Transition Systems

Computational Models for Complex Systems

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Introduction

Assume we are interested in studying whether a system can reach a bad state

We have seen that

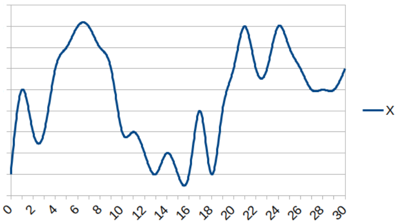
- ODEs allow us to study the "average" behavior of a system
- Stochastic simulation provide us with a number of different possibile behaviors (as many as the simulations we run)

How can we guarantee that the systems will never reach the bad state

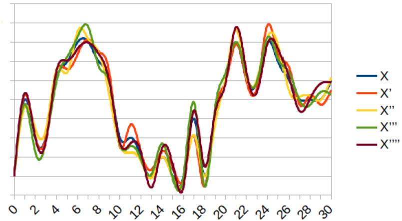
- simulations are not exhaustive

Introduction

For example, assume we want to avoid X to reach value 0



ODEs

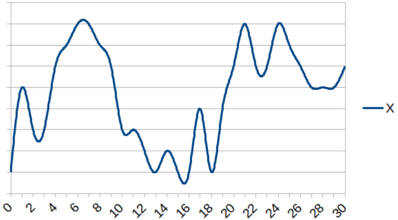


Stochastic Simulation

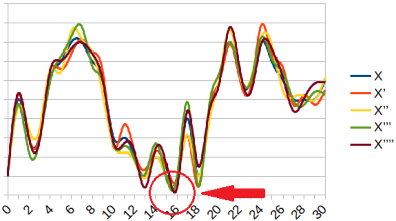
We may run 1000 simulations and conclude that this never happens...

Introduction

For example, assume we want to avoid X to reach value 0



ODEs



Stochastic Simulation

... but maybe this happens very rarely (e.g. once every 10^6 times) and simply it didn't happen in the 1000 simulations!

Introduction

From the 1000 simulations we can conclude that the probability that X reaches 0 is very small (roughly, smaller than $1/1000$), but they cannot guarantee that this will never happen.

In some cases we would like to explore all possible systems behaviors

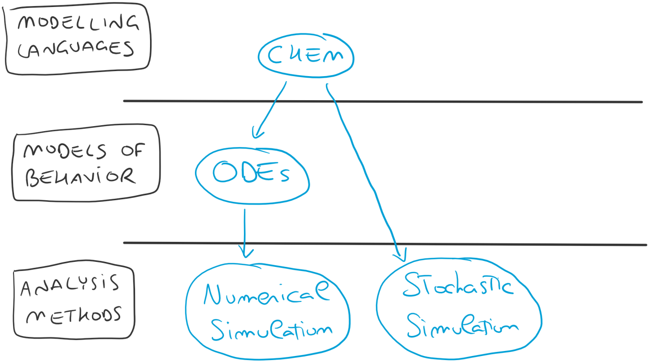
- in order to verify behavioral properties
- e.g. in the study of safety-critical systems

This requires a new way of modeling the system behavior:

- Transition Systems

Roadmap

Where are we going?



Roadmap

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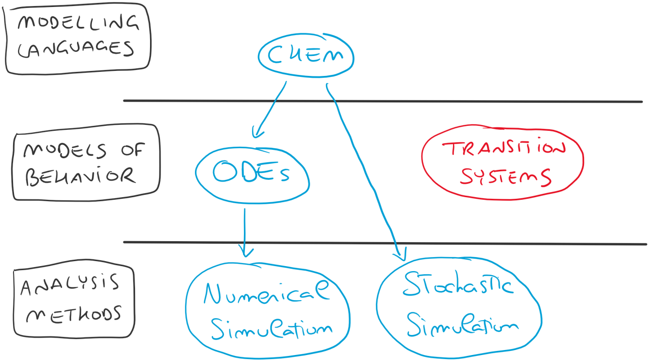

