The Chemical Reaction Metaphor

Computational Models for Complex Systems

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Introduction

We will see how to model and simulate chemical reactions.

WHY?

For several reasons:

- chemical reactions are illustrative examples of complex systems:
    - they exhibit complex dynamics out of very simple interactions
- chemical reactions allow us to interpret some classes of ODEs (reverse engineering)
- chemical reactions allow us to introduce stochastic modeling and to relate it with ODEs
- chemical reactions are a simple and effective metaphor for many kinds of systems (populations, production systems, markets, . . .)

Molecules and chemical solutions

Chemical reactions describe transformation of molecules:

- A group of molecules (reactants) is transformed into another group of molecules (products)

Molecules are represented as symbols:

- A, B, C, H_2, O_2, ...

Molecules are assumed to float in a fluid medium

- called chemical solution
- the quantity of molecules is usually expressed in terms of concentrations (density)
  - $[A]$ denotes the concentration of molecule A
  - usually expressed in mol/L measure unit

Chemical reactions

Usual notation for chemical reactions:

$$\ell_1 S_1 + \ldots + \ell_\rho S_\rho \xrightarrow{k} \ell'_1 P_1 + \ldots + \ell'_\gamma P_\gamma$$

where:

- S_i, P_i are molecules (reactants and products, respectively)
- $\ell_i, \ell'_i \in \mathbb{N}$ are stoichiometric coefficients
- $k \in \mathbb{R}_{\geq 0}$ is the kinetic constant

Example: $2H_2 + O_2 \xrightarrow{5} 2H_2O$

- stochiometric coefficients express the number of reactants/products of each type that are consumed/produced by the reaction
- the kinetic constant is a coefficient used to compute the rate of occurrence of the reaction in a chemical solution

Types of chemical reactions

Type	**Reaction**	**Illustration**
Synthesis	$\xrightarrow{k} P$	
Degradation	$S \xrightarrow{k}$	
Transformation	$S \xrightarrow{k} P$	
Binding	$S_1 + S_2 \xrightarrow{k} P$	
Unbinding	$S \xrightarrow{k} P_1 + P_2$	
Catalysis	$E + S \xrightarrow{k} E + P$	
Other	$\vdots$	$\vdots$

Types of chemical reactions

Catalysis (like other more complex types of reaction)

can be expressed in terms of binding, unbinding and transformation

Synthesis, degradation, transformation, binding and unbinding are a quite "complete" set of reaction types

Chemical reactions

Chemical reactions are often reversible

- they can occur in both directions

The usual notation for reversible chemical reaction is:

$$\ell_1 S_1 + \ldots + \ell_\rho S_\rho \underset{k_{-1}}{\overset{k}{\rightleftharpoons}} \ell'_1 P_1 + \ldots + \ell'_\gamma P_\gamma$$

where $k_{-1} \in \mathbb{R}_{\geq 0}$ is the kinetic constant of the inverse reaction transforming products P_i into reactants S_i

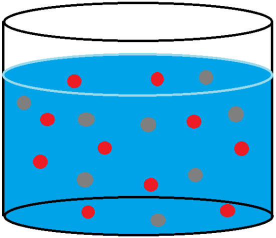
Example: $2H_2 + O_2 \underset{0.5}{\overset{5}{\rightleftharpoons}} 2H_2O$

The mass action kinetics of chemical reactions

The dynamics of chemical reactions is usually based on the following assumption:

molecules float in a (well-stirred) fluid medium (e.g. water)

Hence, they are free to move and randomly meet with each other



When a group of molecules meet, they can react

The dynamics (kinetics) of a chemical solution is modeled by the law of mass action

The mass action kinetics of chemical reactions

$$\ell_1 S_1 + \ldots + \ell_\rho S_\rho \underset{k_{-1}}{\overset{k}{\rightleftharpoons}} \ell'_1 P_1 + \ldots + \ell'_\gamma P_\gamma$$

The rate of a chemical reaction expresses the number of occurrences of such reaction in a given chemical solution in a time unit

The law of mass action is an empirical law for the computation of the rate of a chemical reaction

Definition: Law of mass action

The rate of a chemical reaction is proportional to the product of the concentrations of its reactants.

