Multiobjective optimization

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Optimization Methods
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Pareto order

In a multiobjective optimization problem the objective function f is a vector of p elements: $f(x) = (f_1(x), f_2(x), \dots, f_p(x))$.

There are often conflicting objectives $\longrightarrow$ definition of *optimality* is not obvious. We need to define an order in $\mathbb{R}^p$.

Pareto order

Given $x, y \in \mathbb{R}^p$, we say that

$$x \geq y \quad \Longleftrightarrow \quad x_i \geq y_i \quad \text{for any } i = 1, \dots, p.$$

This relation is a partial order in $\mathbb{R}^p$: it is

- reflexive: $x \geq x$
- asymmetric: if $x \geq y$ and $y \geq x$ then $x = y$
- transitive: if $x \geq y$ and $y \geq z$ then $x \geq z$

but it is not a total order: if $x = (1, 4)$ and $y = (3, 2)$ then $x \not\geq y$ and $y \not\geq x$

Minimum definitions for a set of vectors

Definition Given a subset $A \subseteq \mathbb{R}^p$, we say

- $x \in A$ is a Pareto ideal minimum (or ideal efficient point) of A if $y \geq x$ for any $y \in A$.
- $x \in A$ is a Pareto minimum (or efficient point) of A if there is no $y \in A$, $y \neq x$ such that $x \geq y$.
- $x \in A$ is a Pareto weak minimum (or weakly efficient point) of A if there is no $y \in A$, $y \neq x$ such that $x > y$, i.e., $x_i > y_i$ for any $i = 1, \ldots, p$.

$IMin(A)$, $Min(A)$ and $WMin(A)$ denote the set of ideal minima, minima, weak minima of A, respectively.

Example 1. $A = \{x \in \mathbb{R}^2 : \; x_1 \geq 0, \; x_2 \geq 0, \; x_1 + x_2 \leq 1\}$.
$IMin(A) = Min(A) = \{(0,0)\}$, $WMin(A) = \{x \in A : \; x_1 = 0 \text{ or } x_2 = 0\}$.

Example 2. $B = \{x \in \mathbb{R}^2 : \; 0 \leq x_1 \leq 2, \; 0 \leq x_2 \leq 2, \; x_1 + x_2 \geq 1\}$.
$IMin(B) = \emptyset$, $Min(B) = \{x \in B : x_1 + x_2 = 1\}$,
$WMin(B) = \{x \in B : x_1 = 0 \text{ or } x_2 = 0 \text{ or } x_1 + x_2 = 1\}$.

Proposition. $IMin(A) \subseteq Min(A) \subseteq WMin(A)$.
If $IMin(A) \neq \emptyset$, then $IMin(A) = Min(A) = \{\bar{x}\}$.

Minimum definitions for an optimization problem

Definition

Given a multiobjective optimization problem

$$\begin{cases} \min \ f(x) = (f_1(x), f_2(x), \dots, f_p(x)) \\ x \in \Omega \end{cases} \tag{P}$$

- $x^* \in \Omega$ is a Pareto ideal minimum of (P) if $f(x^*)$ is an Pareto ideal minimum of $f(\Omega)$, i.e., $f(x) \geq f(x^*)$ for any $x \in \Omega$.
- $x^* \in \Omega$ is a Pareto minimum of (P) if $f(x^*)$ is a Pareto minimum of $f(\Omega)$, i.e., if there is no $x \in \Omega$ such that

$$\begin{array}{ll} f_i(x^*) \geq f_i(x) & \text{for any } i = 1, \dots, p, \\ f_j(x^*) > f_j(x) & \text{for some } j \in \{1, \dots, p\}. \end{array}$$

- $x^* \in \Omega$ is a Pareto weak minimum of (P) if $f(x^*)$ is a Pareto weak minimum of $f(\Omega)$, i.e., if there is no $x \in \Omega$ such that

$$f_i(x^*) > f_i(x) \qquad \text{for any } i = 1, \dots, p.$$

Minimum definitions for an optimization problem

Example. Consider

$$\begin{cases} \min\ (x_1 - x_2,\ -2x_1 + x_2) \\ x_1 \leq 1 \\ -x_1 \leq 0 \\ -x_1 + x_2 \leq 2 \\ 2x_1 - x_2 \leq 0 \end{cases} \tag{P}$$