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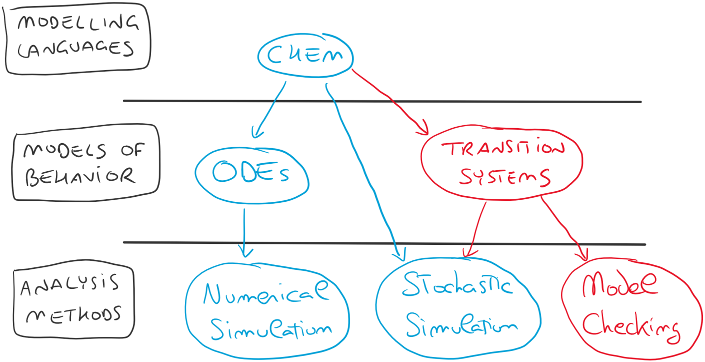
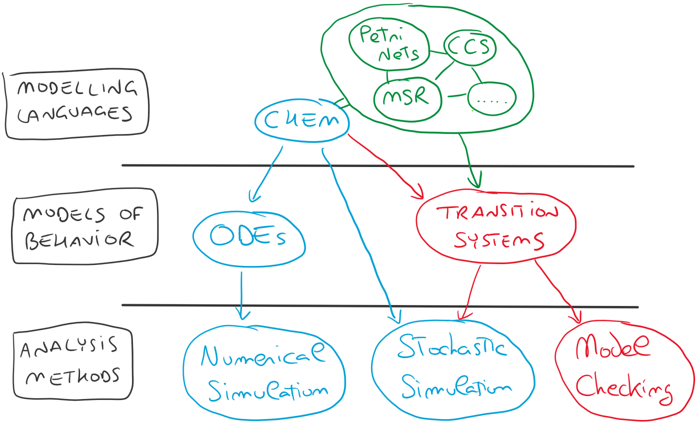


Roadmap

Where are we going?



Roadmap

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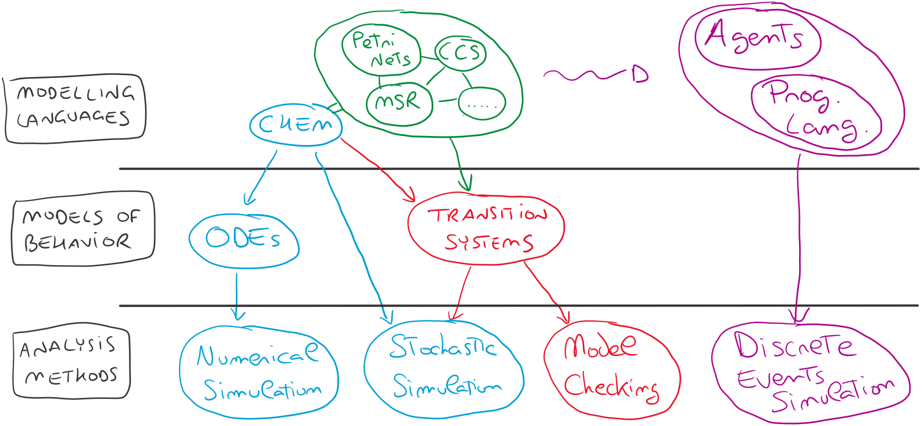


Transition Systems

Definition: Transition System (TS)

A Transition System is a pair $(S, \rightarrow)$ where

- S is a set of states and
- $\rightarrow \subseteq S \times S$ is the transition relation

Given $s, s' \in S$, $(s, s') \in \rightarrow$ is usually denoted as $s \rightarrow s'$. Moreover, usually $s \not\rightarrow$ denotes that there exists no $s' \in S$ such that $s \rightarrow s'$.

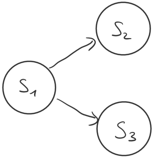
So, a TS is essentially a graph...

A TS aims ad modeling the behavior of a system. Note that:

- the set of states can be infinite (but it is typically assumed to be recursively enumerable)
- transitions describe system state changes
- a state may have more than one outgoing transitions ($s \rightarrow s'$ and $s \rightarrow s''$) capturing non-deterministic behaviors

Transition Systems

Non-deterministic does not mean random!



The fact that $s1$ may evolve either into $s2$ or into $s3$ does not mean that there is a random choice between the two possibilities.