The kinetic constant is the proportionality ratio of the reaction

The mass action kinetics of chemical reactions

$$\ell_1 S_1 + \ldots + \ell_\rho S_\rho \underset{k_{-1}}{\overset{k}{\rightleftharpoons}} \ell'_1 P_1 + \ldots + \ell'_\gamma P_\gamma$$

The rate of a reaction (the left-to-right direction of the reversible reaction above) is defined as follows:

$$k[S_1]^{\ell_1} \cdots [S_\rho]^{\ell_\rho}$$

similarly, the rate of the inverse reaction (the right-to-left direction) is

$$k_{-1}[P_1]^{\ell'_1} \cdots [P_\gamma]^{\ell'_\gamma}$$

The mass action kinetics of chemical reactions

For example, the rates of

$$2H_2 + O_2 \underset{0.5}{\overset{5}{\rightleftharpoons}} 2H_2O$$

are

- $5[H_2]^2[O_2]$ for the left-to-right direction
- $0.5[H_2O]^2$ for the right-to-left direction

In a solution with the following concentrations:

- 10 mol/L of H_2
- 16 mol/L of O_2
- 30 mol/L of H_2O

rates take the following values:

- $5 \cdot 10^2 \cdot 16 = 8000$
- $0.5 \cdot 30^2 = 450$

(namely, the production of water is much faster than its decomposition)

A small comment about kinetic constants

What is the measure unit of a kinetic constant k?

It depends on the number of reactants!

- The measure unit of concentrations is mol/L
- The measure unit of the reaction rate is $mol/(L \cdot sec)$ (it is the concentration of each product produced in one unit of time)

Examples:

- $A \xrightarrow{k} B$: the rate is $r = k[A]$, that is $k = r/[A]$. So the measure unit of k is $(mol/(L \cdot sec))/(mol/L) = 1/sec$
- $A + B \xrightarrow{k} C$: the rate is $r = k[A][B]$, that is $k = r/[A][B]$. So the measure unit of k is $(mol/(L \cdot sec))/(mol/L)^2 = L/(sec * mol)$
- $2A + B \xrightarrow{k} C$: the rate is $r = k[A]^2[B]$, that is $k = r/[A]^2[B]$. So the measure unit of k is $(mol/(L \cdot sec))/(mol/L)^3 = L^2/(sec * mol^2)$

Some kinetic constants have to be changed if you change the measure unit of concentrations (e.g. μmol) or the time unit (e.g. hours)

Dynamic equilibrium

$$\ell_1 S_1 + \ldots + \ell_\rho S_\rho \underset{k_{-1}}{\overset{k}{\rightleftharpoons}} \ell'_1 P_1 + \ldots + \ell'_\gamma P_\gamma$$

A reversible reaction is said to be in dynamic equilibrium when the rates of its two directions are the same, namely:

$$k[S_1]^{\ell_1} \cdots [S_\rho]^{\ell_\rho} = k_{-1}[P_1]^{\ell'_1} \cdots [P_\gamma]^{\ell'_\gamma}$$

When the dynamic equilibrium is reached, often the chemical solution seems stable (it seems that nothing is happening inside it). Instead, reactions happen continuously, compensating each other.

It is easy to see that at the equilibrium we have:

$$\frac{k}{k_{-1}} = \frac{[P_1]^{\ell'_1} \cdots [P_\gamma]^{\ell'_\gamma}}{[S_1]^{\ell_1} \cdots [S_\rho]^{\ell_\rho}}$$

Dynamic equilibrium: simple example

As an example, let's consider:

$$A + B \underset{2.6}{\overset{0.2}{\rightleftharpoons}} C$$

with the following initial concentrations:

- $[A]_0 = 10\ mol/L$
- $[B]_0 = 15\ mol/L$
- $[C]_0 = 2\ mol/L$

the forward rate is $0.2 \cdot 10 \cdot 15 = 30$ and the backward rate is $2.6 \cdot 2 = 5.2$, hence the production of C is faster than the production of A and B.

