 and SIMD-FastPFor [42]; Masked-VByte [72]</td>
</tr>
<tr>
<td>2017</td>
<td>clustered Elias-Fano [68]</td>
</tr>
<tr>
<td>2018</td>
<td>Stream-VByte [44]; ANS-based [53, 54]; Opt-VByte [71]; SIMD-Delta [87]</td>
</tr>
<tr>
<td>2019</td>
<td>DINT [67]; Slicing [65]</td>
</tr>
</tbody>
</table>

---

¹Actually before 1963. See Note 1 on page 61 in the book by Abramson [1].
the list. Secondly, it shows that less flexibility can be exploited to enlarge the “visibility” of the algorithms. For example, algorithms in the second class seek opportunities for better compression by looking for regions of similar integers in the list. Instead, algorithms in the third class seek for such regularities across many lists (or even the whole index).

After the description of the techniques, we provide pointers to further readings (Section 5).

The last part of the article is dedicated to the experimental comparison of the paradigms used to represent the inverted lists (Section 6). In our intention, this part is targeted to more experienced readers who are already familiar with the research field and the practical implementation of the techniques. We release the full experimental suite at https://github.com/jermp/2i_bench, in the hope of spurring further research in the field.

We conclude the survey by summarizing experimental lessons and discussing some future research directions (Section 7).

2 INTEGER CODES

The algorithms we consider in this section compress a single integer. The most classical solution is to assign the integer a uniquely-decodable (or self-delimiting) variable-length code, in order to decode correctly from left to right. Clearly, the aim of such algorithms is to assign the smallest codewords as possible.

In this context, a key notion is represented by the Kraft–McMillan inequality that gives a necessary and sufficient condition for the existence of a uniquely-decodable code for a given set of codeword lengths, where no codeword is a prefix of another one (prefix-free condition). More formally, if \( \ell_x \) is the codeword length of the integer \( x \leq u \) for some universe size \( u \), then it must hold

\[
\sum_{x=0}^{u} 2^{-\ell_x} \leq 1
\]

for the code to be uniquely-decodable. However, it should be clear that no code is optimal for all possible integer distributions. According to Shannon [79], the ideal codeword length \( L(x) \) of an integer \( x \) should be \( \log_2(1/\mathbb{P}(x)) \) bits long, where \( \mathbb{P}(x) \) is the probability of occurrence of the integer \( x \) in the input. By solving the equation

\[
L(x) = \log_2 \frac{1}{\mathbb{P}(x)}
\]

with respect to \( \mathbb{P}(x) \) we derive the distribution for which the considered integer code is optimal.

Sometimes it could also be useful to implement a sub-optimal code if it allows faster decoding for a given application, or to (re)organize the output bit stream in a way that is more suitable for special hardware instructions such as SIMD (Single-Instruction-Multiple-Data) [16]. SIMD is a computer organization that exploits the independence of multiple data objects to execute a single instruction on these objects simultaneously. Specifically, a single instruction (e.g., arithmetic or boolean) is executed for every element of a vector, that is a large(r) machine register containing, e.g., 128 or 256 bits of data, that packs multiple elements together. SIMD is widely used to accelerate the execution of many data-intensive tasks, and (usually) an optimizing compiler is able to automatically “vectorize” code snippets to make them run faster.

Many algorithms that we describe in this article exploit SIMD instructions.

2.1 Unary and Binary

Perhaps the most primitive form of integer representation is unary coding, that is the integer \( x \geq 0 \) is represented by a run of \( x \) zeroes plus a final one: \( U(x) = 0^x 1 \). The presence of the final 1 bit
Table 2. The integers 0..8 as represented with unary, γ, δ and δ’ codes respectively.

<table>
<thead>
<tr>
<th>integer</th>
<th>unary</th>
<th>γ</th>
<th>δ</th>
<th>δ’</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>−</td>
<td>1−0</td>
<td>−</td>
</tr>
<tr>
<td>1</td>
<td>0−1</td>
<td>1</td>
<td>1−1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>00−1</td>
<td>0−10</td>
<td>010−10</td>
<td>010−0</td>
</tr>
<tr>
<td>3</td>
<td>000−1</td>
<td>0−11</td>
<td>010−11</td>
<td>010−1</td>
</tr>
<tr>
<td>4</td>
<td>0000−1</td>
<td>0−100</td>
<td>011−100</td>
<td>011−00</td>
</tr>
<tr>
<td>5</td>
<td>00000−1</td>
<td>0−101</td>
<td>011−101</td>
<td>011−01</td>
</tr>
<tr>
<td>6</td>
<td>000000−1</td>
<td>0−110</td>
<td>011−110</td>
<td>011−10</td>
</tr>
<tr>
<td>7</td>
<td>0000000−1</td>
<td>0−111</td>
<td>011−111</td>
<td>011−11</td>
</tr>
<tr>
<td>8</td>
<td>00000000−1</td>
<td>00−1000</td>
<td>00100−1000</td>
<td>00100−000</td>
</tr>
</tbody>
</table>

implies that the encoding is uniquely-decodable: keep reading zeroes until we hit a 1, then stop and report the number of read zeroes. Because the length in bits of $U(x)$ is exactly $x + 1$, this encoding strongly favours small integers. For example, we can represent 2 with just 3 bits (001) but we would need 1001 bits to represent the integer 1000, whereas the simple binary representation of 1000, $B(1000)$, takes just $\lceil \log_2 1000 \rceil = 10$ bits. The problem of binary coding is that it is not uniquely-decodable unless we know the number of bits that we dedicate to the representation of each integer in the coded stream. Indeed, if the integers in a stream are drawn from a universe $u$ bounded by $2^k$ for some $k > 0$, then each integer can be represented with $\lceil \log_2 (u + 1) \rceil \leq k$ bits. Many compressors that we present in Section 3 exploit this simple strategy.

Since the length of the unary representation of $x$ is $|U(x)| = x + 1$, by solving $\log_2 (1/P(x)) = x + 1$ we derive that the unary code is optimal whenever $P(x) = 2^{-(x+1)}$. Concerning the binary code, we have that $|B(x)| = \lceil \log_2 (x + 1) \rceil$, thus we derive the distribution $P(x) = \frac{1}{x+1}$.

2.2 Gamma and Delta

The two codes we now describe were introduced by Elias [28] and are called universal because the length of these codes is $O(\log x)$ bits for every integer $x$, thus a constant factor away from the optimal binary representation $B(x)$ of length $\lceil \log_2 (x + 1) \rceil$ bits. Additionally, they are uniquely-decodable.

Given an integer $x > 0$, the gamma code of $x$ is $\gamma(x) = \lceil |B(x)| \rceil B(x)$. Therefore, $|\gamma(x)| = 2\lceil \log_2 (x + 1) \rceil − 1$ bits and $P(x) \approx \frac{1}{x^2}$. Decoding $\gamma(x)$ is simple: first count the number of zeroes up to the one, say there are $n$ of these, then read the following $n + 1$ bits and interpret them as $x$. For example, the integer 113 is represented as $\gamma(113) = 0000000 − 1110001$, because $B(113) = 1110001$ is 7 bits long and thus we prefix it with $7 − 1 = 6$ zeroes.

The key inefficiency of the gamma code lies in the use of the unary code for the representation of $|B(x)| − 1$, which may become very large for big integers. To overcome this limitation, the delta code replaces the run $\lceil |B(x)| \rceil$ with $\gamma(|B(x)|)$, therefore we have $\delta(x) = \gamma(|B(x)|)B(x)$. Notice that, since we are representing with $\gamma$ the quantity $|B(x)|$ which is guaranteed to be greater than zero, the delta code can also represent the value zero. The number of bits required by $\delta(x)$ is $|\gamma(|B(x)|)| + |B(x)|$, which is $2\lceil \log (|B(x)| + 1) \rceil + |B(x)| − 1$. We derive a corresponding distribution of $P(x) \approx 1/(2x(\log x)^2)$. The decoding of δ codes follows automatically from the one of γ codes. Again, the integer 113 is represented as $\delta(113) = 00 − 111 − 1110001$. The last part, 1110001, is $B(113)$ whose binary length is 7. Therefore we prefix $B(113)$ by $\gamma(7) = 00 − 111$, which is the first part of the encoding.
Table 2 shows the first 9 codes assigned to the integers 0..8 by γ and δ. The dash symbol highlights the distinction between different parts of the codes, e.g., it separates the binary part on the right of γ from its prefix of zeroes, and it is not included in the final coded representation.

In case we have integers greater than 0, it is possible to omit a bit in the corresponding δ codes. In fact, observe that for the integer \( x > 0 \) the minimal binary length is \( \lceil \log_2 x \rceil - 1 \). In fact, given \( b \) bits, the values that can be represented having a minimal binary length of \( b \) bits are the integers in \([2^{b-1}, 2^b - 1]\) for which their most significant bit is always 1 and, thus, can be omitted. Therefore, our previous example for \( \delta(113) \) becomes 00 – 111 – 110001. In fact, having specified a binary length of 7 bits, the most significant bit is redundant and we only write the last 6 bits (least significant) of 113.

We refer to this variant as \( \delta' \) in Table 2. In case the value 0 is the most frequent, it could be useful to use just 1 bit rather than 2 as assigned by the δ code. In such case, we can set \( \delta(x) = \delta'(x + 1) \), \( x \geq 0 \). During decoding we subtract 1 to the read number. This technique is sometimes referred to as biased-δ and it is possible to generate the analogous biased-γ code in the obvious way [77].

In order to decode gamma codes faster on modern processors, a simple variant of gamma is proposed by Schlegel et al. [78] and called k-gamma. Groups of \( k \) integers are encoded together, with \( k = 2, 3 \) or 4, using the same number of bits. Thus, instead of recording the unary length for each integer, only the length of the largest integer in the group is written. This leads to a higher compression ratio if the \( k \) integers are close to each other, namely they require the same codeword length. On the other hand, if the largest integers in the group requires more bits than the other integers, this encoding is wasteful compared to the traditional gamma. However, decoding is faster: once the binary length has been read, a group of \( k \) integers is decoded in parallel using SIMD instructions.

Similarly, Trotman and Lilly [87] introduced a SIMD version of delta codes. A 512-bit payload is broken down into its 16 \( \times \) 32-bit integers and the base-2 magnitude of the largest integer is written using gamma coding as a selector (writing the selector in unary code gives 16-gamma). Although not as fast as k-gamma, the representation is faster to decode compared to the original scalar decoding algorithm.

### 2.3 Golomb-Rice

In 1966 Golomb introduced a parametric code that is a hybrid between unary and binary codes [35]. Having fixed a positive integer \( b \), the Golomb code consists in two pieces: the quotient \( q = \lfloor x/b \rfloor \) and the remainder \( r = x - q \times b \). The quotient is encoded in unary, i.e., with \( U(q) \). Let \( c \) be \( \lfloor \log_2 b \rfloor \). Then the first \( 2^c - b \) values of \( r \) are encoded using \( c - 1 \) bits; the remaining ones with \( c \) bits. Clearly, the closer \( b \) is to the value of \( x \) the smaller the value of \( q \); this implies a better compression and faster decoding speed.

Golomb first observed that if \( n \) integers are drawn at random from a universe of size \( u \), than the gaps between the integers follow a geometric distribution \( \mathbb{P}(x) = p(1-p)^{x-1} \) with parameter \( p = n/u \) being the probability to find an integer \( x \) among the ones selected. It is now clear that the optimal value for \( b \) depends on \( p \) and it can be shown that this value is the integer closest to \(-1/\log_2(1-p)\), i.e., the value that satisfies \((1-p)^b \approx \frac{1}{2}\). Doing the math we derive \( b = 0.69 \times \frac{1}{p} \), which is a good approximation of the optimal value and can be used to define a Golomb code with parameter \( b \). This code is optimal for the geometric distribution \( \mathbb{P}(x) = p(1-p)^{x-1} \).