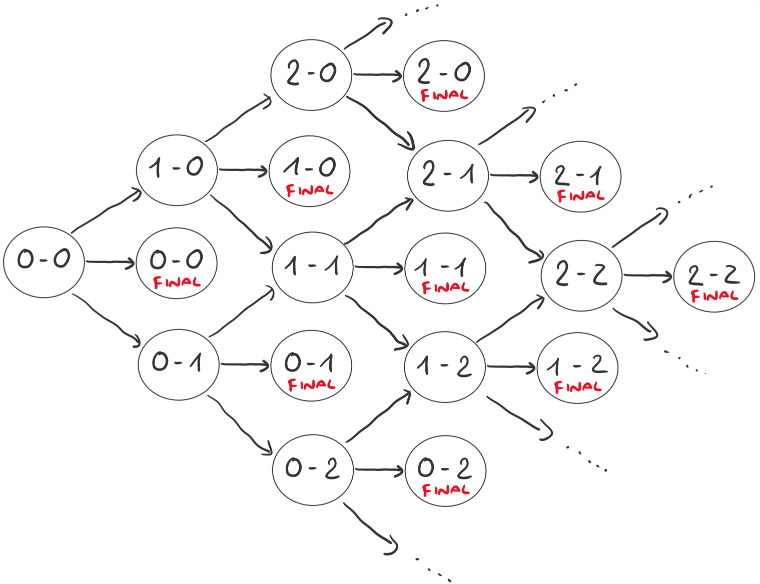
The choice could depend on:

- a timer (e.g. “enter the PIN in 30 sec. or your card will be eaten”)
- a scheduler (e.g. if the two transitions correspond to the progresses of two different concurrent processes)
- a probabilistic choice

In general, non-determinsm is an abstraction of a choice criterion that we simply do not want to model...

Transition Systems

Example: soccer match



Traces

In a TS $(S, \rightarrow)$, often one state $s_0 \in S$ is chosen as initial state

A possibile behavior of the system starting from the initial state s_0 corresponds to a trace of the TS

Definition: Trace

A trace t of a Transition System $(S, \rightarrow)$ with initial state s_0, is a (possibly infinite) sequence of states $t = s_0, s_1, s_2, \ldots$ such that for each s_{i+1} with $i \in \mathbb{N}$ in t it holds $s_i \rightarrow s_{i+1}$.

A few notes:

- s_0 is the minimal trace
- a trace t is maximal if either
  - t is infinite
  - $t = s_0, s_1, \ldots, s_n$ and $s_n \not\rightarrow$

Traces

In the soccer example (with initial state (0-0)):

- (0-0),(1-0),(1-1) is a trace
- (0-0),(1-0),(1-1),(1-1 final) is a maximal trace
- (0-0),(1-0),(2-0),(3-0),... is a maximal trace

Reachability

Traces allow us to define a notion of reachability of states.

Definition: Reachability

A state s of a Transition System $(S, \rightarrow)$ with initial state s_0 is reachable (from the initial state) if there exist $s_1, \ldots, s_n \in S$ such that $s_0, s_1, \ldots, s_n, s$ is a trace.

Very often, reachability of a particular (good or bad) state is the property one wants to verify on a TS

- for example through a Breadth-First Search (BFS), with on-the-fly generation of states (if the graph is huge or infinite)

Kripke Structures

A particular class of Transition Systems are Kripke Structures

In a Kripke Structure, states are characterized by a set of atomic propositions that can be either true or false

Definition: Kripke Structure

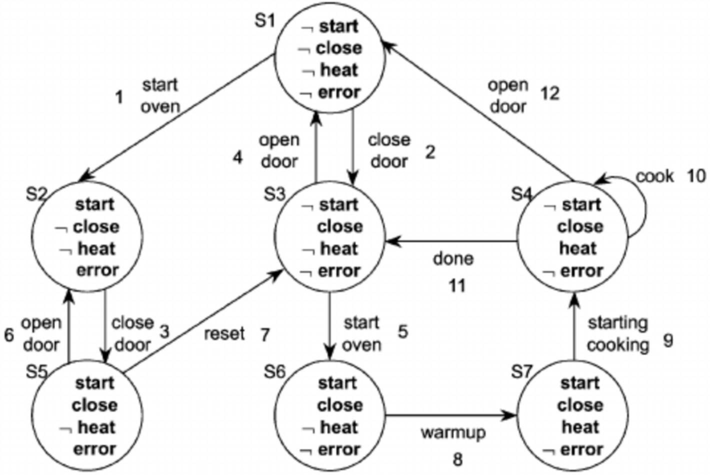
Given a (finite) set of atomic propositions AP, a Kripke Structure K is a Transition System $(S, \rightarrow)$ where $S = \mathcal{P}(AP)$.

Some notes:

- $\mathcal{P}(AP)$ denotes the powerset of AP.
- The interpretation is that an atomic proposition a is contained in a state if and only if it is true in that state.