So, at the beginning the concentrations of A and B decrease, and the concentration of C increases

This causes the forward rate to decrease and the backward rate to increase, until they become equal (dynamic equilibrium), that is when

- $0.2[A][B] = 2.6[C]$

Dynamic equilibrium: simple example

We have more or less this initial dynamics:

Dynamic equilibrium: simple example

$$A + B \underset{2.6}{\overset{0.2}{\rightleftharpoons}} C$$

In order to determine the concentrations at the steady state, we can exploit the following conservation properties:

- $[A] + [C]$ is always constant (equal to $[A]_0 + [C]_0 = 12$)
- $[B] + [C]$ is always constant (equal to $[B]_0 + [C]_0 = 17$)

since each A and each B are transformed into a C, and viceversa)

So, from the formula of dynamic equilibrium, we can derive:

$$\frac{0.2}{2.6} = \frac{[C]}{[A][B]} \implies \frac{1}{13} = \frac{[C]}{(12-[C])(17-[C])} \implies$$

$$\implies [C]^2 - 42[C] + 204 = 0 \implies [C] \simeq 36.395 \text{ or } [C] \simeq 5.605$$

Dynamic equilibrium

Since $[A] = 12 - [C]$ and $[B] = 17 - [C]$, the two obtained values for $[C]$ give the following two candidate solutions:

- $[A] \simeq -24.395$, $[B] \simeq -7.395$ and $[C] \simeq 36.395$
- $[A] \simeq 6.395$, $[B] \simeq 11.395$ and $[C] \simeq 5.605$

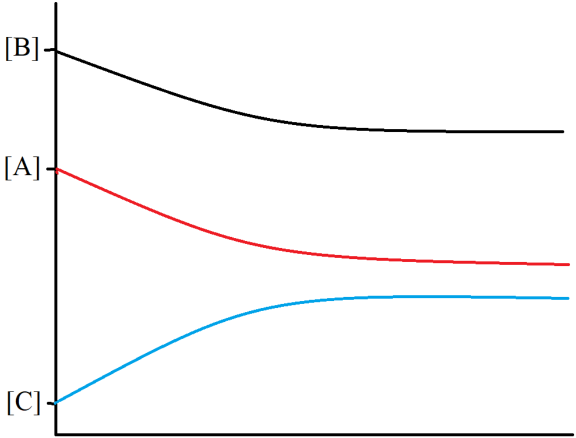
The first one is not acceptable (negative), so the second one is the steady state.

Indeed, those value make the rates of the two reactions equal:

$$0.2[A][B] = 0.2 \cdot 6.395 \cdot 11.395 \simeq 14.574$$
$$2.6[C] = 2.6 \cdot 5.605 \simeq 14.574$$

Dynamic equilibrium: simple example

We have more or less this dynamics:



From reactions to ODEs

Reaction rates can be used to construct a ODE model of the dynamics of a set of chemical reactions

The ODE model of a set of chemical reactions contains one differential equation for each molecular species

For each molecule S, an ODE is constructed by applying these two rules:

- for every reaction R in which S occurs as a reactant, its ODE contains a negative term
  $$-\ell r$$
  where ℓ is the stoichiometric coefficient of reactant S in R and r is the rate of R
- for every reaction R in which S occurs as a product, its ODE contains a positive term
  $$+\ell r$$
  where ℓ is the stoichiometric coefficient of product S in R and r is the rate of R

From reactions to ODEs

From these two generic reactions:

$$\ell_1 S_1 + \ldots + \ell_\rho S_\rho \underset{k_{-1}}{\overset{k}{\rightleftharpoons}} \ell'_1 P_1 + \ldots + \ell'_\gamma P_\gamma$$

we obtain the following ODEs:

$$\begin{cases} \frac{d[S_1]}{dt} = \ell_1 k_{-1}[P_1]^{\ell'_1} \cdots [P_\gamma]^{\ell'_\gamma} - \ell_1 k [S_1]^{\ell_1} \cdots [S_\rho]^{\ell_\rho} \\ \vdots \\ \frac{d[S_\rho]}{dt} = \ldots \\ \frac{d[P_1]}{dt} = \ell'_1 k [S_1]^{\ell_1} \cdots [S_\rho]^{\ell_\rho} - \ell'_1 k_{-1}[P_1]^{\ell'_1} \cdots [P_\gamma]^{\ell'_\gamma} \\ \vdots \\ \frac{d[P_\gamma]}{dt} = \ldots \end{cases}$$