Lastly, Gallager and Van Voorhis [33] showed that the optimal value for \( b \) can be computed as

\[
b = \left\lceil -\frac{\log_2(2-p)}{\log_2(1-p)} \right\rceil.
\]
The minimal binary code of an integer \( x \) is \( h_x \) bits if \( \alpha \) is represented with Fibonacci-based codes. In bold we mark the final control bit.

<table>
<thead>
<tr>
<th>integer</th>
<th>( F_1 = 1 )</th>
<th>( F_2 = 2 )</th>
<th>( F_3 = 3 )</th>
<th>( F_4 = 5 )</th>
<th>( F_5 = 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>[1]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>[1]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>[1]</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>[1]</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>[1]</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>[1]</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>[1]</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The Rice code \([74]\) is actually a special case of the Golomb code for which \( b \) is set to \( 2^k \), for some \( k > 0 \) (sometimes also referred to as Golomb-Rice code). Let \( R_k(x) \) be the Rice code of \( x \) with parameter \( k > 0 \). In this case the remainder \( r \) is always encoded using \( k \) bits. Therefore, \( |R_k(x)| = |U([x/2^k])| + k \) bits. The Rice code is optimal for an exponential distribution, such as the Laplace distribution.

Given the parameter \( k \) and the constant \( 2^k \) that is computed ahead, decoding Rice codes is simple too: count the number of zeroes up to the one, say there are \( q \) of these, then multiply \( 2^k \) by \( q \) and finally add the remainder, by reading the next \( k \) bits. As an example, consider \( R_5(113) = 0001 – 10001 \). Since \( [113/2^5] = 3 \), we first write \( U(3) = 0^3.1 \). Then we write in binary the remainder \( 113 – 3 \times 2^5 = 17 \), using 5 bits. If we would have adopted \( k = 6 \), then \( R_6(113) = 01 – 110001 \). Notice that we are saving one bit with respect to \( R_5(113) \) since \( 2^6 \) is closer to 113 than \( 2^5 \). In fact, the quotient in this case is 1 rather than 3.

The computation of the optimal parameter \( k \) for the Rice code is very complicated and out of the scope of this work: we point the interested reader to the technical report by Kiely \([41]\).

### 2.4 Zeta

Boldi and Vigna \([6, 7]\) introduced the family of zeta codes that is optimal for integers distributed according to a power law with small exponent \( a \) (e.g., less than 2), that is \( P(x) = \frac{1}{\zeta(a) x^a} \), where \( \zeta \) denotes the Reimann’s zeta function.

Having fixed a positive integer \( k \) called the shrinking factor, a positive integer \( x \) in the interval \( [2^{hk}, 2^{(h+1)k} - 1] \) is represented by the zeta code \( \zeta_k \) by writing \( h \) in unary, followed by a minimal binary code assigned to \( x - 2^{hk} \) in the shrunk interval \([0, 2^{(h+1)k} - 2^{hk} - 1] \). The first \( 2^{hk} \) integers in the range \([0, 2^{(h+1)k} - 2^{hk} - 1] \) are encoded using \( k \times (h + 1) - 1 \) bits, the others using \( k(h + 1) \) bits. The minimal binary code of an integer \( x \) in the interval \([0, z - 1] \) is the \( x \)-th binary word of \( \ell - 1 \) bits if \( x < 2^\ell - z \) or is the \((x - z - 2^\ell)\)-th binary word of \( \ell \) bits otherwise, with \( \ell = \lfloor \log z \rfloor \).

For example, \( \zeta_2(10) = 011010 \), \( \zeta_3(10) = 0100010 \) and \( \zeta_4(10) = 11010 \). Also, notice that \( \zeta_4 \) coincides with Elias’ \( \gamma \). A more involved example is the one for 113: \( \zeta_5(113) = 01 – 0001110001 \). In this case, \( h = 1 \) because \( 2^{hk} \) with \( h = 1 \) and \( k = 5 \) is the largest power of two among \( \{2^{1x5}, 2^{2x5}, 2^{3x5}, \ldots \} \) that is less than 113. Therefore, we write \( h = 1 \) in unary. Then, since \( 113 - 2^3 = 81 \) is not less than the left extreme \( 2^3 \), we write 113 in binary using \( k \times (h + 1) = 10 \) bits for \( h = 1 \) and \( k = 5 \).

### 2.5 Fibonacci

Fraenkel and Klein \([32]\) introduced in 1985 a class of codes based on Fibonacci numbers \([59]\) and later generalized by Apostolico and Fraenkel \([4]\). The encoding is a direct consequence of the
Zeckendorf’s theorem: every positive integer can be uniquely represented as the sum of some, non-adjacent, Fibonacci numbers. Let $F_n = F_{n-1} + F_{n-2}$ define the $n$-th Fibonacci number for $n > 2$, with $F_1 = 1$ and $F_2 = 2$. Then we have: $F_3 = 3$, $F_4 = 5$, $F_5 = 8$, $F_6 = 13$, ecc. For example, $F_7 = F_2 + F_4 = 2 + 5$.

The Fibonacci encoding $F(x)$ of an integer $x$ is obtained by emitting a 1 bit if the $n$-th Fibonacci number is used in the sum giving $x$, or emitting a 0 bit otherwise. In our example, $F(7) = 0101 - 1$, where the final 1 bit after the dash is inserted to mark the end of the codeword. In fact, notice that, for the Zeckendorf’s theorem we can not use two adjacent Fibonacci number for the encoding, thus every codeword terminates with 01. Appending an additional 1 bit is sufficient to detect the decoding stopping condition. Table 3 shows the first 8 integers as encoded with Fibonacci, where we highlighted in bold font the final control bit.

There is a closed-form formula for computing the $n$-th Fibonacci number, $n \geq 1$, called Binet’s formula:

$$F_n = \frac{1}{\sqrt{5}} \left[ (1 + \sqrt{5})^n + (1 - \sqrt{5})^n \right] \approx \left( \frac{1 + \sqrt{5}}{2} \right)^n = \phi^n,$$

where $\phi = \frac{1 + \sqrt{5}}{2}$ is the so-called golden ratio. Using the above formula, it can be shown that the codeword length of $F(x)$, i.e., $|F(x)|$, is approximately equal to $1 + \log_\phi x$ bits. Therefore, we derive a corresponding distribution of $P(x) = \frac{1}{2} x^{-1/\log_\phi \delta} \approx \frac{1}{2} x^{-1.44}$. This implies that Fibonacci-based codes are shorter than $\gamma$ for all values except 1; and as good as or even better than $\delta$ for a wide range of practical values ranging from $F_2 = 2$ to $F_{19} = 6765$.

### 2.6 Variable-Byte

The codes described in the previous sections are bit-aligned as they do not represent an integer using a multiple of a fixed number of bits, e.g., a byte. Decoding an integer requires several bitwise instructions that can be expensive. Therefore it could be preferable to use byte-aligned or even word-aligned codes when decoding speed is the main concern.

Variable-Byte (VByte), first described by Thiel and Heaps [84], is the most popular and simplest byte-aligned code: the binary representation of a non-negative integer is split into groups of 7 bits which are represented as a sequence of bytes. In particular, the 7 least significant bits of each byte are reserved for the data whereas the most significant, called the continuation bit, is equal to 1 to signal continuation of the byte sequence. The last byte of the sequence has its 8-th bit set to 0 to signal, instead, the termination of the byte sequence. The main advantage of Variable-Byte codes is decoding speed: we just need to read one byte at a time until we found a value smaller than 2$^7$. Conversely, the number of bits to encode an integer cannot be less than 8, thus Variable-Byte is only suitable for large numbers and its compression ratio may not be competitive with the one of bit-aligned codes for small integers. Variable-Byte uses $\lceil \log_2(x + 1) \rceil / 7 \times 8$ bits to represent the integer $x$, thus it is optimal for the distribution $P(x) \approx \sqrt[8]{1/x^2}$.

For example, the integer 65,790 is represented as

$$\text{VByte}(65,790) = [0]0000100 - [1]0000001 - [1]1111110,$$

where we mark the control bits in bold font. Also notice the padding bits in the first byte starting from the left, inserted to align the binary representation of the number to a multiple of 8 bits.

Nibble coding is a simple variation of this strategy where, instead of 7 bits, 3 bits are used for data, which is optimal for the distribution $P(x) \approx \sqrt[3]{1/x^4}$.

Culppepper and Moffat [18] describe a byte-aligned code with the property that the first byte of each codeword defines the length of the codeword, which makes decoding simpler and faster.
Table 4. The integers 0..15 as represented by (2, 3)- and (4, 4)-dense codes respectively.

<table>
<thead>
<tr>
<th>integer</th>
<th>(2, 3)</th>
<th>(4, 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>011</td>
<td>100</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>101</td>
</tr>
<tr>
<td>2</td>
<td>011−000</td>
<td>110</td>
</tr>
<tr>
<td>3</td>
<td>100−000</td>
<td>111</td>
</tr>
<tr>
<td>4</td>
<td>011−001</td>
<td>100−000</td>
</tr>
<tr>
<td>5</td>
<td>100−001</td>
<td>101−000</td>
</tr>
<tr>
<td>6</td>
<td>011−010</td>
<td>110−000</td>
</tr>
<tr>
<td>7</td>
<td>100−010</td>
<td>111−000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>integer</th>
<th>(2, 3)</th>
<th>(4, 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>011−000−000</td>
<td>100−001</td>
</tr>
<tr>
<td>9</td>
<td>100−000−000</td>
<td>101−001</td>
</tr>
<tr>
<td>10</td>
<td>011−000−001</td>
<td>110−001</td>
</tr>
<tr>
<td>11</td>
<td>100−000−001</td>
<td>111−001</td>
</tr>
<tr>
<td>12</td>
<td>011−000−010</td>
<td>100−010</td>
</tr>
<tr>
<td>13</td>
<td>100−000−010</td>
<td>101−010</td>
</tr>
<tr>
<td>14</td>
<td>011−001−000</td>
<td>110−010</td>
</tr>
<tr>
<td>15</td>
<td>100−001−000</td>
<td>111−010</td>
</tr>
</tbody>
</table>

Various enhancements were proposed to accelerate the sequential decoding speed of Variable-Byte. For example, in order to reduce the probability of a branch misprediction that leads to higher throughput and helps keeping the CPU pipeline fed with useful instructions, the control bits can be grouped together. If we assume that the largest represented integer fits into four bytes, we have to distinguish between only four different byte-lengths, thus two bits are sufficient. In this way, groups of four integers require one control byte only. This optimization was introduced in Google’s Varint-GB format [20], which is faster to decode than the original Variable-Byte code.

Working with byte-aligned codes also opens the possibility of exploiting the parallelism of SIMD instructions to further enhance the sequential decoding speed. This is the approach taken by the proposals Varint-G8IU [82], Masked-VByte [72] and Stream-VByte [44] that we overview below.

Varint-G8IU [82] uses a format similar to the one of Varint-GB: one control byte describes a variable number of integers in a data segment of exactly eight bytes, therefore each group can contain between two and eight compressed integers. Masked-VByte [72] works, instead, directly on the original Variable-Byte format. The decoder first gathers the most significant bits of consecutive bytes using a dedicated SIMD instruction. Then using previously-built look-up tables and a shuffle instruction, the data bytes are permuted to obtain the decoded integers. Stream-VByte [44] separates the encoding of the control bits from the data bits by writing them into separate streams. This organization permits to decode multiple control bits simultaneously and, consequently, to reduce data dependencies that can stop the CPU pipeline execution when decoding the data stream.