Kripke Structures

An example: Microwave Oven



Transition Systems over a set of variables

Very often the definition of the state of a transition system is based on a set of variables describing the features of the state

Definition: Transition System over a set of variables

Given a set of variables $\mathbf{X} = \{X_1, \ldots, X_n\}$ and a set of domains $\{D_1, \ldots, D_n\}$ s.t. D_i is the domain of X_i, a Transition System over $\mathbf{X}$ is a Transition System $(S, \rightarrow)$ with $S = D_1 \times \cdots \times D_n$.

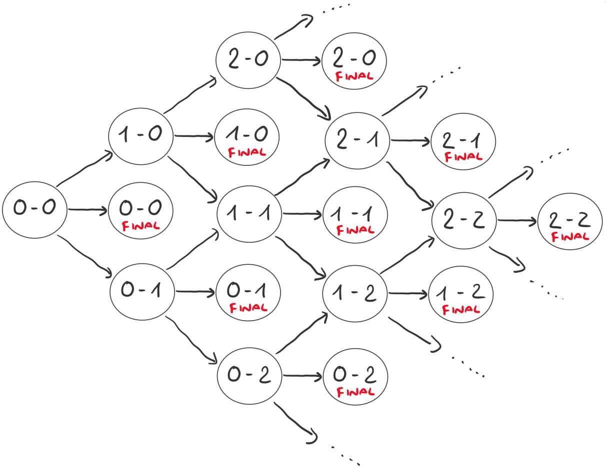
Some notes:

- Each domain D_i should be a recursively enumerable set of values (integers, naturals, rationals, ...) or, better, a finite set of values (bounded integers, bounded rationals, enums, booleans, ...)
- The number of variables impacts significantly on the number of states of the TS (combinatorial explosion)

Transition Systems over a set of variables

Example: soccer match

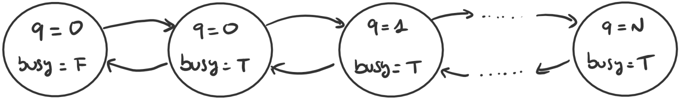
- Variables: `team1` (natural), `team2` (natural), `final` (boolean)



Transition Systems over a set of variables

Example: server with a queue

- if the server is busy, requests are enqueue
- the size of the queue is N
- Variables: `busy` (boolean), `q` (bounded natural)



Example: Kripke structures

- Kripke structures are a particular case of TS over a set of (boolean) variables

Specifying Transition Systems

Transition Systems over a set of variables can be specified by giving a set of transition rules (or if-then rules) having the following form:

```
guard -> update
```

where

- `guard` is a conjunction of conditions on the state variables, each having the form $X_i \ \mathrm{op}\ Exp$ with op a comparison operator.
- `update` is a conjunction of assignements to state variables, each having the form $X_i' = Exp$, with X_i' denoting the new value of X_i

Specifying Transition Systems

```
guard -> update
```

The idea is that the transition relation with contain a transition between each pair of states s_1, s_2 such that:

- s_1 satisfies the `guard`
- s_2 can be obtained by applying to s_1 the assignments described in `update`

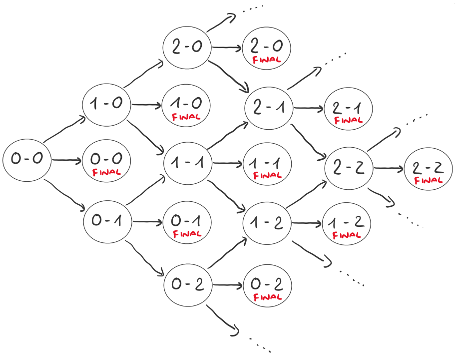
Specifying Transition Systems

Example: soccer match

- Variables: `team1` (natural), `team2` (natural), `final` (boolean)

Specification:

```
final=false -> team1'=team1+1
final=false -> team2'=team2+1
final=false -> final'=true
```



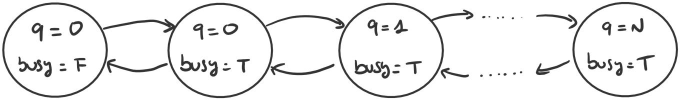
Specifying Transition Systems

Example: server with a queue

- if the server is busy, requests are enqueue
- the size of the queue is N
- Variables: `busy` (boolean), `q` (bounded natural)

Specification:

```
busy=false -> busy'=true
busy=true & q<N -> q'=q+1
q>0 -> q'=q-1
q=0 & busy=true -> busy'=false
```