From reactions to ODEs

$$2H_2 + O_2 \underset{0.5}{\overset{5}{\rightleftharpoons}} 2H_2O$$

ODEs obtained from mass action kinetics:

$$\begin{cases} \frac{d[H_2]}{dt} = 2 \cdot 0.5[H_2O]^2 - 2 \cdot 5[H_2]^2[O_2] \\ \frac{d[O_2]}{dt} = 0.5[H_2O]^2 - 5[H_2]^2[O_2] \\ \frac{d[H_2O]}{dt} = 2 \cdot 5[H_2]^2[O_2] - 2 \cdot 0.5[H_2O]^2 \end{cases}$$

$$E + S \underset{1000}{\overset{0.1}{\rightleftharpoons}} ES \underset{0.01}{\overset{0.3}{\rightleftharpoons}} E + P$$

ODEs obtained from mass action kinetics:

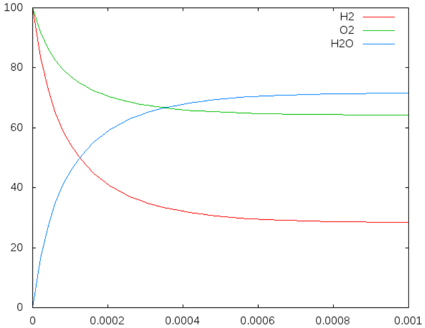
$$\begin{cases} \frac{d[E]}{dt} = 1000[ES] + 0.3[ES] - 0.1[E][S] \\ \frac{d[ES]}{dt} = 0.1[E][S] + 0.01[E][P] - 1000[ES] - 0.3[ES] \\ \frac{d[S]}{dt} = 1000[ES] - 0.1[E][S] \\ \frac{d[P]}{dt} = 0.3[ES] - 0.01[E][P] \end{cases}$$

From reactions to ODEs

We can now simulate the dynamics of the reactions (from given initial concentrations) by using numerical integration

$$2H_2 + O_2 \underset{0.5}{\overset{5}{\rightleftharpoons}} 2H_2O \qquad \begin{cases} \frac{d[H_2]}{dt} = 2 \cdot 0.5[H_2O]^2 - 2 \cdot 5[H_2]^2[O_2] \\ \frac{d[O_2]}{dt} = 0.5[H_2O]^2 - 5[H_2]^2[O_2] \\ \frac{d[H_2O]}{dt} = 2 \cdot 5[H_2]^2[O_2] - 2 \cdot 0.5[H_2O]^2 \end{cases}$$

with initial concentrations $[H_2]_0 = [O_2]_0 = 100$ and $[H_2O]_0 = 0$.



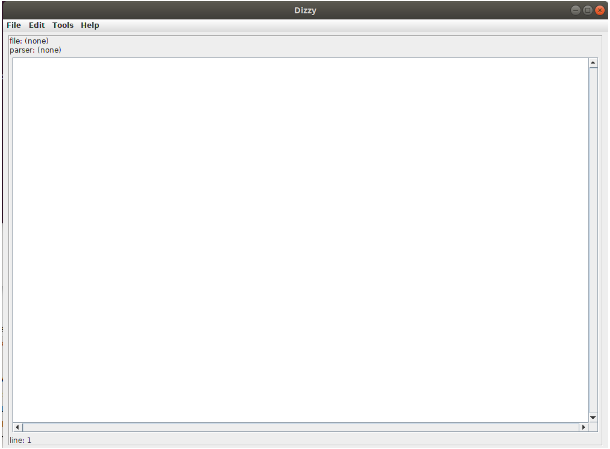
Simulation tools of chemical reactions

Many tools and libraries are available for the numerical simulation of chemical reaction systems:

- COPASI (`http://copasi.org/`)
  Professional (free) tool
- libRoadRunner (`http://libroadrunner.org/`)
  Library for C++/Python
- Dizzy (available on the course web page)
  Multiplatform simulation tool. Unmantained, but very simple...
- Many other tools...
  See `sbml.org/SBML_Software_Guide/SBML_Software_Summary` for a list

Simulating chemical reactions with Dizzy

Dizzy's input (the chemical reactions to be simulated) have to be inserted in text format



Simulating chemical reactions with Dizzy

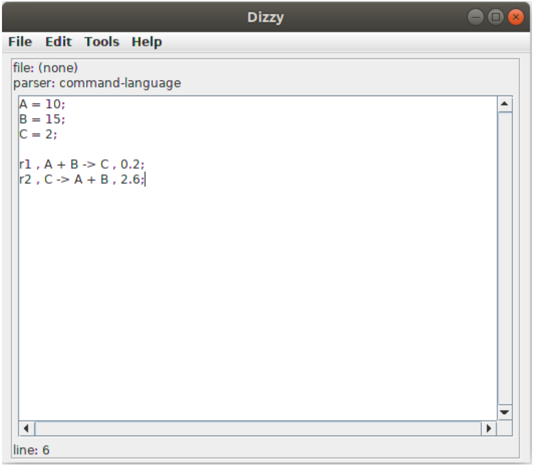
The input language requires:

