Lagrangian duality

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Lagrangian relaxation

Consider the general optimization problem

$$\begin{cases} \min \ f(x) \\ g(x) \leq 0 \\ h(x) = 0 \end{cases} \qquad \text{(P)}$$

where $x \in \mathcal{D}$ and the optimal value is $v(P)$.

Lagrangian function $L : \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ is

$$L(x, \lambda, \mu) := f(x) + \sum_{i=1}^{m} \lambda_i \, g_i(x) + \sum_{j=1}^{p} \mu_j \, h_j(x)$$

Lagrangian relaxation and dual function

Definition

Given $\lambda \geq 0$ and $\mu \in \mathbb{R}^p$, the problem

$$\begin{cases} \min \ L(x, \lambda, \mu) \\ x \in \mathcal{D} \end{cases}$$

is called Lagrangian relaxation of (P) and $\varphi(\lambda, \mu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \mu)$ is the Lagrangian dual function.

Dual function φ

- is concave because inf of linear functions w.r.t (λ, μ)
- can be $-\infty$
- can be not differentiable

Lagrangian relaxation and dual function

Theorem
Given $\lambda \geq 0$ and $\mu \in \mathbb{R}^p$, we have

$$\varphi(\lambda, \mu) \leq v(P).$$

Proof. If $x \in \Omega$, i.e. $g(x) \leq 0, h(x) = 0$, then

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i \, g_i(x) \leq f(x),$$

hence

$$\varphi(\lambda, \mu) = \min_{x \in \mathcal{D}} \; L(x, \lambda, \mu) \leq \min_{x \in \Omega} \; L(x, \lambda, \mu) \leq \min_{x \in \Omega} \; f(x) = v(P)$$

□

Lagrangian dual problem

Definition

The problem

$$\begin{cases} \max \ \varphi(\lambda, \mu) \\ \lambda \geq 0 \end{cases} \tag{D}$$

is called Lagrangian dual problem of (P).

Dual problem consists in finding the best lower bound of $v(P)$.

Dual problem is a convex problem, even if (P) is not convex.

Lagrangian dual problem

Example 1 - Linear Programming.

Primal problem:

$$\begin{cases} \min \ c^{\mathsf{T}} x \\ Ax \geq b \end{cases} \qquad \text{(P)}$$

Lagrangian function: $L(x,\lambda) = c^{\mathsf{T}} x + \lambda^{\mathsf{T}}(b - Ax) = \lambda^{\mathsf{T}} b + (c^{\mathsf{T}} - \lambda^{\mathsf{T}} A)x$
Dual function:

$$\varphi(\lambda) = \min_{x \in \mathbb{R}^n} L(x,\lambda) = \begin{cases} -\infty & \text{if } c^{\mathsf{T}} - \lambda^{\mathsf{T}} A \neq 0 \\ \lambda^{\mathsf{T}} b & \text{if } c^{\mathsf{T}} - \lambda^{\mathsf{T}} A = 0 \end{cases}$$

Dual problem:

$$\begin{cases} \max \ \varphi(\lambda) \\ \lambda \geq 0 \end{cases} \longrightarrow \begin{cases} \max \ \lambda^{\mathsf{T}} b \\ \lambda^{\mathsf{T}} A = c^{\mathsf{T}} \\ \lambda \geq 0 \end{cases} \qquad \text{(D)}$$

is a linear programming problem.

Exercise. What is the dual of (D)?

Lagrangian dual problem

Example 2 - Least norm solution of linear equations.

Primal problem:

$$\begin{cases} \min \ x^{\mathsf{T}}x \\ Ax = b \end{cases} \tag{P}$$

Lagrangian function: $L(x,\mu) = x^{\mathsf{T}}x + \mu^{\mathsf{T}}(Ax - b)$.
Dual function: $\varphi(\mu) = \min_{x\in\mathbb{R}^n} L(x,\mu)$.
$L(x,\mu)$ is quadratic and strongly convex w.r.t x, thus

$$\nabla_x L = 2x + A^{\mathsf{T}}\mu = 0 \Longleftrightarrow x = -\frac{1}{2}A^{\mathsf{T}}\mu,$$

hence $\varphi(\mu) = -\frac{1}{4}\mu^{\mathsf{T}}AA^{\mathsf{T}}\mu - b^{\mathsf{T}}\mu$.
Dual problem:

$$\begin{cases} \max \ -\frac{1}{4}\mu^{\mathsf{T}}AA^{\mathsf{T}}\mu - b^{\mathsf{T}}\mu \\ \mu \in \mathbb{R}^p \end{cases} \tag{D}$$

is an unconstrained convex quadratic programming problem.