Labeled Transition Systems

Labeled Transition Systems are an extended version of Transition Systems in which transitions are enriched with labels

Definition: Labeled Transition System (LTS)

A Labeled Transition System (LTS) is a triple $(S, L, \rightarrow)$ where

- S is a set of states,
- L is a set of labels, and
- $\rightarrow \subseteq S \times L \times S$ is a labeled transition relation

Given $s, s' \in S$ and $\ell \in L$, $(s, \ell, s') \in \rightarrow$ is usually denoted $s \xrightarrow{\ell} s'$.
Moreover, usually $s \stackrel{\ell}{\not\rightarrow}$ denotes that there exists no $s' \in S$ such that $s \xrightarrow{\ell} s'$.

LTSs and Concurrent Interactive Systems

LTSs are usually well suited to model the behavior of CONCURRENT INTERACTIVE SYSTEMS in a COMPOSITONAL WAY

- concurrent interactive systems are systems consisting of a number of independent components which may perform some actions sinchronizing with each other or with the environment
- compositional modeling consist in inferring the model of the behavior of the system from the models of the behaviors of the components
- The idea:
    1. specify the LTS of each component
    2. combine the LTSs by taking synchronizations into account

LTSs and Concurrent Interactive Systems

The role of transition labels:

- transition labels describe the action performed by the system (or component) during the transition
- Label τ describe an internal action
  - performed in isolation, without synchronizing with any other component
- Other labels, $a, b, c, \ldots$ describe potential actions the system (or component) could perform by interacting with some other component

LTSs and Concurrent Interactive Systems

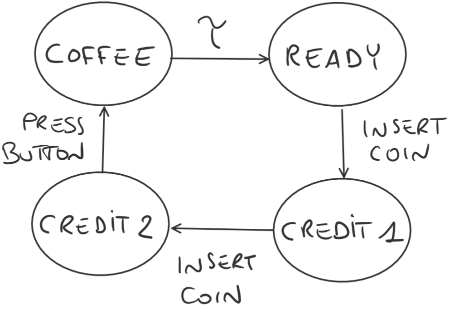
Two approaches to synchronization:

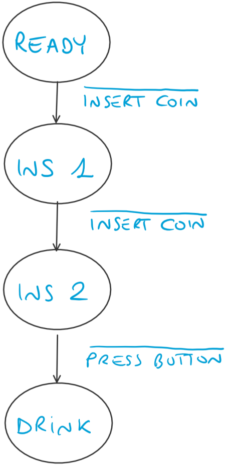
- Binary sinchronization: non-τ actions are split into two sets denoted $\{a, b, c, \ldots\}$ and $\{\overline{a}, \overline{b}, \overline{c}, \ldots\}$.
  - a transition with label a has to be performed together with a transition with label $\overline{a}$ (same symbol, but with overline) of another component
- Global sinchronization: non-τ actions are synchronized among all system components
  - all components having a transition with label a must perform such a transition together

The synchronization of a number of transitions result in a new τ transition.

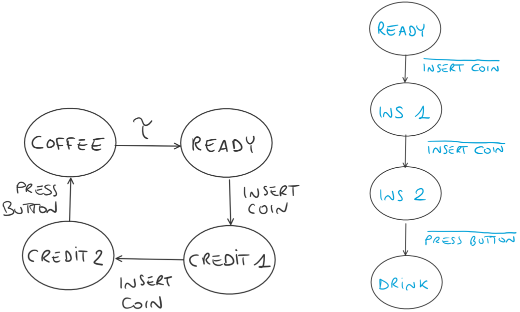
Example: compositional modeling of a coffee machine

LTSs modeling the behaviors of a coffee machine (left) and a user (right)

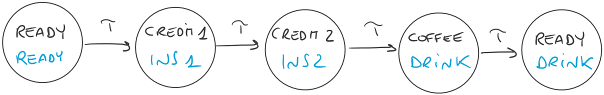




Example: compositional modeling of a coffee machine

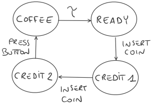


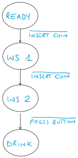
This is the LTS that describes the behaviour of the system consisting of a machine and a user:

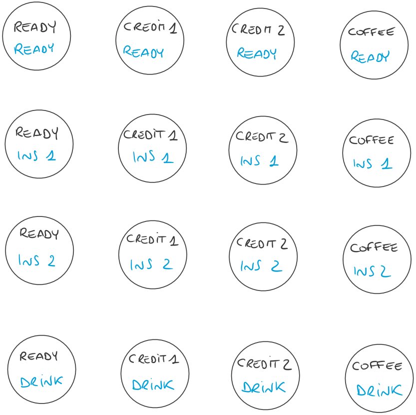


Example: compositional modeling of a coffee machine

How to obtain the LTS of the whole system from the LTSs of the components (machine and user)?