- A sequence of molecule declarations
  (and initializations with concentration values)
  Syntax: `A = 10;`
- A sequence of reactions
  Syntax: `r , A + B -> C , 0.1;`
  where:
  - `r` is a reaction label
  - `A + B -> C` is the reaction
  - `0.1` is the kinetic constant (can be an expression)

Simulating chemical reactions with Dizzy

Example:

$$A + B \underset{2.6}{\overset{0.2}{\rightleftharpoons}} C$$



file: (none)
parser: command-language

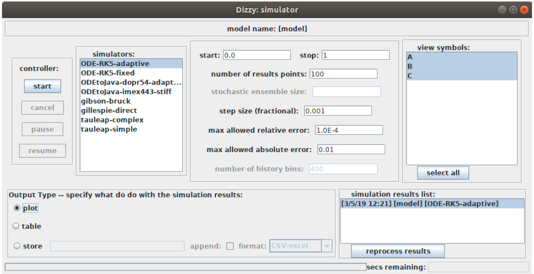
```
A = 10;
B = 15;
C = 2;

r1 , A + B -> C , 0.2;
r2 , C -> A + B , 2.6;
```

line: 6

Simulating chemical reactions with Dizzy

Example:

$$A + B \underset{2.6}{\overset{0.2}{\rightleftharpoons}} C$$



model name: [model]

simulators:
- ODE-RK5-adaptive
- ODE-RK5-fixed
- ODEtoJava-dopr54-adapt…
- ODEtoJava-imex443-stiff
- gibson-bruck
- gillespie-direct
- tauleap-complex
- tauleap-simple

start: 0.0 stop: 1

number of results points: 100

stochastic ensemble size:

step size (fractional): 0.001

max allowed relative error: 1.0E-4

max allowed absolute error: 0.01

number of history bins: 400

view symbols:
- A
- B
- C

Output Type -- specify what do do with the simulation results:

● plot

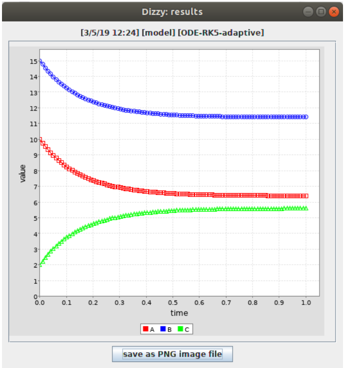
○ table

○ store append: ☐ format: CSV-excel

simulation results list:

[3/5/19 12:21] [model] [ODE-RK5-adaptive]

secs remaining:



[3/5/19 12:24] [model] [ODE-RK5-adaptive]

Reverse engineering ODEs

The law of mass actions allows chemical reaction systems to be "translated" into ODEs

The obtained ODEs are polynomial

Question: can we perform the inverse translation?

- from polynomial ODEs to chemical reactions

Answer: often!

Notation: when we start from ODEs, we omit the $[\cdot]$ brackets representing concentrations

- The idea is to start form a generic system of ODEs, possibly not describing chemical solutions

Reverse engineering ODEs

Let us consider the following example:

$$\begin{cases} \frac{dX}{dt} = 6X - 0.2XY \\ \frac{dY}{dt} = 0.4XY - 2Y \end{cases}$$

We can translate it back to a system of chemical reactions by constructing one reaction for each term of each equation

- we have to match each term with the pattern $\ell_i k [S_1]^{\ell_1} \cdots [S_1]^{\ell_\gamma}$

For example, $6X$ in the equation of X tells us that

- there is a reaction producing X (since the sign of the term is positive)
- the reaction has X as (its only) reactant
- $\ell_i k = 6$, where ℓ_i is the number of new X to be produced

Infinite solutions (all good and equivalent from the ODE viewpoint):

$$X \stackrel{6}{\rightarrow} 2X \qquad X \stackrel{3}{\rightarrow} 3X \qquad X \stackrel{0.1}{\rightarrow} 61X \qquad \ldots$$