The image $f(\Omega) = \{(y_1, y_2) :\ y_1 = x_1 - x_2,\ y_2 = -2x_1 + x_2,\ x \in \Omega\}$.
We obtain $x_1 = -y_1 - y_2$ and $x_2 = -2y_1 - y_2$, hence
$f(\Omega) = \{(y_1, y_2) :\ -y_1 - y_2 \leq 1,\ y_1 + y_2 \leq 0,\ -y_1 \leq 2,\ -y_2 \leq 0\}$.
$IMin(f(\Omega)) = \emptyset$. $Min(f(\Omega)) = \{y \in f(\Omega) :\ -y_1 - y_2 = 1\}$, thus

$$\{\text{minima of (P)}\} = \{x \in \Omega :\ -x_1 + x_2 + 2x_1 - x_2 = 1\} = \{x \in \Omega :\ x_1 = 1\}.$$

$WMin(f(\Omega)) = \{y \in f(\Omega) :\ -y_1 - y_2 = 1 \text{ or } y_1 = -2 \text{ or } y_2 = 0\}$, thus

$$\{\text{weak minima of (P)}\} = \{x \in \Omega :\ x_1 = 1 \text{ or } x_1 - x_2 = -2 \text{ or} - 2x_1 + x_2 = 0\}.$$

Existence results

Generalized Weierstrass Theorem

If f_i is continuous for any $i = 1 \dots, p$ and Ω is closed and bounded, then there exists a minimum of (P).

Theorem

If f_i is continuous for any $i = 1 \dots, p$, Ω is closed and there are $v \in \mathbb{R}$ and $j \in \{1, \dots, p\}$ such that the sublevel set

$$\{x \in \Omega : \ f_j(x) \leq v\}$$

is nonempty and bounded, then there exists a minimum of (P).

Corollary. If f_i is continuous for any $i = 1 \dots, p$, Ω is closed and f_j is coercive for some $j \in \{1, \dots, p\}$, then there exists a minimum of (P).

Optimality conditions

Theorem

$x^* \in \Omega$ is a minimum of (P) if and only if the auxiliary optimization problem

$$
\begin{cases}
\max \ \sum_{i=1}^{p} \varepsilon_i \\
f_i(x) + \varepsilon_i \leq f_i(x^*) & \forall \ i = 1, \dots, p \\
x \in \Omega \\
\varepsilon \geq 0
\end{cases}
$$

has optimal value equal to 0.

Theorem

$x^* \in \Omega$ is a weak minimum of (P) if and only if the auxiliary optimization problem

$$
\begin{cases}
\max \ v \\
v \leq \varepsilon_i & \forall \ i = 1, \dots, p \\
f_i(x) + \varepsilon_i \leq f_i(x^*) & \forall \ i = 1, \dots, p \\
x \in \Omega \\
\varepsilon \geq 0
\end{cases}
$$

has optimal value equal to 0.

Optimality conditions

Exercise 1. Consider the linear multiobjective problem

$$\begin{cases} \min\ (x_1 + 2x_2 - 3x_3\ ,\ -x_1 - x_2 - x_3\ ,\ -4x_1 - 2x_2 + x_3) \\ x_1 + x_2 + x_3 \leq 10 \\ x_3 \leq 5 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Check if the points $u = (5, 0, 5)$, $v = (4, 4, 2)$ and $w = (1, 4, 4)$ are minima or weak minima by solving the corresponding auxiliary problems.

First-order optimality conditions: unconstrained problems

Consider an unconstrained multiobjective problem

$$\begin{cases} \min \ f(x) = (f_1(x), f_2(x), \dots, f_p(x)) \\ x \in \mathbb{R}^n \end{cases} \tag{P}$$

where f_i is continuously differentiable for any $i = 1, \dots, p$.

Necessary optimality condition

If x^* is a weak minimum of (P), then there exists $\xi^* \in \mathbb{R}^p$ such that

$$\begin{cases} \sum_{i=1}^{p} \xi_i^* \nabla f_i(x^*) = 0 \\ \xi^* \geq 0, \quad \sum_{i=1}^{p} \xi_i^* = 1 \end{cases} \tag{S}$$

Sufficient optimality condition

If the problem (P) is convex, i.e., f_i is convex for any $i = 1, \dots, p$, and (x^*, ξ^*) is a solution of the system (S), then x^* is a weak minimum of (P).

First-order optimality conditions: unconstrained problems

Exercise 2. Consider the nonlinear multiobjective problem

$$\begin{cases} \min\ (x_1^2 + x_2^2,\ (x_1 - 1)^2 + (x_2 - 1)^2) \\ x \in \mathbb{R}^2 \end{cases}$$

1. Find the set of weak minima exploiting the first-order optimality conditions.
2. Find the set of minima.

First-order optimality conditions: constrained problems

Consider a constrained multiobjective problem

$$\begin{cases} \min \ f(x) = (f_1(x), f_2(x), \ldots, f_p(x)) \\ g_j(x) \leq 0 \qquad \forall\ j = 1, \ldots, m \\ h_k(x) = 0 \qquad \forall\ k = 1, \ldots, q \end{cases} \tag{P}$$

where f_i, g_j and h_k are continuously differentiable for any i, j, k.