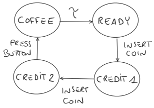


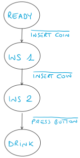


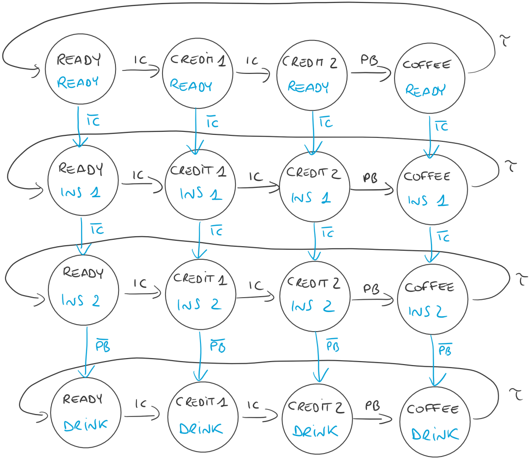
First: consider all possible states by combining states of the components

Example: compositional modeling of a coffee machine

How to obtain the LTS of the whole system from the LTSs of the components (machine and user)?



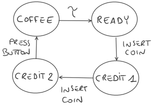


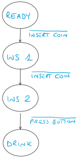


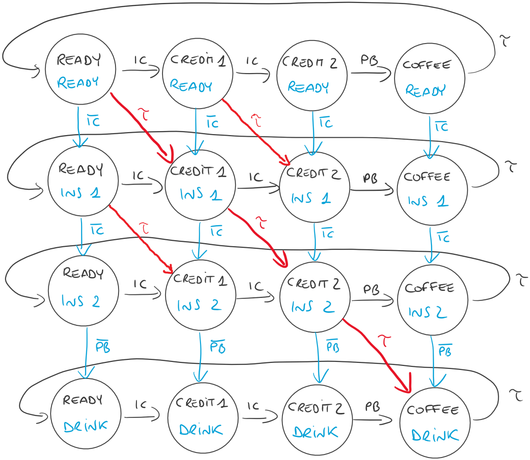
Second: "copy" transitions of the two components

Example: compositional modeling of a coffee machine

How to obtain the LTS of the whole system from the LTSs of the components (machine and user)?



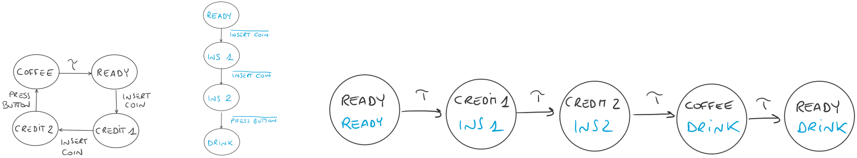




Third: add τ transitions to states in which different components can perform co-actions. The result is the LTS of the whole system

Example: compositional modeling of a coffee machine

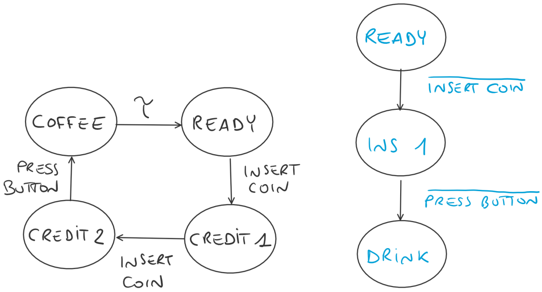
How to obtain the LTS of the whole system from the LTSs of the components (machine and user)?