Reverse engineering ODEs

Let's translate the whole system:

$$\begin{cases} \frac{dX}{dt} = 6X - 0.2XY \\ \frac{dY}{dt} = 0.4XY - 2Y \end{cases}$$

Reactions:

$$X \xrightarrow{6} 2X$$

Reverse engineering ODEs

Let's translate the whole system:

$$\begin{cases} \frac{dX}{dt} = 6X - 0.2XY \\ \frac{dY}{dt} = 0.4XY - 2Y \end{cases}$$

Reactions:

$$X \xrightarrow{6} 2X$$

$$X + Y \xrightarrow{0.2} Y$$

Reverse engineering ODEs

Let's translate the whole system:

$$\begin{cases} \frac{dX}{dt} = 6X - 0.2XY \\ \frac{dY}{dt} = 0.4XY - 2Y \end{cases}$$

Reactions:

$$X \xrightarrow{6} 2X$$

$$X + Y \xrightarrow{0.2} Y$$

$$X + Y \xrightarrow{0.4} X + 2Y$$

Reverse engineering ODEs

Let's translate the whole system:

$$\begin{cases} \frac{dX}{dt} = 6X - 0.2XY \\ \frac{dY}{dt} = 0.4XY - 2Y \end{cases}$$

Reactions:

$$X \xrightarrow{6} 2X$$

$$X + Y \xrightarrow{0.2} Y$$

$$X + Y \xrightarrow{0.4} X + 2Y$$

$$Y \xrightarrow{2}$$

Reverse engineering ODEs

Let's translate the whole system:

$$\begin{cases} \frac{dX}{dt} = 6X - 0.2XY \\ \frac{dY}{dt} = 0.4XY - 2Y \end{cases}$$

Reactions:

$$X \xrightarrow{6} 2X$$

$$X + Y \xrightarrow{0.2} Y$$

$$X + Y \xrightarrow{0.4} X + 2Y$$

$$Y \xrightarrow{2}$$

Reverse engineering ODEs

Let's re-translate to ODEs to check what we did:

$$X \xrightarrow{6} 2X$$

$$X + Y \xrightarrow{0.2} Y$$

$$X + Y \xrightarrow{0.4} X + 2Y$$

$$Y \xrightarrow{2}$$

Reactions:

$$\begin{cases} \frac{dX}{dt} = -6X + 12X \\ \frac{dY}{dt} = \end{cases}$$

Reverse engineering ODEs

Let's re-translate to ODEs to check what we did:

$$X \xrightarrow{6} 2X$$

$$X + Y \xrightarrow{0.2} Y$$

$$X + Y \xrightarrow{0.4} X + 2Y$$

$$Y \xrightarrow{2}$$

Reactions:

$$\begin{cases} \frac{dX}{dt} = 6X \\ \frac{dY}{dt} = \end{cases}$$

Reverse engineering ODEs

Let's re-translate to ODEs to check what we did:

$$X \xrightarrow{6} 2X$$

$$X + Y \xrightarrow{0.2} Y$$

$$X + Y \xrightarrow{0.4} X + 2Y$$

$$Y \xrightarrow{2}$$

Reactions:

$$\begin{cases} \frac{dX}{dt} = 6X - 0.2XY \\ \frac{dY}{dt} = -0.2XY + 0.2XY \end{cases}$$

Reverse engineering ODEs

Let’s re-translate to ODEs to check what we did:

$$X \xrightarrow{6} 2X$$

$$X + Y \xrightarrow{0.2} Y$$

$$X + Y \xrightarrow{0.4} X + 2Y$$

$$Y \xrightarrow{2}$$

Reactions:

$$\begin{cases} \frac{dX}{dt} = 6X - 0.2XY \\ \frac{dY}{dt} = \end{cases}$$

Reverse engineering ODEs

Let's re-translate to ODEs to check what we did:

$$X \xrightarrow{6} 2X$$

$$X + Y \xrightarrow{0.2} Y$$

$$X + Y \xrightarrow{0.4} X + 2Y$$

$$Y \xrightarrow{2}$$

Reactions:

$$\begin{cases} \frac{dX}{dt} = 6X - 0.2XY - 0.4XY + 0.4XY \\ \frac{dY}{dt} = -0.4XY + 0.8XY \end{cases}$$