2.7 SC-Dense

In Variable-Byte coding the value $2^7$ acts as a separator between continuers, i.e., all values in $[0, 2^7 − 1]$, and stoppers, i.e., all values in $[2^7, 2^8 − 1]$. A generalization of the encoding can be obtained by changing the separator value, thus enlarging or restricting the cardinalities of the set of continuers and stoppers. In general, the values from 0 to $c−1$ are reserved for continuers and the values from $c$ to $c+s−1$ to the stoppers, provided that $c+s=2^8$. Intuitively, changing the separating value can better adapt to the distribution of the integers to be encoded. For example, if most integers are larger than (say) 127, then it is convenient to have more stoppers. This is the main idea behind the $(s, c)$-dense coding algorithm introduced by Brisaboa, Farina, Navarro, and Esteller [9].

Given the integer $x$, its $(s, c)$-dense representation is obtained as follows. Let $k(x) \geq 1$ be the number of $\lceil \log_2(s+c) \rceil$-bit words needed by the representation of $x$. This value $k(x)$ will be such
that

\[ \frac{c^{k(x)-1} - 1}{c - 1} \leq x < \frac{c^{k(x)} - 1}{c - 1}. \]

If \( k(x) = 1 \), then the representation is just the stopper \( c + x \). Otherwise, the representation is the value \([x'/s]\) written in base \( c \), plus the final \( c + (x' \mod s) \), where \( x' = x - \frac{c^{k(x)+1}-2}{c-1} \).

Since the number of bits required by \( x \) is \( x(k(x) \times \lceil \log_2(s + c) \rceil) \), it follows that \( P(x) \approx (s + c)^{-k(x)} \).

Table 4 shows the codewords assigned to the integers 0..15 by the dense codes (2, 3) and (4, 4) respectively, assuming words of 3 bits rather than 8. Let us consider the integer \( x = 13 \) as represented by the (2, 3) code. In this case, \( k(13) = 3 \) and \( x' = 5 \). Since the value \([5/2]\) = 2 is 2 in base 3 and \( 3 + (5 \mod 2) = 4 \), the final representation is (from left to right) 2 - 0 - 4 or, in binary words of 3 bits each, 100-000-010 (from right to left). The same integer 13 under a (4, 4) dense representation will be as follows, instead. Now \( k(13) = 2 \) and \( x' = 9 \). The value \([9/4]\) = 2 is still 2 in base 4 and \( 4 + (9 \mod 4) = 5 \). Therefore, the final representation is (from left to right) 2 - 5 or, in binary words of 3 bits each, 101-010 (from right to left).

Lastly, it is possible to design an algorithm that finds the optimal values for \( s \) and \( c \) given a probability distribution of the symbols from the input source \([9]\).

### 2.8 Entropy coding: Shannon-Fano and Huffman

In this section we overview the famous Huffman’s algorithm \([38]\). Although some authors adopted this code for compressing sparse bitmaps \([8, 40, 58]\) – that are essentially inverted lists – its use for inverted index compression is limited because of reasons that will become clear in the following exposition. However, Huffman coding represents a milestone in data compression and, hence, no work surveying compression techniques would be complete without – at least – mentioning it. For an in-depth discussion of Huffman coding and how it can be implemented efficiently in practice, we point the interested reader to the survey by Moffat \([51]\).

We first recall the definition of entropy, a tool introduced by Shannon \([79]\). He was concerned with the problem of defining the information content of a discrete random variable \( X : \Sigma \rightarrow \mathbb{R} \), with distribution \( P(s) = P(X = s), s \in \Sigma \). He defined the entropy of \( X \) as

\[ H(X) = - \sum_{s \in \Sigma} P(s) \log_2 P(s) \text{ bits}. \]

The quantity \(- \log_2 P(s) \) bits is also called the self-information of the symbol \( s \) and \( H(X) \) represents the number of bits we need to encode each value of \( \Sigma \). Let now \( S \) be a sequence of \( n \) symbols drawn from an alphabet \( \Sigma \). Let also \( n_s \) denote the number of times the symbol \( s \) occurs in \( S \). Assuming empirical frequencies as probabilities \([63]\) (the larger is \( n \), the better the approximation), i.e., \( P(s) \approx n_s/n \), we can consider \( S \) as a random variable assuming value \( s \) with probability \( P(s) \). In this setting, the entropy of the sequence \( S \) is

\[ H_0(S) = -\frac{1}{n} \sum_{s \in \Sigma} n_s \log_2 \frac{n_s}{n} \text{ bits,} \]

also known as the 0-th order (or empirical) entropy of \( S \). In particular, the quantity \( nH_0(S) \) gives a theoretic lower bound on the average number of bits we need to represent \( S \), and consequently to the output size of any compressor that encodes each symbol of \( S \) with a fixed-length codeword. Observe that in case of equally-probable symbols, i.e., \( P(s) = 1/|\Sigma| \) for all \( s \), then \( H_0(S) = \log_2 |\Sigma| \) and hence assigning \( \log_2 |\Sigma| \)-bit codes is optimal.

**Shannon-Fano.** The precursor of the Huffman’s algorithm is the less-known Shannon-Fano algorithm that was independently proposed by Shannon \([79]\) and Fano \([29]\). The Shannon-Fano
Fig. 1. Shannon-Fano and Huffman codes built for the 6 most frequent symbols of the English alphabet. In this case, both codes are optimal with the same average codeword length of 2.54851 bits.

algorithm is greedy and builds the prefix code in a top-down manner. The symbols of $\Sigma$ are first sorted in decreasing order of probability. Then the alphabet is partitioned into two sets, such that these two have the closest as possible sum of probabilities. We keep partitioning the sets recursively until only a single symbol remains. This partitioning of $\Sigma$ can be visualized as a binary tree, whose internal nodes correspond to the alphabet subset being split into its children and whose leaves correspond to the individual symbols of $\Sigma$. The codeword $C(s)$ assigned to the symbol $s$ is obtained by traversing the tree from the root to the leaf of $s$. During the traversal, we append to $C(s)$ a 0 bit every time we go left and a 1 bit every time we go right. The code is prefix-free because we have one symbol per leaf and two siblings are distinguished using 0 and 1, thus no codeword can possibly be a prefix of another one. Fig. 1a shows the Shannon-Fano code built for the top-6 most frequent letters from the English alphabet (we rescaled the probabilities of the letters for make them sum up to 1.0).

However, the Shannon-Fano algorithm does not always produce an optimal prefix code (for example, see Table 5). Aware of this, Fano left the design of an optimal algorithm as an exercise for his students: one of them, David Huffman, proposed the elegant solution \cite{38} we now illustrate.

Huffman. The Huffman algorithm is greedy, like Shannon-Fano, but it builds the code in a bottom-up manner instead of top-down. In particular, the Huffman algorithm maintains a candidate set of tree nodes from which, at each step: (1) the two nodes with smallest probability are selected; (2) they are merged together into a new parent node whose probability is the sum of the probabilities of the two children; (3) the parent node is added to the candidate set. The algorithm repeats this merging step until only the root of the tree (whose probability is 1) is left in the candidate set. Fig. 1b shows the Huffman code built on the same alphabet used in Fig. 1a.

Since the Huffman tree is complete, it has exactly $2|\Sigma| - 1$ nodes. Using the same labelling procedure as in Shannon-Fano, it follows that the code is prefix-free. In case we have more symbols with the same probability, the choice of what nodes to merge is not unique. In fact, the Huffman code for the symbols of $\Sigma$ is not unique and different codes have a different maximum codeword length. Nonetheless, the average codeword length is the same and actually minimum among all possible prefix codes. A strategy used to minimize the maximum codeword length is to merge the
two “oldest” nodes, these being either leaves or two internal nodes that have been created farthest in the past.

Most importantly, if $H_0$ is the empirical entropy of a sequence of symbols drawn from $\Sigma$ and $L$ is the average Huffman codeword length, then $L$ satisfies $H_0 \leq L < H_0 + 1$. The meaning of this inequality is that Huffman can lose up to 1 bit compared to the entropy. In fact, as any other prefix-free code, the Huffman code requires at least 1 bit to encode a symbol, thus if $H_0$ is large, then the extra bit lost is negligible in practice; otherwise the distribution of probabilities is skewed and Huffman loses a significant space compared to the entropy of the source.

In Table 5 we show the Shannon-Fano and Huffman codes built for the whole 26-letter English alphabet (probabilities taken from the book by Salomon [77], at page 174). In this case, the Shannon-Fano code results slightly sub-optimal with an average codeword length of 4.16677 bits compared to the one of Huffman that is 4.15506 bits.

It is also important to mention that, in practice, the decoding of Huffman codes does not traverse the logical tree illustrated before. An elegant variation of the algorithm – known as canonical Huffman – allows fast decoding by using lookup tables. Again, Moffat [51] provides all details.

Now, it should also be clear why Huffman may not be an appropriate choice for inverted index compression. Applying Huffman to the compression of the integers in inverted lists means that its alphabet of representation is too large, thus making the mere description of the code outweigh the cost of representing very sparse inverted lists. The same reason applies if we try to use the code to compress the representation of the differences between successive integers: the largest difference could be as large as the largest integer in the sequence.

2.9 Concluding remarks

In the context of inverted indexes, we can exploit the fact that inverted lists are sorted and, typically, strictly increasing to achieve better compression. In particular, given a sequence $S[0..n]$ of this form, we can transform the sequence into $S'$ where $S'[i] = S[i] - S[i-1] - 1$ for $i > 0$ and $S'[0] = S[0]$. In the literature, $S'$ is said to be formed by the so-called delta-gaps (or just d-gaps) of $S$. Using
the codes described in this section on the gaps of $S$ is a very popular strategy for compressing inverted indexes, with the key requirement of performing a prefix-sum during decoding. Clearly, compressing $d$-gaps is far more convenient than compressing the integers of $S$ because $d$-gaps are smaller, thus less bits are required for their codes. In this case, the compressibility of the inverted index critically depends on the distribution of the $d$-gaps: fortunately, most $d$-gaps are small.

Fig. 2 shows the distribution of the $d$-gaps for three large textual collection that we will introduce in Section 6 (see Table 9, at page 28, for their basic statistics). The plot highlights the skewed
distribution of the \( d \)-gaps: the most frequent symbol is the 0 so that, for better visualization, we cut the percentage to 16% but report the actual value in bold. For example, on the ClueWeb09 dataset 49% of the \( d \)-gaps are just 0. The other values have decreasing frequencies. We divide the distribution into buckets of exponential size: the bucket whose value is \( y \) comprises all \( d \)-gap values in \((x, y]\) where \( x \) is the bucket before \( y \). In particular, the last bucket also comprises all other \( d \)-gap values larger than \( 2^{13} \) (the “tail” of the distribution).

As an illustrative comparison between the codes described in this section, we report in Fig. 3 the number of bits taken by their encoding when representing the integers 0..64, knowing that such values covers most of the \( d \)-gaps we have in inverted index data. In particular, we show the comparison between the codes \( \gamma, \delta \), Variable-Byte (V), Rice (R), \( \zeta \), Fibonacci (F) and binary (B) as an illustrative “single-value” lower bound. In the plots, data points corresponding to different methods that have the same coordinates have been stacked vertically for better visualization otherwise these would have been undistinguishable. For example, in Fig. 3a, all methods except V and B take 5 bits to represent the values 5 and 6.

Not surprisingly, a tuned parametric code such as Rice or \( \zeta \) may be the best choice. However, tuning is not always possible and a single parameter has to be specified for the encoding of all integers in the list. Then, for the smallest values in Fig. 3a the universal codes \( \gamma \) and \( \delta \) are good, whereas are not competitive immediately for slightly larger integers, e.g., the ones in Fig. 3b. On such values and the ones larger, a simple byte-aligned strategy such as Variable-Byte performs well.

3 LIST COMPRESSORS

Differently from the compressors described in the previous section, the algorithms we now consider encode an integer list, instead of representing each single integer separately.

