Exam of Optimization Methods (fac-simile)

1. Consider the following optimization problem:

$$\begin{cases} \min \ -x_1^2 - x_2^2 + 12x_1 + 4x_2 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ x_1 + x_2 \leq 10 \end{cases}$$

   a) Do global optimal solutions exist? Why?
   b) Is it a convex problem? Why?
   c) Does the Abadie constraints qualification hold in any feasible point?
   d) Is the point $(4, 4)$ a local minimum? Why?
   e) Find all the solutions of the KKT system.
   f) Find all local minima and global minima.

2. Consider the following optimization problem:

$$\begin{cases} \min \ x_1^2 + x_1x_2 + x_2^2 + x_3^2 - 9x_1 - 8x_2 - 2x_3 \\ 0 \leq x_1 \leq 2 \\ 0 \leq x_2 \leq 1 \\ 0 \leq x_3 \leq 2 \end{cases}$$

   a) Do global optimal solutions exist? Why?
   b) Is it a convex problem? Why?
   c) Solve the problem by using the active-set method starting from the point $(2, 1, 0)$. What is the global minimum? How many iterations are needed?
   d) Solve the problem by using the Frank-Wolfe method with exact line search, tolerance 10^{-3} and starting from the point $(0, 0, 0)$. What is the global minimum? How many iterations are needed?
   e) Solve the problem by means of the penalty method with $\tau = 0.5$, $\varepsilon_0 = 1$ and $\min(b - Ax) > -10^{-4}$ as stopping criterion. What is the global minimum? How many iterations are needed?
   Hint: Solve the penalized problems with the `fminunc` *function, use* $(0, 0, 0)$ *as starting point and the following options:*
   ```
   options = optimoptions('fminunc','GradObj','on',...
   'Algorithm','quasi-newton','Display','off').
   ```

3. Consider the following unconstrained multiobjective problem:

$$\begin{cases} \min \ \left((x_1+1)^2+(x_2-7)^2 \ , \ (x_1-3)^2+(x_2-8)^2 \ , \ (x_1-4)^2+(x_2-2)^2 \right) \\ x \in \mathbb{R}^2 \end{cases}$$

a) Is it a convex problem? Why?

b) Do Pareto minima exist? Why?

c) Find all the weak Pareto minima by using the first-order optimality conditions.

d) Find all the Pareto minima by using the scalarization method.

e) Find the ideal point.

4. Consider the following bimatrix game:

$$C_1 = \begin{pmatrix} 1 & 8 & 5 \\ 2 & 9 & 6 \\ 7 & 3 & 4 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 5 & 4 & 6 \\ 2 & 8 & 9 \\ 1 & 7 & 3 \end{pmatrix}$$

a) Find strictly dominated strategies (if any) and reduce the cost matrices accordingly.

b) Do pure strategies Nash equilibria exist? Why?

c) Evaluate the gap function at the point (x, y), where $x = (1/2, 1/2)$ and $y = (1/3, 2/3)$, in order to check if it is a Nash equilibrium.

d) Plot the polyhedra P and Q related to the game and find all the mixed strategies Nash equilibria.

e) Find the KKT multipliers related to the mixed strategies Nash equilibrium.

Solutions

1. a) Yes. The existence of a global minimum is guaranteed by the Weierstrass theorem.

   b) No. The objective function is not convex because its hessian matrix is not positive semidefinite.

   c) Yes, because the constraints are linear.

   d) No, because $(4, 4)$ does not solve the KKT system.

   e) The solutions of the KKT system are:
   $x = (6, 0)$, $\lambda = (0, 4, 0)$,
   $x = (6, 2)$, $\lambda = (0, 0, 0)$,
   $x = (0, 0)$, $\lambda = (12, 4, 0)$,
   $x = (0, 2)$, $\lambda = (12, 0, 0)$,
   $x = (7, 3)$, $\lambda = (0, 0, 2)$,
   $x = (10, 0)$, $\lambda = (0, 12, 8)$,
   $x = (0, 10)$, $\lambda = (28, 0, 16)$.

   f) The global minimum is $(0, 10)$. The local minima are $(0, 0)$ and $(10, 0)$.

2. a) Yes. The existence of a global minimum is guaranteed by the Weierstrass theorem.

   b) Yes. The constraints are linear and the objective function is convex because its hessian matrix is positive semidefinite.

   c) After 2 iterations the active-set method finds the global minimum $(2, 1, 1)$.

   d) After 2 iterations the Frank-Wolfe method finds the global minimum $(2, 1, 1)$.

   e) After 16 iterations the penalty method finds the approximated global minimum $(2.000061, 1.000061, 1.000000)$

3. a) Yes, the objective functions are convex.

   b) Pareto minima exist because the objective functions are strongly convex (and hence coercive).

   c) The set of weak Pareto minima is the triangle with vertices $(-1, 7)$, $(3, 8)$ and $(4, 2)$.

   d) The set of Pareto minima coincides with the set of weak Pareto minima.

   Since the problem is convex, any weak Pareto minimum is the optimal solution of a scalarized problem (P_α) for some $\alpha \geq 0$. Moreover, for any $\alpha \geq 0$ the scalarized problem (P_α) has a unique optimal solution since its objective function is strongly convex, hence such optimal solution is a Pareto minimum. Therefore, any weak Pareto minimum is a Pareto minimum.

   e) The ideal point is $(0, 0, 0)$.

4. a) Row 2 and column 3 can be deleted from matrices C_1 and C_2 because they are strictly dominated strategies.

   b) No.

   c) The value of the gap function is $7/3$, hence $x = (1/2, 1/2)$, $y = (1/3, 2/3)$ is not a Nash equilibrium.

   d) There is a unique mixed strategies Nash equilibrium: $x = (6/7, 1/7)$, $y = (5/11, 6/11)$.

   e) The KKT multipliers are $\mu^1 = -53/11$, $\mu^2 = -31/7$ [by using the matrices C_1 and C_2 obtained at point a)].