Reverse engineering ODEs

Let's re-translate to ODEs to check what we did:

$$X \xrightarrow{6} 2X$$

$$X + Y \xrightarrow{0.2} Y$$

$$X + Y \xrightarrow{0.4} X + 2Y$$

$$Y \xrightarrow{2}$$

Reactions:

$$\begin{cases} \frac{dX}{dt} = 6X - 0.2XY \\ \frac{dY}{dt} = 0.4XY \end{cases}$$

Reverse engineering ODEs

Let's re-translate to ODEs to check what we did:

$$X \xrightarrow{6} 2X$$

$$X + Y \xrightarrow{0.2} Y$$

$$X + Y \xrightarrow{0.4} X + 2Y$$

$$Y \xrightarrow{2}$$

Reactions:

$$\begin{cases} \frac{dX}{dt} = 6X - 0.2XY \\ \frac{dY}{dt} = 0.4XY - 2Y \end{cases}$$

Reverse engineering ODEs

Let's re-translate to ODEs to check what we did:

$$X \xrightarrow{6} 2X$$

$$X + Y \xrightarrow{0.2} Y$$

$$X + Y \xrightarrow{0.4} X + 2Y$$

$$Y \xrightarrow{2}$$

Reactions:

$$\begin{cases} \frac{dX}{dt} = 6X - 0.2XY \\ \frac{dY}{dt} = 0.4XY - 2Y \end{cases}$$

Which is exactly the system of ODEs we started from!

Reverse engineering ODEs

Alternative translation. We can observe that two terms are very similar:

$$\begin{cases} \frac{dX}{dt} = 6X - 0.2XY \\ \frac{dY}{dt} = 0.4XY - 2Y \end{cases}$$

Since $0.4XY = 2 \cdot 0.2XY$ we can assume that the two terms are obtained from the same reaction, hence:

$$X \xrightarrow{6} 2X$$

$$X + Y \xrightarrow{0.2} 3Y$$

$$Y \xrightarrow{2}$$

This is another possible set of chemical reactions correponding to the considered system of ODEs

Reverse engineering ODEs

A system of ODEs may correspond to infinite different sets of reactions

- infinite in the number of possible parameters
- the number of possible combinations of reactants and products is limited

All these sets of reactions are equivalent under the point of view of the ODEs

- but they will be different under other viewpoints (we will see...)

The choice of the set of reactions should be guided by knowledge on the system

- E.g. is $X + Y \overset{0.2}{\rightarrow} 3Y$ more realistic than $X + Y \overset{0.2}{\rightarrow} Y$ and $X + Y \overset{0.4}{\rightarrow} X + 2Y$?

Reverse engineering ODEs

Question:

Does the reverse translation work for all systems of polynomial ODEs?

Answer: No!

Let's see a counterexample:

$$\begin{cases} \frac{dX}{dt} = 6X - 0.2XY - Y \\ \frac{dY}{dt} = 0.4XY - 2Y \end{cases}$$

No reaction can reduce the concentration of X without having X among its reactants!

Conjecture: The translation should work for all systems of polynomial ODEs in which each negative term contains the variable of its equation

Reverse engineering the Lotka-Volterra model

$$\begin{cases} \dot{V} = rV - aVP \\ \dot{P} = -sP + abVP \end{cases}$$

Where V are preys, P are predators and

- a denote the portion of meetings resulting in a hunting
- b denote the number of offsprings produced for each hunting

We obtain:

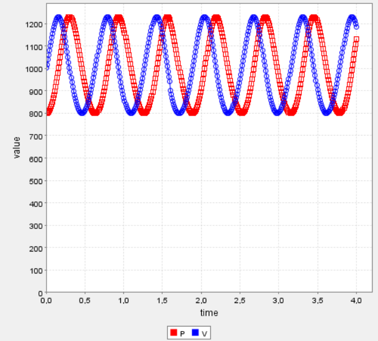
$$V \xrightarrow{r} 2V \qquad P \xrightarrow{s} \qquad V + P \xrightarrow{a} (1+b)P$$

provided $b \in \mathbb{N}$, otherwise two separate reactions have to be considered for hunting and predator reproduction

$$V \xrightarrow{r} 2V \qquad P \xrightarrow{s} \qquad V + P \xrightarrow{a} P \qquad V + P \xrightarrow{ab} V + 2P$$