A useful tool to analyze the space effectiveness of these compressors is the combinatorial information-theoretic lower bound, giving the minimum number of bits needed to represent a list of \( n \) strictly increasing integers drawn at random from a universe of size \( u \geq n \), that is \[62\] (\( e = 2.718 \) is the base of the natural logarithm):

\[
\left\lceil \log_2 \left( \frac{u}{n} \right) \right\rceil = n \log_2 \left( \frac{eu}{n} \right) - \Theta(n^2 / u) - O(\log n)
\]

which is approximately

\[
n \left( \log_2 \frac{u}{n} + \log_2 e \right) = n \log_2 \frac{u}{n} + 1.443n \text{ bits, for } n = o(\sqrt{u}).
\]

The compressors we describe in this section often outperform the ones introduced before for sufficiently long sequences because they take advantage of the fact that inverted lists often contain clusters of close integers, e.g., runs of consecutive integers, that are far more compressible than highly scattered regions. Refer, again, to Fig. 2: the most frequent symbol is the 0 number. The reason for the presence of such clusters is that the indexed documents themselves tend to be clustered, i.e., there are subsets of documents sharing the very same set of terms. As a meaningful example, consider all the Web pages belonging to a certain domain: since their topic is likely to be the same, they are also likely to share a lot of terms. Therefore, not surprisingly, list compressors greatly benefit from docID-reordering strategies that focus on re-assigning the docIDs in order to form larger clusters. When the indexed documents are Web pages, a simple and effective strategy is to assign identifiers to documents according to the lexicographical order of their URLs [80]. Another approach uses a recursive graph bisection algorithm to find a suitable re-ordering of docIDs [23]. In this model, the input graph is a bipartite graph in which one set of vertices represents the terms
Table 6. The 9 different ways of packing integers in a 28-bit segment.

<table>
<thead>
<tr>
<th>selector</th>
<th>integers</th>
<th>bits per integer</th>
<th>wasted bits</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>28</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
<td>3</td>
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<tr>
<td>6</td>
<td>4</td>
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<tr>
<td>7</td>
<td>3</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>14</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>28</td>
<td>0</td>
</tr>
</tbody>
</table>

of the index and the other set represents the docIDs. A graph bisection identifies a permutation of the docIDs and, thus, the goal is the one of finding, at each step of recursion, the bisection of the graph which minimizes the size of the graph compressed using delta coding.

### 3.1 Binary packing

A simple way to improve both compression ratio and decoding speed is to encode a block of integers, instead of the whole sequence. This line of work finds its origin in the so-called frame-of-reference (FOR) [34]. Once the sequence has been partitioned into blocks (of fixed or variable length), then each block is encoded separately. An example of this approach is binary packing. Given a block, we can compute the bit width \( b = \lceil \log_2(\text{max} + 1) \rceil \) of the max element in the block and then represent all integers in the block using \( b \)-bit codes. Clearly the bit width \( b \) must be stored prior to the representation of the block. Moreover, \( d \)-gaps can be computed to lower the value of \( b \). Plenty of variants of this simple approach has been proposed [22, 42, 81].

**Vector of Splits Encoding (VSE).** Dividing a list into fixed-size partitions may be suboptimal given that regions of close identifiers may be contained in a block containing a much larger value. Thus, it would be preferable to partition the list into variable-size blocks (or, splits) in order to better adapt to the distribution of the integers in the list. Silvestri and Venturini [81] present a dynamic programming algorithm that finds the optimal splitting of a list of size \( n \) in time \( O(k \times n) \), where \( k \) is the maximum block size allowed, in order to minimize the overall encoding cost. Lemire and Boytsov [42] report that the fastest implementation of this approach is the one using splits of size 1..14, 16 and 32 integers.

**BP32 and SIMD-BP128.** Lemire and Boytsov [42] propose word-aligned versions of binary packing for fast decoding. In the scheme called BP32, 4 groups of 32 bit-packed integers each are stored together in a meta block. Each meta block is aligned to 32-bit boundaries and a 32-bit word is used as descriptor of the meta block. The descriptor stores the 4 bit widths of the 4 blocks in the meta block (8-bit width for each block). The variant called SIMD-BP128 combines 16 blocks of 128 integers each that are aligned to 128-bit boundaries. The use of SIMD instructions provides fast decoding speed.

### 3.2 Simple

Rather than splitting the sequence into blocks of integers as in binary packing, we can split the sequence into fixed-memory units and ask how many integers can be packed in a unit. This is the key idea of the Simple family of encoders introduced by Anh and Moffat [2]: pack as many
Techniques for Inverted Index Compression

integers as possible in a memory word, i.e., 32 or 64 bits. This approach typically provides good compression and high decompression speed.

For example, Simple9 [2] (sometimes also referred to as Simple4b [3]) adopts 32-bit memory words and dedicates 4 bits to the selector code and 28 bits for data. The selector bits provide information on how many elements are packed in the data segment using equally-sized codewords. A selector 1 may correspond to 28 1-bit integers; 2 to 14 2-bit integers; 3 to 9 3-bit integers (1 bit unused), and so on, as we can see in Table 6. The four bits distinguish from 9 possible configurations. Similarly, Simple16 [94] has 16 possible configurations using 32-bit words. Simple8b [3], instead, uses 64-bit words with 4-bit selectors. Leaving 60 bits available for data offers 14 different combinations rather than just 9, with only 2 configurations having wasted bits rather than 3.

Anh and Moffat [2] also describe two variations of the Simple9 mechanism, named Relative10 and Carryover12. The idea behind Relative10 is to just use 2 bits for the selector, thus allowing 10 packing configurations with 30 bits. In order to make use of more than 4 options, the selector code is combined with the one of the previous word, hence enabling the whole range of 10 possibilities. However, when packing $7 \times 4$-bit integers or $4 \times 7$-bit integers, two bits per word are wasted (only 28 out of the 30 available bits are used). Therefore, in the Carryover12 approach these two bits are used to define the selector code of the following word configuration that makes use of the full 32 bits for packing the integers.

QMX. A similar approach to the one of the Simple family is used in the QMX mechanism, introduced by Trotman [86]. Considering memory words larger than 64 bits is a popular strategy for exploiting the parallelism of SIMD instructions. QMX packs as many integers as possible into 128- or 256-bit words (Quantities) and stores the selectors (eXtractors) separately in a different stream. The selectors are compressed (Multipliers) with run-length encoding, that is with a stream of pairs (value, length). For example, given the sequence $[12, 12, 12, 5, 7, 7, 7, 7, 9, 9]$, its corresponding RLE representation is $[(12, 3), (5, 1), (7, 4), (9, 2)]$.

### 3.3 PForDelta

The biggest limitation of block-based strategies is their space-inefficiency whenever a block contains just one large value, because this forces the compressor to use a universe of representation as large as that value. This is the main motivation for the introduction of PForDelta (PFor), proposed by Zukowski et al. [96]. The idea is to choose a proper value $k$ for the universe of representation of the block, such that a large fraction, e.g., 90%, of its integers can be written in $k$ bits each. All integers that do not fit in $k$ bits, are treated as exceptions and encoded in a separate stream using another compressor, e.g., Variable-Byte or Simple. This strategy is called patching.

More precisely, two configurable parameters are chosen: a base value $b$ and a universe of representation $k$, so that most of the values fall in the range $[b, b + 2^k - 1]$ and can be encoded with $k$ bits each by shifting them (delta-encoding) in the range $[0, 2^k − 1]$. To mark the presence of an exception, we also need a special escape symbol, thus we have $[0, 2^k − 2]$ available configurations.

For example, the sequence $[3, 4, 7, 21, 9, 12, 5, 16, 6, 2, 34]$ will be represented using PForDelta with parameters $b = 2$ and $k = 4$ as $[1, 2, 5, *, 7, 10, 3, *, 4, 0, *] − [21, 16, 34]$. The special symbol $*$ marks the presence of an exception that is written in a separate sequence, reported in bold after the dash.
3.4 Elias-Fano

The optimized variant Opt-PFor devised by Yan, Ding, and Suel [93], which selects for each block the values of $b$ and $k$ that minimize its space occupancy, it is more space-efficient and only slightly slower than the original PFor.

Lemire and Boytsov [42] proposed another variant called FastPFor where exceptions are compressed in pages, i.e., groups of blocks of integers. For example, a page may be 32 consecutive blocks of 128 integers each, for a total of 4096 integers. In this scheme, all the $b$-bit exceptions from all the blocks in a page are stored contiguously, for $b = 1..32$. What makes this organization faster to decode is the fact that exceptions are decoded in bulk at a page level, rather than at a (smaller) block level as in Opt-PFor.

<table>
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<th>13</th>
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Fig. 4. An example of Elias-Fano encoding. The missing 3-bit configuration 101 is highlighted in bold.

The optimized variant Opt-PFor devised by Yan, Ding, and Suel [93], which selects for each block the values of $b$ and $k$ that minimize its space occupancy, it is more space-efficient and only slightly slower than the original PFor.

Lemire and Boytsov [42] proposed another variant called FastPFor where exceptions are compressed in pages, i.e., groups of blocks of integers. For example, a page may be 32 consecutive blocks of 128 integers each, for a total of 4096 integers. In this scheme, all the $b$-bit exceptions from all the blocks in a page are stored contiguously, for $b = 1..32$. What makes this organization faster to decode is the fact that exceptions are decoded in bulk at a page level, rather than at a (smaller) block level as in Opt-PFor.

3.4 Elias-Fano

The encoder we now describe was independently proposed by Elias [27] and Fano [30]. Let $S(n, u)$ indicate a sorted sequence of $n$ integers drawn from a universe of size $u \geq S[n − 1]$. We write each $S[i]$ in binary using $\lceil \log_2 u \rceil$ bits. The binary representation of each integer is then split into two parts: a low part consisting in the right-most $\ell = \lceil \log_2 \frac{u}{n} \rceil$ bits that we call low bits and a high part consisting in the remaining $\lceil \log_2 u \rceil − \ell$ bits that we similarly call high bits. Let us call $\ell_l$ and $h_l$ the values of low and high bits of $S[i]$ respectively. The Elias-Fano encoding of $S(n, u)$ is given by the encoding of the high and low parts.

The integers $L = \{l_0, \ldots, l_{n-1}\}$ are written explicitly in $n\lceil \log_2 \frac{u}{n} \rceil$ bits and they represent the encoding of the low parts. Concerning the high bits, we represent them in negated unary using a bitvector of $n + 2^\lceil \log_2 n \rceil \leq 2n$ bits as follows. We start from a 0-valued bitvector $H$ and set the bit in position $h_i + i$, for all $0 \leq i < n$. It is easy to see that the $k$-th unary value $m$ of $H$ indicates that $m$ integers of $S$ have high bits equal to $k$, $0 \leq k \leq \lceil \log_2 n \rceil$. Finally the Elias-Fano representation of $S$ is given by the concatenation of $H$ and $L$ and overall takes

$$\text{EF}(S(n, u)) \leq n \left\lceil \log_2 \frac{u}{n} \right\rceil + 2n \text{ bits.} \quad (1)$$

Although we can opt for an arbitrary split into high and low parts, ranging from 0 to $\lceil \log_2 u \rceil$, it can be shown that $\ell = \lceil \log_2 \frac{u}{n} \rceil$ minimizes the overall space occupancy of the encoding [27].

Moreover, given that the information-theoretic lower bound is approximately $n \log_2 \frac{u}{n} + n \log_2 e$ bits, it can be shown [27] that less than half a bit is wasted per element by Formula 1.

Fig. 4 shows a graphical example of encoding for the sequence $S(12, 62) = [3, 4, 7, 13, 14, 15, 21, 25, 36, 38, 54, 62]$. The missing high bits embody the representation of the fact that using $\lceil \log_2 n \rceil$ bits to represent the high part of an integer, we have at most $2^\lceil \log_2 n \rceil$ distinct high parts because not all of them could be present. In Fig. 4, we have $\lceil \log_2 12 \rceil = 3$ and we can form up to 8 distinct
high parts. Notice that, for example, no integer has high part equal to 101 which are, therefore, “missing” high bits.

