Exercises on Nonlinear Optimization Theory

Answer the following questions for any of the nonlinear optimization problems reported below:

a) Is it a convex optimization problem?

b) Do global optimal solutions exist? Why?

c) Does the Abadie constraints qualification hold? Why?

d) Find all the solutions of the KKT system associated with the problem.

e) Find local and global optimal solutions exploiting the optimality conditions.

f) Write the Lagrangian dual problem and try to solve it.

1. $$\begin{cases} \min\ -x_1 + x_2^2 \\ -x_1^2 - x_2^2 + 4 \leq 0 \end{cases}$$

2. $$\begin{cases} \min\ x_1^3 + x_2^3 \\ -x_1 - 1 \leq 0 \\ -x_2 - 1 \leq 0 \end{cases}$$

3. $$\begin{cases} \min\ -x_1^2 - x_2^2 \\ x_1^2 + x_2^2 - 1 \leq 0 \end{cases}$$

4. $$\begin{cases} \min\ -3\,x_1^2 - 3\,x_2^2 + 4\,x_1\,x_2 \\ -x_1 - x_2 + 2 \leq 0 \\ x_1^2 + x_2^2 - 4 \leq 0 \end{cases}$$

5. $$\begin{cases} \min\ x_2 - x_1^2 \\ x_1^2 + x_2^2 - 4 \leq 0 \\ x_2 \leq 0 \end{cases}$$

6. $$\begin{cases} \min\ 4\,x_1^2 + x_2^2 - x_1\,x_2 - 14\,x_1 - 2\,x_2 \\ -x_1 \leq 0 \\ -x_2 \leq 0 \\ x_1 + x_2 - 1 \leq 0 \end{cases}$$

7. $$\begin{cases} \min\ -(x_1 - 3)^2 - 4\,(x_2 - 2)^2 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$$

8. $$\begin{cases} \min\ -3\,x_1^2 - 3\,x_2^2 + 4\,x_1\,x_2 \\ -x_1 - x_2 + 2 \leq 0 \\ x_1^2 + x_2^2 - 4 \leq 0 \end{cases}$$

9. $$\begin{cases} \min\ -x_1^2 - 2\,x_2^2 \\ -x_1 + 1 \leq 0 \\ -x_2 + 1 \leq 0 \\ x_1 + x_2 - 6 \leq 0 \end{cases}$$

10. Find the largest circle contained in the triangle

$$T := \{x \in \mathbb{R}^2 : \; x_1 \geq 0, \quad x_2 \geq 0, \quad 3\,x_1 + 4\,x_2 \leq 12\}.$$

11. Find the distance between the circle $\mathcal{C} = \{x \in \mathbb{R}^2 : \; x_1^2 + x_2^2 \leq 1\}$ and the line $r = \{x \in \mathbb{R}^2 : \; x_1 - x_2 = 2\}$.

12. Given the line

$$r = \{x \in \mathbb{R}^3 : \; x_1 - x_2 + x_3 = 4, \quad x_1 + x_2 + x_3 = 8\},$$

find the point in r closest to the origin.

13. Find the projection of a point $z \in \mathbb{R}^n$ on the ball with center a and radius R.

14. Find the projection of a point $z \in \mathbb{R}^n$ on the box $\{x \in \mathbb{R}^n : \; \ell_i \leq x_i \leq u_i\}$.

15. Find the distance between two parallel hyperplanes

$$H_1 = \{x \in \mathbb{R}^n : \; a^\mathsf{T} x = b_1\}, \qquad H_2 = \{x \in \mathbb{R}^n : \; a^\mathsf{T} x = b_2\}, \qquad b_1 \neq b_2.$$

16. Find the distance between two balls

$$B_1 = \{x \in \mathbb{R}^n : \; \|x - a\| \leq R_1\}, \qquad B_2 = \{x \in \mathbb{R}^n : \; \|x - b\| \leq R_2\}.$$