Reverse engineering the Lotka-Volterra model

```
V = 1000;
P = 800;

r1, V -> V + V, 10;
r2, P -> , 10;
r3, V + P -> P + P, 0.01;
```



Dizzy source code

Numerical simulation

Reverse engineering the SIR model

$$\begin{cases} \dot{S} = \mu - \beta S I - \mu S \\ \dot{I} = \beta S I - \gamma I - \mu I \\ \dot{R} = \gamma I - \mu R \end{cases}$$

Where S are susceptible, I are infected, R are recovered and:

- β is the infection coefficient
- γ is the recovery coefficient
- μ is the birth and death coefficient

We obtain:

$$\xrightarrow{\mu} S \qquad S \xrightarrow{\mu} \qquad I \xrightarrow{\mu} \qquad R \xrightarrow{\mu}$$

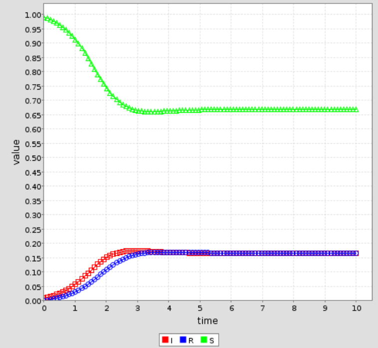
$$S + I \xrightarrow{\beta} 2I \qquad I \xrightarrow{\gamma} R$$

that is a very natural (and elegant) description of the epidemic system

Reverse engineering the SIR model

```
S = 0.99;
I = 0.01;
R = 0;

r1 , S + I -> I + I , 6;
r2 , I -> R , 2;
r3 , -> S , 2;
r4 , S -> , 2;
r5 , I -> , 2;
r6 , R -> , 2;
```

Dizzy source code



Numerical simulation

Reverse engineering the logistic equation

$$\dot{N} = r_c N \left(1 - \frac{N}{K}\right)$$

Where r_c is the birth rate and K is the carrying capacity of the environment.

The equation can be rewritten as:

$$\dot{N} = r_c N - \frac{r_c}{K} N^2$$

We obtain:

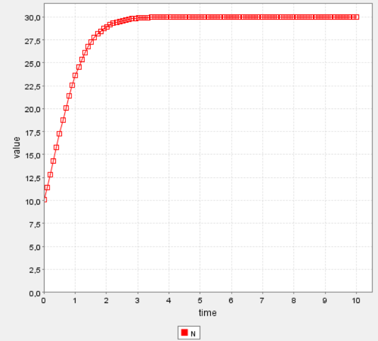
$$N \xrightarrow{r_c} 2N \qquad 2N \xrightarrow{\frac{r_c}{K}} N$$

which is a reversible reaction! By the dynamic equilibrium we obtain

$$r_c N = \frac{r_c}{K} N^2 \qquad \Rightarrow \qquad r_c \frac{K}{r_c} = N^2/N \qquad \Rightarrow \qquad K = N$$

Reverse engineering the logistic equation

```
N = 10;

r1 , N -> N + N , 2;
r2 , N + N -> N , 2/30;
```



Dizzy source code

Numerical simulation

Lessons learnt

Summing up:

- Chemical reactions are a simple language for the modeling of interactions among individuals of a population (or components of a system)
- The law of mass action allows us to describe the dynamics of a set of chemical reactions in terms of (polynomial) ODEs
- In many cases polynomial ODEs (describing the dynamics of any kind of system) can be translated into a chemical reaction representation

Lessons learnt

We are using chemical reactions as a modeling language and ODEs as the semantics of such language

- the modeling language focuses on the description of the events (e.g. infection)

$$S + I \xrightarrow{6} 2I$$

- the semantics focuses on the description of the behaviour (e.g. the variation of infected in time)

$$\dot{I} = \beta S I - \gamma I - \mu I$$

Next step

Stochastic simulation of chemical reactions

- Chemistry provides us algorithms for the stochastic simulation of chemical reactions proved to be consistent with the law of mass action
- Such algorithms can be used also outside of chemistry...

Exercises

- Have a look at the Brussellator system:
  `https://en.wikipedia.org/wiki/Brusselator`
  and try to implement and study it with Dizzy
- Consider the males/females fish population model we studied in the lesson on ODEs. Is it possibile to translate it back into chemical reactions?