Despite the elegance of the encoding, it is possible to support random access to individual integers without decompressing the whole sequence. Formally, we are interested in implementing the operation \( \text{Access}(i) \) that returns \( S[i] \). The operation is supported by using an auxiliary data structure that is built on the bitvector \( H \) and efficiently answers \( \text{Select}_1 \) queries. The answer to a \( \text{Select}_b(i) \) query over a bitvector is the position of the \( i \)-th bit set to \( b \). This auxiliary data structure is succinct in the sense that it is negligibly small in asymptotic terms, compared to \( \text{EF}(S(n, u)) \), requiring only \( o(n) \) additional bits [46, 90].

Using the \( \text{Select}_1 \) primitive, it is possible to implement Access in \( O(1) \). We basically have to re-link together the high and low bits of an integer, previously split up during the encoding phase. The low bits \( \ell_i \) are trivial to retrieve as we need to read the range of bits \( \ell_i = L[i, (i + 1)\ell] \). The retrieval of the high bits is, instead, more complicated. Since we write in negated unary how many integers share the same high part, we have a bit set for every integer in \( S \) and a zero for every distinct high part. Therefore, to retrieve the high bits of the \( i \)-th integer, we need to know how many zeros are present in \( H[0, \text{Select}_1(i)] \). This quantity is evaluated on \( H \) in \( O(1) \) as \( \text{Select}_1(i) - i \). Finally, linking the high and low bits is as simple as: \( \text{Access}(i) = ((\text{Select}_1(i) - i) << \ell_i) \mid \ell_i \), where \( \lfloor \rfloor \) indicates the left shift operator and \( \lfloor \rfloor \) is the bitwise OR.

The query \( \text{Successor}(x) \), returning the smallest integer \( y \) of \( S \) such that \( y \geq x \), is supported in \( O(1 + \log \frac{x}{u}) \) time as follows. Let \( h_x \) be the high bits of \( x \). Then for \( h_x > 0 \), \( i = \text{Select}_0(h_x) - h_x + 1 \) indicates that there are \( i \) integers in \( S \) whose high bits are less than \( h_x \). On the other hand, \( j = \text{Select}_0(h_x + 1) - h_x \) gives us the position at which the elements having high bits greater than \( h_x \) start. The corner case \( h_x = 0 \) is handled by setting \( i = 0 \). These two preliminary operations take \( O(1) \). Now we can conclude the search in the range \( S[i, j] \), having skipped a potentially large range of elements that, otherwise, would have required to be compared with \( x \). We therefore determine the successor of \( x \) by binary searching in this range which contains up to \( u/n \) integers. The time bound follows. As an example, consider the query \( \text{Successor}(30) \) over the sequence illustrated in Fig. 4. Since \( h_{30} = 3 \), we have \( i = \text{Select}_0(3) - 3 + 1 = 7 \) and \( j = \text{Select}_0(4) - 3 = 8 \). Therefore we conclude our search in the range \( S[7, 8] \) by returning \( \text{Successor}(30) = S[8] = 36 \).

In the specific context of inverted indexes, the query \( \text{Successor} \) is called NextGEQ (Next Greater-than or Equal-to).

**Partitioning the integers by cardinality.** One of the most relevant characteristics of the Elias-Fano space bound, given by Formula 1, is that it only depends on two parameters, i.e., the size \( n \) and universe \( u \) of the sequence \( S(n, u) \). As already mentioned, inverted lists often present clusters of very similar integers and Elias-Fano fails to exploit them for better compression because it always uses a number of bits per integer at most equal to \( \lceil \log_2(u/n) \rceil + 2 \), thus proportional to the logarithm of the average gap \( u/n \) between the integers and regardless any skewed distribution. However, in order to better adapt to the distribution of the gaps between the integers, we can partition the sequence, therefore having the so-called partitioned Elias-Fano (PEF) representation [61].

The sequence \( S(n, u) \) is partitioned into \( k \) blocks of variable length. The first level of representation stores two sequences compressed with plain Elias-Fano: (1) the sequence made up of the last elements \( \{u_1, \ldots, u_k\} \) of the blocks, the so-called upper-bounds and (2) the prefix-summed sequence of the sizes of the blocks. The second level is formed, instead, by the representation of the blocks themselves, that can be again encoded with Elias-Fano. The main advantage of this two-level representation, is that now the integers in the \( i \)-th block are encoded with a smaller universe, i.e., \( u_i - u_{i-1} - 1, i > 0 \), thus improving the space with respect to the original Elias-Fano representation.
More precisely, each block in the second level is encoded with one among three different strategies. As already stated, one of them is Elias-Fano. The other two additional strategies come into play to overcome the space inefficiencies of Elias-Fano when representing dense blocks.

Let consider a block and call $b$ its size, $m$ its universe respectively. Vigna [90] first observed that as $b$ approaches $m$ the space bound $b \lceil \log_2 (m/b) \rceil + 2b$ bits becomes close to $2m$ bits. In other words, the closer $b$ is to $m$, the denser the block. However, we can always represent the block with $m$ bits by writing the characteristic vector of the block, that is a bitvector where the $i$-th bit is set if the integer $i$ belongs to the block. Therefore, besides Elias-Fano, two additional encodings can be chosen to encode the block, according on the relation between $m$ and $b$. The first one addresses the extreme case in which the block covers the whole universe, i.e., when $b = m$: in such case, the first level of the representation (upper-bound and size of the block) trivially suffices to recover each element of the block which is, therefore, encoded with 0 bits. The second case is used whenever the number of bits used by the Elias-Fano representation of the block is larger than $m$ bits: by doing the math, it is not difficult to see that this happens whenever $b > m/4$. In this case we can encode the block with its characteristic bitvector using $m$ bits. The choice of the proper encoding for a block is rather fundamental for the practical space effectiveness of the partitioned Elias-Fano representation.

Fig. 5 illustrates such three cases for a universe $m = 40$. In the sparse case (a), we can see that Elias-Fano takes $\log_2(40/5) + 2 = 5$ bits per element, whereas a characteristic vector representation would have taken $40/5 = 8$ bits per element. Conversely, in the dense case (b), a bitmap just takes $40/30 = 1.33$ bits per element, whereas Elias-Fano would have taken 3 bits per element.

Splitting the sequence $S(n,u)$ into equally-sized block is clearly sub-optimal, since we cannot expect clusters of similar integers to be aligned with uniform partitions. For such reason, an algorithm based on dynamic programming is presented by Ottaviano and Venturini [61] that yields a partition whose cost in bits is at most $(1 + \epsilon)$ times away from the optimal one taking $O(n \log \frac{1}{\epsilon})$ time and $O(n)$ space for any $0 < \epsilon < 1$. Notice that the time complexity becomes $\Theta(n)$ when $\epsilon$ is constant. In fact, the problem of determining the partition of minimum encoding cost can be seen as the problem of finding the path of minimum cost (shortest) in a complete, weighted and directed acyclic graph (DAG). This DAG has $n$ vertices, one for each integer of $S$, and $\Theta(n^2)$ edges where the cost $w(i,j)$ of edge $(i,j)$ represents the number of bits needed to represent $S[i,j]$. Each edge cost $w(i,j)$ is computed in $O(1)$ by just knowing the universe and size of the chunk $S[i,j]$. By pruning the DAG it is possible to attain to the mentioned complexity by preserving the approximation guarantees [61].

**Partitioning the integers by universe.** As already mentioned, we can opt for an arbitrary split between the high and the low part of the Elias-Fano representation. Partitioning the universe $u$
Fig. 6. Partitioned-by-universe strategies. In Roaring (a), a universe of size \( u \leq 2^{32} \) is partitioned into at most \( 2^{16} \) chunks, each spanning \( 2^{16} \) integers. Chunks are classified as sparse (s) or dense (d). In Slicing (b), sparse chunks are further partitioned into at most \( 2^8 \) blocks, each spanning \( 2^8 \) integers. Full chunks (f) are represented implicitly.

into chunks containing at most \( 2^\ell \) integers each, with \( \ell = \lceil \log_2(u/n) \rceil \), minimizes the space of the encoding [27] but a non-parametric split – independent from the values of \( u \) and \( n \) – is also possible.

For example, Roaring [14, 43, 45], partitions \( u \leq 2^{32} \) into chunks spanning \( 2^{16} \) values each and represents all the integers of the sequence falling into a chunk in two different ways according to the cardinality of the chunk: if the chunk contains less than 4096 elements, then it is considered to be sparse and represented as a sorted array of 16-bit integers; otherwise it is considered dense and encoded as a bitmap of \( 2^{16} \) bits. Lastly, very dense chunks can also be encoded with runs if advantageous. A run is represented as a pair \((v, \ell)\) meaning that all the integers \( v \leq x < v + \ell \) belong to the chunk. Refer to Fig. 6a for a graphical example.

Inspired by the van Emde Boas tree [88, 89], the Slicing [65] data structure recursively slices the universe of representation in order to better adapt to the distribution of the integers being compressed. Differently from Roaring, a sparse sparse chunk is further partitioned into at most \( 2^8 \) blocks of \( 2^8 \) elements each. Therefore, a non-empty universe slice of \( 2^{16} \) elements can be either: represented with a bitmap of \( 2^{16} \) bits (dense case); represented implicitly if the slice contains all the possible \( 2^{16} \) elements (full case); or it is recursively partitioned into smaller slices of \( 2^8 \) elements each. Finally, each non-empty slice of \( 2^8 \) elements is encoded with a sorted array of 8-bit integers (sparse case); or with a bitmap of \( 2^8 \) bits (dense case). Fig. 6b shows an example of the data structure.

It should be noted that all the partitioning strategies we have described in this section, namely partitioned Elias-Fano (PEF), Roaring and Slicing, exploit the same idea to attain to good space effectiveness: look for dense regions to be encoded with bitmaps and use a different mechanism for sparse regions. While PEF achieves this goal by splitting the sequence optimally by cardinality, Roaring and Slicing partition the universe of representation greedily, hence maintaining the property that all partitions are represented using the same universe. As we will better see in Section 6, these different partitioning paradigms achieve different space/time trade-offs.

### 3.5 Interpolative

The Binary Interpolative Code (BIC) invented by Moffat and Stuiver [55, 56] represents a sorted integer sequence without requiring the computation of its \( d\)-gaps. The key idea of the algorithm is to exploit the order of the already-encoded elements to compute the number of bits needed to represent the elements that will be encoded next.
At the beginning of the encoding phase, suppose we are specified two quantities $lo \leq S[0]$ and $hi \geq S[n-1]$. Given such quantities, we can encode the element in the middle of the sequence, i.e., $S[m]$ with $m = \lceil n/2 \rceil$, in some appropriate manner, knowing that $lo \leq S[m] \leq hi$. For example, we can write $S[m] - lo - m$ using just $\lceil \log_2(hi - lo - n + 1) \rceil$ bits. After that, we can apply the same step to both halves $S[0, m)$ and $S[m + 1, n)$ with updated knowledge of lower and upper values ($lo, hi$) that are set to $(lo, S[m] - 1)$ and $(S[m] + 1, hi)$ for the left and right half respectively.

We now show and encoding example applied to the sequence $S[0..12] = [3, 4, 7, 13, 14, 15, 21, 25, 36, 38, 54, 62]$. As it is always safe to choose $lo = 0$ and $hi = S[n-1]$, we do so, thus at the beginning of the encoding phase we have $lo = 0$ and $hi = 62$. Since we set $hi = S[n-1]$, the last value of the sequence is first encoded and we process every value outside this range is assigned a longer one on every value.

