Exam of Optimization Methods – June 10, 2019

1. Consider the following optimization problem:

$$\begin{cases} \min \ x_1 - x_2 \\ x_1^2 + x_2^2 - 8 \leq 0 \\ -x_1^2 - x_2^2 + 2 \leq 0 \end{cases}$$

   a) Do global optimal solutions exist? Why?
   b) Is it a convex problem? Why?
   c) Does the Abadie constraints qualification hold in any feasible point?
   d) Find all the solutions of the KKT system.
   e) Find all local minima and global minima.
   f) Find the objective function and constraints of the Lagrangian dual problem.
   g) Solve the Lagrangian dual problem. Is the strong duality property satisfied?

2. Consider the following unconstrained optimization problem:

$$\begin{cases} \min \ x_1^2 + 2x_2^2 + 2x_3^2 + 3x_4^2 - x_1x_2 + x_1x_3 + x_2x_4 + 2x_3x_4 - x_1 + x_2 - x_3 + x_4 \\ \\ x \in \mathbb{R}^4 \end{cases}$$

   a) Is it a convex problem? Why?
   b) Do global minima exist? Why?
   c) Is the global minimum unique? Why?
   d) Solve the problem by means of the gradient method with exact line search starting from the point $(0,0,0,0)$ and using $\|\nabla f(x)\| < 10^{-6}$ as stopping criterion. What is the global minimum? What is the optimal value? How many iterations are needed?
   e) Solve the problem by means of the gradient method with inexact line search setting $\alpha = 0.1$, $\gamma = 0.5$, $\bar{t} = 1$, starting from the point $(0,0,0,0)$ and using $\|\nabla f(x)\| < 10^{-6}$ as stopping criterion. What is the global minimum? How many iterations are needed?
   f) Solve the problem by means of the conjugate gradient method starting from the point $(0,0,0,0)$ and using $\|\nabla f(x)\| < 10^{-6}$ as stopping criterion. What is the global minimum? Write the vector x found at each iteration.

3. Consider the following multi-objective problem:

$$\begin{cases} \min \ \big(x_1 + 3x_2 \ , \ -2x_1 - 2x_2\big) \\ x_1 + x_2 \leq 10 \\ -x_1 + x_2 \leq 10 \\ -x_2 \leq 0 \end{cases}$$

a) Is it a convex problem? Why?

b) Do Pareto minima exist? Why?

c) Is the point $(0, 0)$ a Pareto minimum? Why?

d) Find all the Pareto minima by using the scalarization method.

e) Find all the weak Pareto minima by using the scalarization method.

f) Find the ideal point and apply the goal method with $\| \cdot \|_2$.

4. Consider the following bimatrix game:

$$C_1 = \begin{pmatrix} 4 & 5 & 3 \\ 5 & 1 & 2 \\ 5 & 6 & 1 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 7 & 8 & 8 \\ 2 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix}$$

a) Find strictly dominated strategies (if any) and reduce the cost matrices accordingly.

b) Do pure strategies Nash equilibria exist? Why?

c) Given the point (x, y), where $x = (2/3, 1/3)$ and $y = (2/3, 1/3)$, evaluate the gap function ψ at (x, y) in order to check if it is a Nash equilibrium.

d) Plot the polyhedra P and Q related to the game and find all the mixed strategies Nash equilibria.

e) Find the KKT multipliers related to the mixed strategies Nash equilibrium.