Necessary optimality condition

If x^* is a weak minimum of (P) and ACQ holds at x^*, then there exist $\xi^* \in \mathbb{R}^p$, $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^q$ such that $(x^*, \xi^*, \lambda^*, \mu^*)$ solves the KKT system

$$\begin{cases} \sum_{i=1}^{p} \xi_i^* \nabla f_i(x^*) + \sum_{j=1}^{m} \lambda_j^* \nabla g_j(x^*) + \sum_{k=1}^{q} \mu_k^* \nabla h_k(x^*) = 0 \\ \xi^* \geq 0, \quad \sum_{i=1}^{p} \xi_i^* = 1 \\ \lambda^* \geq 0 \\ \lambda_j^* \, g_j(x^*) = 0 \qquad \forall\ j = 1, \ldots, m \end{cases}$$

Sufficient optimality condition

If (P) is convex, i.e., f_i convex, g_j convex and h_k affine, and $(x^*, \xi^*, \lambda^*, \mu^*)$ solves the KKT system, then x^* is a weak minimum of (P).

First-order optimality conditions: constrained problems

Exercise 3. Consider the nonlinear multiobjective problem

$$\begin{cases} \min \ (x_1 + x_2, \ -x_1 + x_2) \\ x_1^2 + x_2^2 - 1 \leq 0 \end{cases}$$

1. Find the set of weak minima by solving the KKT system.
2. Find the set of minima.

Scalarization method

Define a vector of weights associated to the objectives:

$$\alpha = (\alpha_1, \dots, \alpha_p) \geq 0 \quad \text{such that} \quad \sum_{i=1}^{p} \alpha_i = 1$$

and consider the following scalar optimization problem

$$\begin{cases} \min \displaystyle\sum_{i=1}^{p} \alpha_i f_i(x) \\ x \in \Omega \end{cases} \qquad (\mathrm{P}_\alpha)$$

Let S_α be the set of optimal solutions of (P_α).

Theorem

- $\displaystyle\bigcup_{\alpha \geq 0} S_\alpha \subseteq \{ \text{ weak minima of (P) } \}$
- $\displaystyle\bigcup_{\alpha > 0} S_\alpha \subseteq \{ \text{ minima of (P) } \}$
- If $\alpha \geq 0$ and x^* is the unique optimal solution of (P_α), then x^* is a minimum of (P).

Scalarization method

Solving (P_α) for any possible choice of α does not allow finding all the minima and weak minima.

Example. Consider

$$\begin{cases} \min\ (x_1,\ x_2) \\ x_1^2 + x_2^2 - 4 \le 0 \\ -x_1^2 - x_2^2 + 1 \le 0 \\ x_1 \ge 0,\ x_2 \ge 0 \end{cases}$$

$$\bigcup_{\alpha \ge 0} S_\alpha = \{(0, x_2):\ x_2 \in [1,2]\} \cup \{(x_1, 0):\ x_1 \in [1,2]\},$$

while

$$\{ \text{ weak minima of (P) } \} =$$
$$\{(0, x_2):\ x_2 \in [1,2]\} \cup \{(x_1, 0):\ x_1 \in [1,2]\} \cup \{x \in \mathbb{R}^2_+ :\ x_1^2 + x_2^2 = 1\}.$$

Furthermore,

$$\bigcup_{\alpha > 0} S_\alpha = \{(0,1), (1,0)\},$$

while

$$\{ \text{ minima of (P) } \} = \{x \in \mathbb{R}^2_+ :\ x_1^2 + x_2^2 = 1\}.$$

Scalarization method

Theorem

- If (P) is convex, then $\bigcup_{\alpha \geq 0} S_\alpha = \{$ weak minima of (P) $\}$
- If (P) is linear, then $\bigcup_{\alpha > 0} S_\alpha = \{$ minima of (P) $\}$

Exercise 4. Consider the linear multiobjective problem

$$\begin{cases} \min \ (x_1 - x_2 \ , \ x_1 + x_2) \\ -2\, x_1 + x_2 \leq 0 \\ -x_1 - x_2 \leq 0 \\ 5\, x_1 - x_2 \leq 6 \end{cases}$$

Find the set of minima and weak minima by means of the scalarization method.