We identify the range of smaller codewords, delimited by the values $r_{lo}$ and $r_{hi}$, such that every value $x \leq r$ such that $r_{lo} < x < r_{hi}$ is assigned a shorter $(\lceil \log_2(r+1) \rceil - 1)$-bit codeword and every value outside this range is assigned a longer one on $(\lceil \log_2(r+1) \rceil)$ bits. To maintain unique decodability, we first always read $(\lceil \log_2(r+1) \rceil)$ bits and interpret these as the value $x$. Then we check if condition $r_{lo} < x < r_{hi}$ is satisfied: if so, we are done; otherwise, the codeword must be extended by 1 bit.

Fig. 7 shows the sequence of recursive calls performed by the Binary Interpolative Coding algorithm when applied to the sequence $[3, 4, 7, 13, 14, 15, 21, 25, 36, 38, 54, 62]$. In bold font we highlight the middle element being encoded.

![Fig. 7](image-url)
Table 7. Minimal binary codeword assignment to the values in 0..12 for a left-most assignment rule (a) and a centered assignment rule (b). In bold we highlight the codewords delimiting the range of the shorter codewords.

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In a left-most minimal binary code assignment, the first \( c \) values are assigned the shorter codewords, thus \( r_{hi} = 2^{\lceil \log_2(r+1) \rceil} - r - 1 \) (and we only check whether \( x < r_{hi} \)).

In a centered minimal binary code assignment, the values in the centre of the range are assigned the shorter codewords, thus \( (r_{lo}, r_{hi}) = ([r/2] - [c/2] - 1, [r/2] + [c/2] + 1) \) if \( r \) is odd, or \( (r_{lo}, r_{hi}) = ([r/2] - [c/2], [r/2] + [c/2] + 1) \) if \( r \) is even.

It should be noted that the exact assignment of codewords is irrelevant and many assignments could be possible: what matters is to assign correct lengths and maintain the property of unique decodability.

For example, when \( r = 12 \), we have that \( \lceil \log_2(12 + 1) \rceil = 4 \) and \( 2^4 - 12 - 1 = 3 \) codewords can be made 3-bit long and the remaining 10 codewords have a standard length of 4 bits. In the left-most assignment, these 3 shorter codewords are assigned to the values 0, 1 and 2; in the centered assignment, these are assigned to the values 5, 6 and 7. Table 7 shows the comparison between these two codeword allocation assignments (see also Table 2 from [56] for another example of centered minimal codes assigned using a different rule than the one used here).

Continuing our example in Fig. 7, using a left-most minimal assignment we have the following sequence of codeword length \([5, 4, 3, 2, 3, 3, 5, 4, 3, 5, 4]\), resulting in a code of 41 bits (ignoring the encoding of \( n \) and \( S[n-1] \)), thus 5 bits shorter than the one using a simple binary assignment. Lastly, the centered minimal assignment produces a further saving of 1 bit on that example sequence.

3.6 Directly-addressable codes

Brisaboa, Ladra, and Navarro [10] introduced a representation for a list of integers that supports random access to individual integers – called directly-addressable code (DAC) – noting that this is not generally possible for many of the representations described in Section 2 and 3. They reduced the problem of random access to the one of ranking over a bitmap. Given a bitmap \( B[1..n] \) of \( n \) bits, the query \( \text{Rank}_b(B, i) \) returns the number of \( b \) bits in \( B[1, i] \), for \( i \leq n \). For example, if \( B = 010001101110 \) then \( \text{Rank}_b(6) = 2 \) and \( \text{Rank}_b(8) = 5 \). Rank queries can be supported in \( O(1) \) by requiring only \( o(n) \) additional bits [15, 36, 39].

Each integer in the list is partitioned into \((b+1)\)-bit chunks. Similarly to Variable-Byte, \( b \) bits are dedicated to the representation of the integer and the control bit indicates whether another chunk follows or not. All the first \( n \) \( b \)-bit chunks of every integer are grouped together in a codeword stream \( C_1 \) and the control bits form a bitmap \( B_1[1..n] \) of \( n \) bits. If the \( i \)-th bit is set in such bitmap, then the \( i \)-th integer in the sequence needs a second chunk, otherwise a single chunk is sufficient. Proceeding recursively, all the second \( m \leq n \) chunks are concatenated together in \( C_2 \) and the control bits in a bitmap \( B_2[1..m] \) of \( m \) bits. Again, the \( i \)-th bit of such bitmap is set if the \( i \)-th integer
with at least two chunks needs a third chunk of representation. In general, if \( u \) is the maximum integer in the list, there are at most \( \lceil \log_2(u+1)/b \rceil \) levels (i.e., streams of chunks).

Fig. 8 shown an encoding example for the sequence [2, 7, 12, 5, 13, 142, 61, 129] using \( b = 3 \). Accessing the integer \( x \) in position \( i \) reduces to a sequence of \( c - 1 \) \( \text{Rank}_k \) operations over the levels’ bitmaps, where \( c \geq 1 \) is the number of \( (b+1) \)-bit chunks of \( x \), that is \( \lceil \log_2(x+1)/b \rceil \). Now, for \( k = 1..c \), we repeat the following step: (1) retrieve the \( i \)-th chunk from the \( C_k \) in constant time given that all chunks are \( b \) bits long; (2) if \( B_k[i] = 0 \), we are done; otherwise \( j = \text{Rank}_k(B_k, i) \) gives us the number of integers (in the level \( k+1 \)) that have more than \( k \) chunks, so we set \( i = j \) and repeat.

For example, \( \text{Access}(5) \) is resolved as follows. We retrieve \( C_1[5] = 101 \); since \( B_1[5] = 1 \), we compute \( \text{Rank}_1(B_1, 5) = 2 \). Now we retrieve \( C_2[2] = 001 \) and given \( B_2[2] = 0 \), we stop by returning the integer \( C_2[2]C_1[5] = 001 - 101 \), that is 13.

Lastly, nothing prevents from changing the value of \( b \) at each level of the data structure. For this purpose, the authors of DAC present an algorithm, based on dynamic programming, that finds such optimal values for a given list.

### 3.7 Entropy coding: Arithmetic and Asymmetric Numeral Systems

In this section we reprise the overview of entropy coding, by first introducing Arithmetic coding and then describing the idea of Asymmetric Numeral Systems (ANS). Similarly for the case of Huffman, no work advocated the use of Arithmetic coding for inverted index compression but some ideas based on ANS were proposed by Moffat and Petri [53, 54] and that we describe later in Section 4.2. Therefore, this section is meant to provide background notions to understand those proposals.

**Arithmetic.** The first concept of Arithmetic coding was introduced by Elias before 1963 according to Note 1 on page 61 in the book by Abramson [1]. However, the method requires infinite precision arithmetic and, because of this, it remained unpublished. The first practical implementation were designed during 1976 by Rissanen [76] and Pasco [64], and later refined by Rissanen [75]. A more recent efficient implementation is described by Moffat et al. [52].

This method offers higher compression ratios than Huffman’s, especially on highly skewed distributions, because it is not a prefix-free code, so it does not require at least one bit to encode a symbol. Indeed a single bit may correspond to more than one input symbol. However, Huffman codes are faster to decode; Arithmetic does not permit to decode an output stream starting from an arbitrary position, but only sequential decoding is possible.

Given a sequence of symbols \( S = s_1s_2...s_n \), the main idea behind the method works as follows. The interval \([0, 1)\) is partitioned into \( |\Sigma| \) segments of length proportional to the probabilities of the symbols. Then the subinterval corresponding with \( s_1 \), say \([\ell_1, r_1)\), is chosen and the partitioning step is applied to it. The process stops when all input symbols have been considered, thus after \( n \) steps, and outputs a single real number \( x \) in \([\ell_n, r_n)\), that is the interval associated to the last input

\[
\begin{array}{ccccccccc}
B_1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
C_1 & 010 & 111 & 100 & 101 & 101 & 110 & 011 & 001 \\
B_2 & 0 & 0 & 1 & 0 & 1 \\
C_2 & 001 & 001 & 001 & 110 & 000 \\
B_3 & 0 & 0 \\
C_3 & 010 & 010 \\
\end{array}
\]

Fig. 8. DAC representation of the sequence [2, 7, 12, 5, 13, 142, 61, 129] using \( b = 3 \).
symbol $s_n$. Then the pair $(x, n)$ suffices to decode the original input sequence $S$. Fig. 9 shows an encoding example for the sequence $S = abbaac$, with $P(a) = 1/2$, $P(b) = 1/3$ and $P(c) = 1/6$.

It can be shown that Arithmetic coding takes at most $nH_0 + 2$ bits to encode a sequence $S$ of length $n$. This means that the overhead with respect to the empirical entropy $H_0$ is only of $2/n$ bits per symbol, thus negligible for basically all practical values of $n$. As already pointed out, Arithmetic coding requires infinite precision that can be very costly to be approximated. In fact, a practical implementation [92] using approximated arithmetic can take up to $0.02$ bits of loss per symbol rather than $2/n$.

**Asymmetric Numeral Systems.** Asymmetric Numeral Systems (ANS) is a family of entropy coding algorithms, originally developed by Duda [24, 25], which approaches the compression ratio of Arithmetic coding with a decompression speed comparable with the one of Huffman [26].

The basic idea of ANS coding is to represent a *sequence* of symbols with a natural number $x$. In general, if each symbol $s$ is a bit, then appending $s$ to the end of the binary string representing $x$ will generate a new integer $x' = 2x + s$. This code is optimal whenever $P(0) = P(1) = 1/2$. ANS generalizes this fact by adapting it to a general distribution of symbols. In this case, appending a symbol $s$ to $x$ increases the information content from $\log_2 x$ bits to $\log_2 x - \log_2 P(s) = \log_2 (x/P(s))$ bits, thus the new generated natural number will be $x' \approx x/P(s)$.

Let us consider a more concrete example [54], with an alphabet of 3 symbols only, namely \{a, b, c\} and assuming $P(a) = 1/2$, $P(b) = 1/3$ and $P(c) = 1/6$. In order to derive the encoding of a sequence of symbols, a *frame* $f[1..m]$ of symbols is constructed, having $1/2$ of the entries equal to $a$, $1/3$ equal to $b$ and $1/6$ equal to $c$. For example, one such frame could be $f[1..6] = [aaabbc]$, but other symbol permutations with possibly larger $m$ are possible as well. The frame determines a table that is used to map a sequence of symbols to an entry in the table. Refer to Fig. 10a for an example with the frame $f[1..6] = [aaabbc]$. The entries in the table are the natural numbers assigned incrementally in the order determined by the frame. For example, since the first three symbols in the frame are $aaa$, the first numbers assigned to $a$’s row are 1, 2 and 3. The next two symbols are $bb$, so $b$’s row gets 4 and 5. The last symbol in the frame is $c$, so the first entry in $c$’s row is 6. The process now proceed in cycles, thus placing 7, 8 and 9 in $a$’s row; 10 and 11 in $b$’s row, a final 12 in $c$’s row, and so on (first 10 columns are shown in the figure). Fig. 10b shows an example for another distribution of the symbols, constructed using a frame $f[1..4] = [caba]$. 

Fig. 9. Arithmetic coding example applied to the input sequence $abbaac$. The final interval is $[149/432, 49/144]$. 

\[\begin{align*}
\text{Input} & \quad \text{a} & \quad \text{b} & \quad \text{b} & \quad \text{a} & \quad \text{a} & \quad \text{c} \\
0 & 1/2 & 1/2 & 5/12 & 7/18 & 13/36 & 25/72 \\
& 5/6 & & & & & \\
& & 1/4 & & & & \\
& & & 1/3 & & & \\
& & & & 1/3 & & \\
& & & & & 1/3 & \\
& & & & & & 1/3 \\
\end{align*}\]
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Now, consider the sequence caa and let us determine its ANS code with the table in Fig. 10a. We make use of the transition function defined by the table \( T \) itself as \( \text{state}' = T[s, \text{state}] \) which, given a symbol \( s \) and a state value, produces the next state' of the encoder. At the beginning we set \( \text{state} = 0 \), thus for the given sequence caa the state variable assumes values \( 0 \rightarrow 6 \rightarrow 13 \rightarrow 26 \) (last value not shown in the figure). The code assigned to the sequence is, therefore, the integer 26. For the sequence acb under the encoding table in Fig. 10b we generate, instead, the transitions \( 0 \rightarrow 2 \rightarrow 9 \rightarrow 39 \), thus the assigned code is 39. Decoding reverts this process. For example, given 39 we know that the last symbol of the encoded sequence must have been b because 39 is found on the second row of the table. The value is in column 9, which is found in column 2 in the third row that corresponds to the c symbol. Finally, the column number 2 is found in a’s row, thus we emit the message acb.