Lagrangian dual problem

Exercise. Find the dual problem of a generic quadratic programming problem

$$\begin{cases} \min \ \frac{1}{2} x^{\mathsf{T}} Q x + c^{\mathsf{T}} x \\ Ax \leq b \end{cases} \tag{P}$$

where Q is a symmetric positive definite matrix.

Weak duality

Theorem (weak duality)

For any optimization problem we have $v(D) \leq v(P)$.

Strong duality, i.e., $v(D) = v(P)$, does not hold in general.

Example 1.

$$\begin{cases} \min \ -x^2 \\ x - 1 \leq 0 \\ -x \leq 0 \end{cases} \qquad v(P) = -1$$

$L(x, \lambda) = -x^2 + \lambda_1(x-1) - \lambda_2 x,$

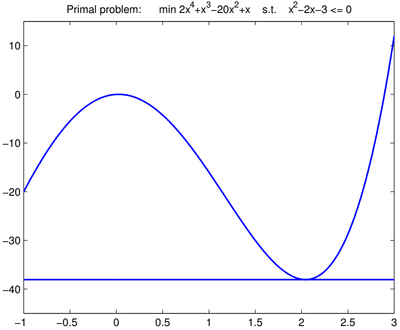
$$\varphi(\lambda) = \min_{x \in \mathbb{R}} L(x, \lambda) = -\infty \qquad \forall\ \lambda \in \mathbb{R}^2,$$

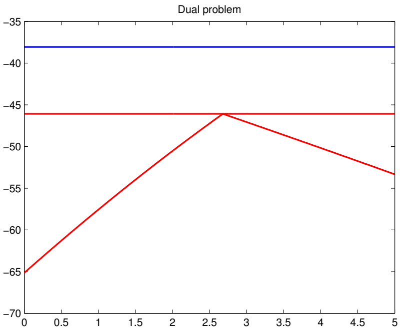
hence $v(D) = -\infty$.

Weak duality

Example 2.

$$\begin{cases} \min \ 2x^4 + x^3 - 20x^2 + x \\ x^2 - 2x - 3 \leq 0 \end{cases}$$





Primal optimal solution $x^* \simeq 2.0427$, $v(P) \simeq -38.0648$.
Dual optimal solution $\lambda^* \simeq 2.68$, $v(D) \simeq -46.0838$.

Strong duality

Theorem (strong duality)

If the problem

$$\begin{cases} \min \ f(x) \\ g(x) \leq 0 \\ h(x) = 0 \end{cases}$$

is convex, there exists an optimal solution x^*, and ACQ holds at x^*, then KKT multipliers (λ^*, μ^*) associated to x^* are an optimal solution of the dual problem and $v(D) = v(P)$.

Proof. $L(x, \lambda, \mu)$ is convex with respect to x, thus

$$v(D) \geq \varphi(\lambda^*, \mu^*) = \min_x L(x, \lambda^*, \mu^*) = L(x^*, \lambda^*, \mu^*) = f(x^*) = v(P) \geq v(D)$$

□

Strong duality

Strong duality can hold also for some nonconvex problems.

Example.

$$\begin{cases} \min \ -x_1^2 - x_2^2 \\ x_1^2 + x_2^2 - 1 \leq 0 \end{cases} \qquad v(P) = -1$$

$$L(x,\lambda) = -x_1^2 - x_2^2 + \lambda(x_1^2 + x_2^2 - 1) = (\lambda - 1)x_1^2 + (\lambda - 1)x_2^2 - \lambda.$$

$$\varphi(\lambda) = \begin{cases} -\infty & \text{if } \lambda < 1 \\ -\lambda & \text{if } \lambda \geq 1 \end{cases}$$

hence $\lambda^* = 1$ is the dual optimum and $v(D) = -1$.

Exercises

Exercise 1. Consider the problem

$$\begin{cases} \min \ \sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} x_i \geq 1 \end{cases}$$

- Discuss existence and uniqueness of optimal solutions
- Find the optimal solution and the optimal value
- Write the dual problem
- Solve the dual problem and check whether strong duality holds

Exercise 2. Given $a, b \in \mathbb{R}$ with $a < b$, consider the problem

$$\begin{cases} \min \ x^2 \\ a \leq x \leq b \end{cases}$$

- Find the optimal solution and the optimal value for any a, b
- Solve the dual problem and check whether strong duality holds