Scalarization method

Exercise 5. Consider the nonlinear multiobjective problem

$$\begin{cases} \min\ (x_1^2 + x_2^2 + 2x_1 - 4x_2\ ,\ x_1^2 + x_2^2 - 6x_1 - 4x_2) \\ -x_2 \leq 0 \\ -2x_1 + x_2 \leq 0 \\ 2x_1 + x_2 \leq 4 \end{cases}$$

a) Find a subset of minima by means of the scalarization method.

b) Find the set of all weak minima by means of the scalarization method.

c) Find the set of all minima.

Goal method

In the objective space $\mathbb{R}^p$ define the ideal point z as

$$z_i = \min_{x \in \Omega} f_i(x), \qquad \forall\ i = 1, \dots, p.$$

Since very often (P) has no ideal minimum, i.e., $z \notin f(\Omega)$, we want to find the point of $f(\Omega)$ which is as close as possible to z:

$$\begin{cases} \min \ \|f(x) - z\|_s \\ x \in \Omega \end{cases} \qquad \text{with} \ \ s \in [1, +\infty]. \tag{G}$$

Theorem

- If $s \in [1, +\infty)$, then any optimal solution of (G) is a minimum of (P).
- If $s = +\infty$, then any optimal solution of (G) is a weak minimum of (P).

Goal method

Assume that (P) is a linear multiobjective optimization problem, i.e.,

$$\begin{cases} \min \ Cx \\ Ax \leq b \end{cases} \tag{P}$$

where C is a $p \times n$ matrix.

If $s = 2$, then (G) is equivalent to a quadratic programming problem:

$$\begin{cases} \min \ \frac{1}{2}\|Cx - z\|_2^2 = \frac{1}{2}x^{\mathsf{T}}C^{\mathsf{T}}Cx - x^{\mathsf{T}}C^{\mathsf{T}}z + \frac{1}{2}z^{\mathsf{T}}z \\ Ax \leq b \end{cases} \tag{G$_2$}$$

Goal method

If $s = 1$, then (G) is equivalent to the linear programming problem

$$\begin{cases} \min\limits_{x,y} \ \sum\limits_{i=1}^{p} y_i \\ y_i \geq C_i x - z_i & \forall\ i = 1, \dots, p \\ y_i \geq z_i - C_i x & \forall\ i = 1, \dots, p \\ Ax \leq b \end{cases} \tag{G_1}$$

If $s = +\infty$, then (G) is equivalent to the linear programming problem

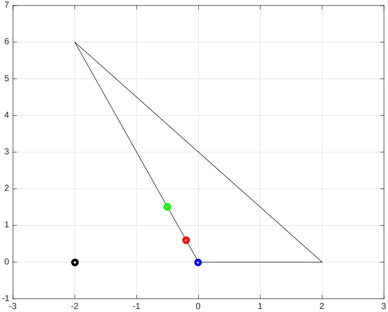
$$\begin{cases} \min\limits_{x,y} \ y \\ y \geq C_i x - z_i & \forall\ i = 1, \dots, p \\ y \geq z_i - C_i x & \forall\ i = 1, \dots, p \\ Ax \leq b \end{cases} \tag{G_∞}$$

Goal method

Example. Consider

$$\begin{cases} \min\ (x_1 - x_2\ ,\ x_1 + x_2) \\ -2\,x_1 + x_2 \leq 0 \\ -x_1 - x_2 \leq 0 \\ 5\,x_1 - x_2 \leq 6 \end{cases}$$

The set $f(\Omega)$ is shown in the figure below.
The ideal point is $z = (-2, 0)$ (black point).
The optimal solution of (G_2) is $x^* = (1/5, 2/5)$ and $f(x^*) = (-1/5, 3/5)$.
The optimal solution of (G_1) is $\widetilde{x} = (0, 0)$ and $f(\widetilde{x}) = (0, 0)$.
The optimal solution of (G_∞) is $\bar{x} = (1/2, 1)$ and $f(\bar{x}) = (-1/2, 3/2)$.



Goal method

Exercise 6. Consider the linear multiobjective problem

$$\begin{cases} \min\ (x_1 + 2x_2 - 3x_3\ ,\ -x_1 - x_2 - x_3\ ,\ -4x_1 - 2x_2 + x_3) \\ x_1 + x_2 + x_3 \leq 10 \\ x_3 \leq 5 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

a) Find the ideal point.

b) Apply the goal method with $s = 1$.

c) Apply the goal method with $s = 2$.

d) Apply the goal method with $s = +\infty$. Is the found point a minimum?