3.8 Hybrid approaches

Hybrid approaches are possible by using different compressors to represent the blocks of a list. For example, given a query log, we can collect access statistics at a block-level granularity, namely how many times a block is accessed during query processing, and represent rarely-accessed blocks with more space-efficient compressor; vice versa frequently-accessed blocks are encoded with more time-efficient compressor [60]. This hybrid strategy produces good space/time trade-offs.

Optimal partitioning for point-wise encoders. Pibiri and Venturini [71] show that a list of \( n \) sorted integers can be optimally partitioned into variable-length blocks whenever the chosen representation for each block is given by either: (1) any compressor described in Section 2, namely a point-wise encoder, or (2) the characteristic vector of the block. From Section 3.4 we recall that, given a block of universe \( m \), the characteristic vector representation of the block is given by a bitmap of \( m \) bits where the \( i \)-th bit is set if the integer \( i \) belongs to the block.

By exploiting the fact that the chosen encoder is point-wise, i.e., the number of bits needed to represent an integer solely depends on the integer itself and not on the block to which it belongs to, it is possible to devise an algorithm that finds an optimal partitioning in \( \Theta(n) \) time and \( O(1) \) space. The constant factor hidden by the asymptotic notation is very small and, in practice, the algorithm is very fast.

4 INDEX COMPRESSORS

This section is devoted to approaches that look for regularities among all the list in the inverted index, thus usually obtaining higher compression ratios. As already motivated at the beginning of Section 3, this is possible because the inverted index naturally presents some amount of redundancy, namely many integers are shared between the lists.

Fig. 10. Two example of ANS encoding table, with respectively frame \( f[1..6] = [aaabbc] \) and \( f[1..4] = [caba] \).
Techniques for Inverted Index Compression

4.1 Clustered

Pibiri and Venturini [68] propose a clustered index representation. The inverted lists are grouped into clusters of “similar” lists, i.e., the ones sharing as many integers as possible. Then for each cluster, a reference list is synthesized with respect to which all lists in the cluster are encoded. More specifically, the integers belonging to the intersection between the cluster reference list and a list in the cluster are represented as the positions they occupy within the reference list. This makes a big improvement for the cluster space, since each intersection can be re-written in a much smaller universe. Refer to Fig. 11 for a pictorial example of this technique.

Although any compressor can be used to represent the intersection and the residual segment of each list, the authors adopt partitioned Elias-Fano; by varying the size of the reference lists, different time/space trade-offs can be obtained.

4.2 ANS-based

In Section 3.7 we have illustrated the ANS method developed by Duda [24, 25]. In the specific case of inverted indexes, the alphabet size may be too large, as we already observed in Section 2.8. Even after computing the d-gaps, the largest value to represent might be equal to the number of documents in the collection, which is usually several order to magnitudes larger than, for example, the (extended) ASCII alphabet.

For this reason, Moffat and Petri [53] describe several adaptation of this base mechanism tailored for effective index compression. In order to reduce the alphabet size, they perform a pre-processing step with Variable-Byte to reduce the input list to a sequence of bytes and then apply ANS (VByte+ANS). Local variability can be instead captured by using 16 different ANS models, each selected using a 4-bit selector in the spirit of the Simple approach described in Section 3.2 (Simple+ANS). Another variant is obtained by dividing a list into blocks and encoding each block with the most suitable model, chosen among 16 possibilities according to a selected block statistic e.g., its maximum value (Packed+ANS).

4.3 Dictionary-based

Pibiri, Petri, and Moffat [67] show that inverted indexes can be effectively compressed using a dictionary-based approach. Their technique – named Dictionary of INTeger sequences (DINT) – builds on the observation that patterns of d-gaps are highly repetitive across the whole inverted index. Therefore, a dictionary storing the most frequent $2^b$ patterns can be constructed, for some $b > 0$, noting that in general, the problem of building a dictionary that minimizes the number of output bits when sequences symbols are coded as references to its entries is NP-hard [83]. More specifically, an integer list can be modelled as a sequence of $b$-bit codewords, each codeword corresponding to a dictionary pattern. Fig. 12 illustrates this approach. This representation has the

![Fig. 11. Shaded boxes mark the integers falling in the intersection between the list and the reference (map segment); all the others form the residual part (residual segment).](image-url)
Fig. 12. A dictionary-based encoded stream example, where dictionary entries are labelled with different shades as they correspond to \{1, 2, 4, 8, 16\}-long integer patterns and special symbols, such as runs and exceptions. In particular, a run is a pattern made of all 0s and the dictionary can have reserved entries for specific run lengths, e.g., \{32, 64, 128, 256\}. Lastly, a dictionary entry must be reserved for signalling the presence of an exception, i.e., a single integer that is not sufficiently frequent to be assigned a dictionary pattern. Once provision has been made for such a dictionary structure, a sequence of \(d\)-gaps can be modelled as a sequence of codewords \(\{c_k\}\), each being a reference to a dictionary entry. Note that, for example, codeword \(c_9\) signals an exception, therefore the next exception symbol \(e\) is decoded using an escape mechanism.

The twofold advantage of: (1) requiring \(b\) bits to represent a pattern (thus, potentially, several integers); (2) decoding of a pattern requires just a lookup in the dictionary. In their investigation, patterns of size 1, 2, 4, 8 and 16 are considered, with \(b = 16\) to avoid expensive bit-level manipulations and allow extremely fast decoding. Moreover, compacting the dictionary has the potential of letting the data structure fit in the cache, hence speeding up the decoding process thanks to the reduced number of cache misses. Lastly, once the dictionary is built, a shortest-path computation suffices to find the optimal encoding of a list for that specific dictionary.

5 FURTHER READINGS

Besides the individual papers listed in the bibliography, we here mention previous efforts in summarizing encoding techniques for integer sequences. The book by Witten, Moffat, and Bell [91] is the first, to the best of our knowledge, that treats compression and indexing data as a unified problem, by presenting techniques to solve it efficiently. Fenwick [31] and Salomon [77] provide a vast and deep coverage of variable-length codes. The survey by Zobel and Moffat [95] covers more than 40 years of academic research in Information Retrieval and gives an introduction to the field. Moffat and Turpin [57], Moffat [50] and Pibiri and Venturini [70] describe several of the techniques illustrated in this article; Trotman [85] experimentally evaluates many of them.

We also point the interested reader to books offering a broader overview of the research field in Information Retrieval (thus, not only related to inverted indexes), such as the ones by Baeza-Yates, Ribeiro, et al. [5], Büttcher, Clarke, and Cormack [12], Croft, Metzler, and Strohman [17], Manning, Raghavan, and Schütze [49] and Cambazoglu and Baeza-Yates [13].
6 EXPERIMENTS

In this section of the article, we are interested in reporting on the space effectiveness and time efficiency of inverted index representations. Specifically, space effectiveness is measured as the average number of bits dedicated to the representation of a document identifier; time efficiency is assessed in the time needed to perform: sequential decompression; the \( \text{NextGEQ}_S \) query that, given an integer \( x \), returns the integer \( z \geq x \) from the sequence \( S \); intersection and union of two inverted lists. For the latter operations, we focus on materializing the full results set, without any ranking or dynamic pruning mechanism being applied [11, 47].

Due to page limits, we do not aim at being exhaustive here but rather compare some selected representations and point the interested reader to the GitHub repository at https://github.com/jermp/2i_bench for further comparisons. Other resources providing extensive comparisons include the first author’s Ph.D. Thesis (Chapter 7) [66] and the work by Mallia et al. [48].

Tested index representations. We compare the following configurations, summarized in Table 8.

- Elias’ \( \delta \) (Section 2.2) as a classic bit-aligned universal code;
- Variable-Byte (VByte, Section 2.6) as the most used byte-aligned code, reported to achieve the highest decoding speed in the literature when paired with the SIMD-ized decoding algorithm devised by Plaisance et al. [72] and called Masked-VByte;
- Simple16 (Section 3.2) as representative of the paradigm variable-to-fixed, i.e., a variable number of integers is fit into a fixed quantum, e.g., a 64-bit word;
- optimized PForDelta (Opt-PFor, Section 3.3) as representative of the paradigm fixed-to-variable, i.e., a fixed number of integers is compressed using a variable number of bits;
- partitioned Elias-Fano (PEF, Section 3.4) with variable-size partitions as a solution yielding excellent space/time trade-offs;
- Roaring and Slicing (Section 3.4) as solutions partitioning the inverted lists by universe rather than by cardinality as all the other mechanisms.
- Interpolative (BIC, Section 3.5) with leftmost minimal binary codes as the technique giving the most compact representations in the literature;
- optimally-partitioned Variable-Byte (Opt-VByte, Section 3.8) as a refinement of plain Variable-Byte;
- DINT (Section 4.3) as representative of the dictionary-based approach to compression, also named a fixed-to-fixed paradigm by the authors, given that a fixed number of bytes is always consumed and output during a decoding operation.

We use the C++ implementations listed in Table 8 (the prefix https://github.com/ should be prepended to every link). All the mentioned libraries are included in our benchmarking suite as dependencies.

Datasets. We perform the experiments on the following standard test collections.

- Gov2 is the TREC 2004 Terabyte Track test collection, consisting in roughly 25 million .gov sites crawled in early 2004. The documents are truncated to 256 KB.
- ClueWeb09 is the ClueWeb 2009 TREC Category B test collection, consisting in roughly 50 million English web pages crawled between January and February 2009.
- CCNews is a dataset of news freely available from CommonCrawl. Precisely, the datasets consists of the news appeared from 09/01/16 to 30/03/18.

Identifiers were assigned to documents according to the lexicographic order of their URLs [80]. From the original collections we retain all lists whose size is larger than 4096. Table 9 reports the basic statistics for the collections.
Table 8. The different tested index representations.