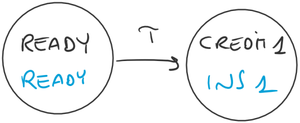
Fourth: if all systems components have been considered, remove the non-τ transitions (and τ transitions unreachable from the initial state)

Example: compositional modeling of a coffee machine

Let's try with a "wrong" user (only one coin)



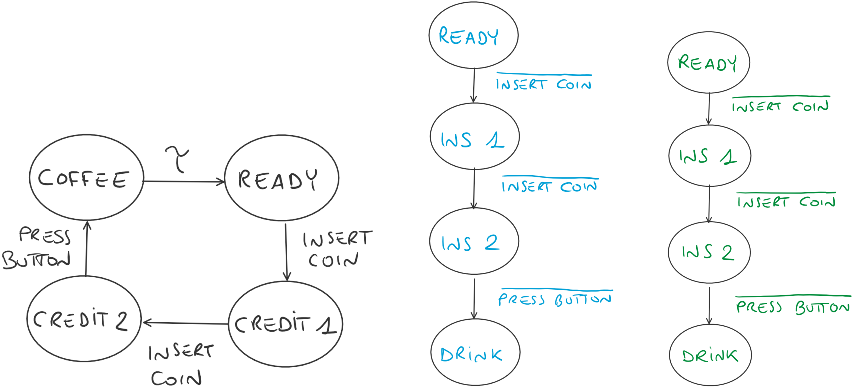
We obtain:



The system reaches a deadlock with the machine waiting for a second coin

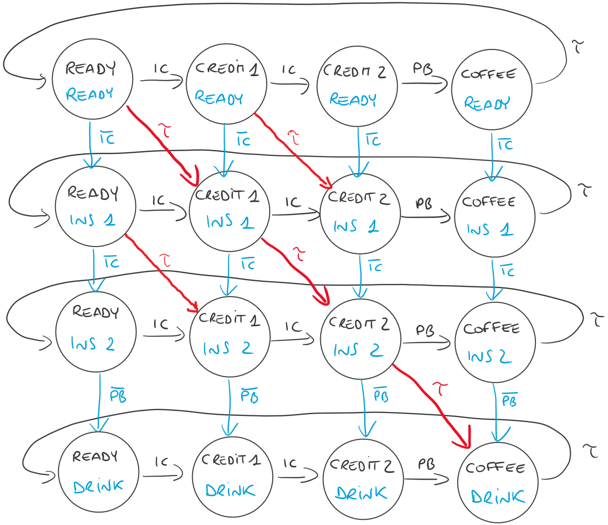
Example: compositional modeling of a coffee machine

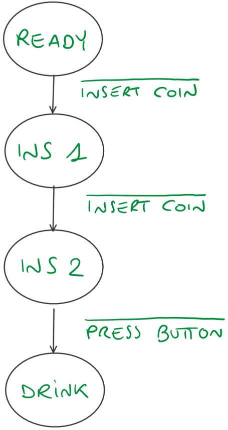
Let's try with two users competing for the machine



Example: compositional modeling of a coffee machine

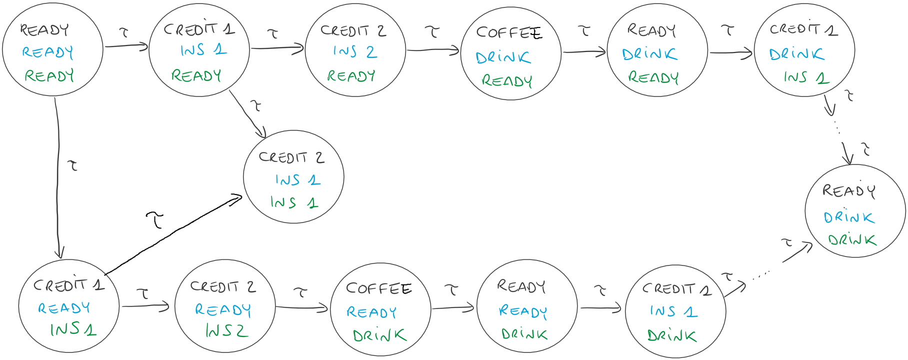
First, we "merge" the machine with the first user:





Example: compositional modeling of a coffee machine

Then we "merge" also the second user and remove non-τ transitions:



Depending on the order of interactions, either both users are served or the system reaches a deadlock state (non deterministic behavior)

Lesson Learnt

Transition Systems are a way to model the behavior/dynamics of a system in an exhaustive way

- all the possibile behaviors are taken into account
- problem state explosion problem

Non-determinism is the most abstract description of alternative behaviors

- it is simple, but misses important quantitave informations on the system behavior such as probailities and rates
  - We will see probabilistic/stochastic transition systems

Reachability of states is the often the property of interest

- but sometimes we may be interested in properties which deal with a sequence of states, such as behavioral patterns (steady states, oscillations, ...)
  - We will see model checking