<table>
<thead>
<tr>
<th>Method</th>
<th>Partitioned by</th>
<th>SIMD</th>
<th>Alignment</th>
<th>Description</th>
<th>Implementation</th>
</tr>
</thead>
<tbody>
<tr>
<td>VByte</td>
<td>cardinality</td>
<td>yes</td>
<td>byte</td>
<td>fixed-size partitions</td>
<td>lemire/MaskedVByte</td>
</tr>
<tr>
<td>Opt-VByte</td>
<td>cardinality</td>
<td>yes</td>
<td>bit</td>
<td>variable-size partitions</td>
<td>jermp/opt_vbyte</td>
</tr>
<tr>
<td>BIC</td>
<td>cardinality</td>
<td>no</td>
<td>bit</td>
<td>fixed-size partitions</td>
<td>ot/ds2i</td>
</tr>
<tr>
<td>δ</td>
<td>cardinality</td>
<td>no</td>
<td>bit</td>
<td>fixed-size partitions</td>
<td>ot/ds2i</td>
</tr>
<tr>
<td>PEF</td>
<td>cardinality</td>
<td>no</td>
<td>bit</td>
<td>variable-size partitions</td>
<td>ot/ds2i</td>
</tr>
<tr>
<td>DINT</td>
<td>cardinality</td>
<td>no</td>
<td>16-bit word</td>
<td>fixed-size partitions</td>
<td>jermp/dint</td>
</tr>
<tr>
<td>Opt-PFor</td>
<td>cardinality</td>
<td>no</td>
<td>32-bit word</td>
<td>fixed-size partitions</td>
<td>lemire/FastPFor</td>
</tr>
<tr>
<td>Simple16</td>
<td>cardinality</td>
<td>no</td>
<td>64-bit word</td>
<td>fixed-size partitions</td>
<td>lemire/FastPFor</td>
</tr>
<tr>
<td>Roaring</td>
<td>universe</td>
<td>yes</td>
<td>byte</td>
<td>single-span</td>
<td>RoaringBitmap/CRoaring</td>
</tr>
<tr>
<td>Slicing</td>
<td>universe</td>
<td>yes</td>
<td>byte</td>
<td>multi-span</td>
<td>jermp/s_indexes</td>
</tr>
</tbody>
</table>

Table 9. Basic statistics for the test collections.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Gov2</th>
<th>ClueWeb09</th>
<th>CCNews</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lists</td>
<td>39,177</td>
<td>96,722</td>
<td>76,474</td>
</tr>
<tr>
<td>Universe</td>
<td>24,622,347</td>
<td>50,131,015</td>
<td>43,530,315</td>
</tr>
<tr>
<td>Integers</td>
<td>5,322,883,266</td>
<td>14,858,833,259</td>
<td>19,691,599,096</td>
</tr>
</tbody>
</table>

Experimental setting and methodology. All the experiments are performed on a server machine with 4 Intel i7-7700 cores (@3.6 GHz), 64 GB of RAM DDR3 (@2.133 GHz) and running Linux 4.4.0 (64 bits). Caches have the following sizes: 32 K for both instruction and data L1 cache; 256 K for L2 cache; 8192 K for L3 cache. We compiled the code with gcc 7.3.0 using the highest optimization setting, i.e., with compilation flags -O3 and -march=native.

We build the indexes in internal memory and write the corresponding data structures to a file on disk. To perform the queries, the data structure is memory mapped from the file and a warming-up run is executed to fetch the necessary pages from disk. To sequentially decode the indexes, the kernel is also instructed to access the memory mapped area sequentially using posix_madvise with flag POSIX_MADV_SEQUENTIAL. From the TREC 2005 and TREC 2006 Efficiency Track topics, we selected all queries whose two terms are both in the lexicons of the tested collection. To test the speed of intersection and union, we use a random sampling of 1000 such queries.

Each experiment was repeated 10 times to smooth fluctuations during measurements. The time reported is the average among these runs.

Compression effectiveness. In Table 10 we report the compression effectiveness of each method expressed as total GiB and bit-per-integer rate. Pretty much consistently across the three tested datasets:

- the most effective method is BIC with PEF being close second;
- the least effective methods are VByte and Roaring (in particular, Roaring is sensibly better than VByte on Gov2 but performs worse on the other two datasets);
- Opt-VByte, δ, DINT, Opt-PFor and Simple16 are all similar in space, taking roughly 3 ÷ 4, 5 ÷ 6 and 6 ÷ 6.5 bits/int for Gov2, ClueWeb09 and CCNews respectively;
- Slicing stands in a middle position between the former two classes of methods.
Two notes are in order. While techniques for inverted index compression are generally the fastest, thanks to their "simpler" design involving byte-aligned codes and bitmaps, and the use of SIMD instructions; BIC and \( \delta \) are the slowest as they only decode one symbol at a time, requiring several bitwise instructions (note that BIC is almost 2\( \times \) slower because of its recursive implementation); the other mechanisms PEF, DINT, Opt-PFor and Simple16 provide similar efficiency, roughly decoding an integer in 1 \( \div \) 1.5 nanoseconds.

<table>
<thead>
<tr>
<th>Method</th>
<th>Gov2 GiB</th>
<th>Gov2 bits/int</th>
<th>Gov2 ns/int</th>
<th>ClueWeb09 GiB</th>
<th>ClueWeb09 bits/int</th>
<th>ClueWeb09 ns/int</th>
<th>CCNews GiB</th>
<th>CCNews bits/int</th>
<th>CCNews ns/int</th>
</tr>
</thead>
<tbody>
<tr>
<td>VByte</td>
<td>5.46</td>
<td>8.81</td>
<td>0.51</td>
<td>15.92</td>
<td>9.20</td>
<td>0.67</td>
<td>21.29</td>
<td>9.29</td>
<td>0.62</td>
</tr>
<tr>
<td>Opt-VByte</td>
<td>2.41</td>
<td>3.89</td>
<td>0.76</td>
<td>9.89</td>
<td>5.72</td>
<td>0.91</td>
<td>14.73</td>
<td>6.42</td>
<td>0.77</td>
</tr>
<tr>
<td>BIC</td>
<td>1.82</td>
<td>2.94</td>
<td>6.37</td>
<td>7.66</td>
<td>4.43</td>
<td>7.96</td>
<td>12.02</td>
<td>5.24</td>
<td>8.92</td>
</tr>
<tr>
<td>( \delta )</td>
<td>2.32</td>
<td>3.74</td>
<td>4.01</td>
<td>8.95</td>
<td>5.17</td>
<td>4.15</td>
<td>14.58</td>
<td>6.36</td>
<td>4.26</td>
</tr>
<tr>
<td>PEF</td>
<td>1.93</td>
<td>3.12</td>
<td>0.91</td>
<td>8.63</td>
<td>4.99</td>
<td>1.32</td>
<td>12.50</td>
<td>5.45</td>
<td>1.57</td>
</tr>
<tr>
<td>DINT</td>
<td>2.19</td>
<td>3.53</td>
<td>0.82</td>
<td>9.26</td>
<td>5.35</td>
<td>1.24</td>
<td>14.76</td>
<td>6.44</td>
<td>1.43</td>
</tr>
<tr>
<td>Opt-PFor</td>
<td>2.25</td>
<td>3.63</td>
<td>1.06</td>
<td>9.45</td>
<td>5.46</td>
<td>1.53</td>
<td>13.92</td>
<td>6.07</td>
<td>1.19</td>
</tr>
<tr>
<td>Simple16</td>
<td>2.59</td>
<td>4.19</td>
<td>1.15</td>
<td>10.13</td>
<td>5.85</td>
<td>1.54</td>
<td>14.68</td>
<td>6.41</td>
<td>1.55</td>
</tr>
<tr>
<td>Roaring</td>
<td>4.11</td>
<td>6.63</td>
<td>0.57</td>
<td>16.92</td>
<td>9.78</td>
<td>0.82</td>
<td>21.75</td>
<td>9.49</td>
<td>0.71</td>
</tr>
<tr>
<td>Slicing</td>
<td>2.67</td>
<td>4.31</td>
<td>0.69</td>
<td>12.21</td>
<td>7.06</td>
<td>0.86</td>
<td>17.83</td>
<td>7.78</td>
<td>0.83</td>
</tr>
</tbody>
</table>

**Sequential decoding.** Table 10 also reports the average nanoseconds spent per decoded integer, measured after decoding all lists in the index. For all the different methods, the result of decoding is materialized into an output buffer of 32-bit integers. Again, results are consistent across the different datasets:

- VByte, Opt-VByte, Roaring and Slicing are the fastest, thanks to their "simpler" design involving byte-aligned codes and bitmaps, and the use of SIMD instructions;
- BIC and \( \delta \) are the slowest as they only decode one symbol at a time, requiring several bitwise instructions (note that BIC is almost 2\( \times \) slower because of its recursive implementation);
- the other mechanisms PEF, DINT, Opt-PFor and Simple16 provide similar efficiency, roughly decoding an integer in 1 \( \div \) 1.5 nanoseconds.

Two notes are in order. While BIC, PEF, Roaring and Slicing decode the actual integers, the other methods decode their \( d \)-gaps and we skipped the final prefix sum operation in this experiment, thus results compare more favourably for such methods.

We optimized the decoding of PEF codes by decoding its partitions separately according to their three possible types, rather than calling a decoding function for each single integer in a sequence.

**NextGEQ query.** Fig. 13 shows the timings of NextGEQ for exponentially increasing jump size on Gov2 (we obtained similar shapes for the other two datasets). When the last two values returned by NextGEQ are \( S[i] \) and \( S[j] \) with \( j \geq i \) and \( j - i = d \), we say that NextGEQ makes a jump of size \( d \).

- We see that BIC and \( \delta \) are generally the slowest for \( d \geq 2^3 \) and with \( \delta \) being roughly 2\( \times \) faster than BIC.
- VByte is the fastest on very small jumps, e.g., less than 2\( ^3 \), but Slicing soon wins out and is significantly faster than all other methods.
- Among all other techniques performing similarly, PEF provides generally better efficiency across all jump sizes thanks to the skipping abilities of Elias-Fano.

Note how – for the techniques using uniform partitions of 128 integers – the curves have a generally flatter pace for jump sizes \( d \geq 2^7 = 128 \) because on such longer jumps we basically pay a block decompression.
Boolean AND/OR queries. We now consider the fundamental list operations of intersection and union. Table 11 reports the timings, with Fig. 14 and Fig. 15 displaying the same data but plotted along space/time trade-off curves (thus, also incorporating the space information brought by Table 10). When considering the general trade-off, especially highlighted by the plots, we see
that the trend of the trade-off is the same for both intersections and unions, even across three different datasets. Therefore we can make some general points.

- For methods partitioned by cardinality, the efficiency of intersection is – not surprisingly – strictly correlated to the efficiency of NextGEQ; the efficiency of union is correlated to the one of sequential decoding. This is not necessarily true for Roaring and Slicing that, being partitioned by universe rather than cardinality, employ an intersection algorithm that does not use NextGEQ.
- There is a cluster of techniques providing similar efficiency/effectiveness trade-offs, including PEF, DINT, Opt-VByte, Simple16 and Opt-PFor.
- BIC and $\delta$ are always the slowest and dominated by the aforementioned techniques.
- Roaring and Slicing outperform in efficiency all techniques by a wide margin\(^2\), at the price of being larger: Slicing stands in a middle position between the mentioned cluster and Roaring.

\(^2\)Concerning Roaring, an important testing detail to report is that we modify the original implementation in order to avoid memory allocations during intersections and unions. Therefore, the result is always materialized into an uncompressed buffer rather than being a new Roaring data structure.
VByte is dominated by Slicing since the latter is always faster and smaller but, however, VByte is still easier to be implemented (as far as SIMD instructions are not concerned).

7 CONCLUSIONS AND FUTURE RESEARCH DIRECTIONS

The problem of introducing a compression format for sorted integer sequences, with good practical intersection/union performance, is well-studied and important, given its fundamental application to large-scale retrieval systems such as Web search engines. As a matter of fact, inverted index compression is still a very active field of research that began several decades ago.

With this article, we aimed at surveying the encoding algorithms suitable to solve this problem. However, identifying a single solution to the problem is not generally easy, rather the many space/time trade-offs available can satisfy different application requirements and the best solution should always be determined by considering the actual data distribution. To this end, we also offer an experimental comparison between many of the techniques described in this article. We feel that the different space/time trade-offs assessed by this analysis are summarized by Fig. 14 and Fig. 15.

Because of the maturity reached by the state-of-the-art and the specificity of the problem, identifying future research directions is not immediate. We mention some promising ones.

In general, devising "simpler" compression formats that can be decoded with algorithms using low-latency instructions (e.g., bitwise) and with as few branches as possible, is a profitable line of research, as demonstrated by the experimentation in this article. Such algorithms favour the super-scalar execution of modern CPUs and are also suitable for SIMD instructions.

Another direction could look at devising dynamic and compressed representations for integer sequences, able of also supporting additions and deletions. This problem is actually a specific case of the more general dictionary problem, which is a fundamental textbook problem. While a theoretical solution already exists with all operations supported in optimal time and compressed space [69], an implementation with good practical performance could be of great interest for dynamic inverted indexes.

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