A multifunctional approach to extremum problems having infinite-dimensional image. Necessary conditions for unilateral constraints

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Abstract. In Ref.1, extremum problems having infinite-dimensional image have been considered and some preliminary properties have been established. Here we carry on the investigation of such problems and study an optimality condition for the case of unilateral constraints, which partially extends the results of [2,3] to the present type of problems. This is done by associating to the feasible set a special multifunction. It turns out that the classic Lagrangian multitplier functions can be factorized into a constant term and a variable one; the former is the gradient of a separating hyperplane as introduced in [2,3]; the latter plays the role of selector of the above multifunction. Finally, the need of enlarging the class of Lagrangian multiplier functions is discussed.

Key Words. Nonsmooth optimization; Image space; Necessary optimality conditions; Lagrange multipliers; Multifunctions.

1. Introduction

Let the positive integers *n,m,p,* with $p \leq m$, the interval $T := [a, b] \subset \mathbb{R}$ with $-\infty < a < b < +\infty$ and the functions $\psi_i : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad i = 0, ..., m$, be given. Let Y be the set of all continuous functions $x(t) := (x_1(t), ..., x_n(t)), \quad t \in T$, with continuous derivatives $x'(t) := (x'_1(t), ..., x'_n(t)), \quad t \in T$, except at most a finite number of points $\bar{t}$ at which there exist and are finite $\lim_{t \downarrow \bar{t}} x'(t)$ and $\lim_{t \uparrow \bar{t}} x'(t)$. Let us define $x'(\bar{t}) = \lim_{t \downarrow \bar{t}} x'(t)$. The set Y forms a vector space on the set of real numbers. The space Y will be equipped with the norm

$$||x||_\infty := \max_{t \in T} ||x(t)||, \qquad x \in Y,$$

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where $||\bullet||$ denotes the Euclidean norm of $\mathbb{R}^n$. Set $\mathcal{I} := \{1, ..., m\}$.

We take for granted the results of [1]. Here we continue the study of problem (2) of [1] in the case of unilateral constraints only; hence now we consider a particular problem belonging to the following class of problems:

$$\min\ f(x), \quad \text{s.t.}\ \ g_i(x) \geq 0, \quad i \in \mathcal{I}, \quad x \in X, \tag{1.1}$$

namely, the problem:

$$\min\ f(x) := \int_T \psi_0(t, x(t), x'(t))\ dt, \tag{1.2a}$$

subject to:

$$\psi_i(t, x(t), x'(t)) \geq 0, \quad \forall t \in T, \quad i \in \mathcal{I}, \tag{1.2b}$$

$$x \in X := \{x \in Y :\ x_i(a) = x_i(b) = 0, \quad i \in \mathcal{I}\}. \tag{1.2c}$$

Consider the multifunction $F : Y \rightrightarrows \mathbb{R}^{1+m}$, defined by

$$F(x) := (f(\overline{x}) - f(x),\ \psi_i(t, x(t), x'(t)),\ i \in \mathcal{I})).$$

We will assume that the following condition holds.

Condition C: There exists a continuous function $\alpha : X \to \mathbb{R}^{1+m}$ such that:

$$\alpha(x) \in F(x) \quad \text{and} \quad \alpha(x) \notin \mathcal{H}, \quad \forall x \in N(\overline{x}), \tag{1.3}$$

where $N(\overline{x})$ is a neighbourhood of $\overline{x} \in Y$ and

$$\mathcal{H} := \{(u, v) \in \mathbb{R} \times \mathbb{R}^m :\ u > 0,\ v \geq 0\}.$$

If we define

$$K_\alpha(\overline{x}) := \{(u, v) \in F(X) :\ (u, v) = \alpha(x),\ x \in N(\overline{x})\},$$

then (1.3) is equivalent to

$$K_\alpha(\overline{x}) \cap \mathcal{H} = \varnothing \tag{1.4}$$

We observe that the existence of a not necessarily continuous function α is necessary and sufficient for $\overline{x}$ to be a minimum point. To suppose the continuity of α allows us to have a further tool in order to develop the analysis. It is simple to prove the following:

Proposition 1.1. (1.4) holds if and only if $\overline{x}$ is a local minimum point of the problem:

$$\max\ \alpha_0(x), \quad \text{s.t.}\ \ \alpha_i(x) \geq 0, \quad i = 1, ..., m, \quad x \in X. \tag{1.5}$$

Proof. $\overline{x}$ is a minimum point of (1.5) iff the system

$$\alpha_0(x) - \alpha_0(\overline{x}) > 0, \quad \alpha_i(x) \geq 0, \quad i = 1, ..., m, \quad x \in N(\overline{x})$$

is impossible. Since $\alpha_0(\overline{x}) = 0$, the impossibility of the previous system is equivalent to (1.4). □

Remark 1.1. Since, given $x \in X$, the first component U of the vector $(u, v) \in F(X)$ is uniquely defined, then necessarily it is

$$\alpha_0(x) = f(\overline{x}) - f(x)$$

A direct consequence of Proposition 1.1 is the following result.

Proposition 1.2. Suppose that Condition C holds. Then, any necessary optimality condition for (1.5) is a necessary condition for $\overline{x}$ to be a local minimum point of the given problem (1.2a-c).

A fundamental aspect of our analysis lies in the possibility of considering well-behaved functions α which fulfil condition C. The function α is a local continuous selection of F(x) in a neighbourhood of $\overline{x}$. Later in the paper we will consider suitable assumptions which ensure that Condition C be fulfilled.

Let Ω be a given set of parameters and consider a function $\Phi : X \times \Omega \to \mathbb{R}^{1+m}$, defined, $\forall x \in X$, by:

$$\Phi(x, \omega) := \int_{F(x)} \omega \, dt = \left(f(\overline{x}) - f(x), \int_T \omega_i(t) \psi_i(t, x, x') \, dt, \quad i \in \mathcal{I} \right),$$

where the 1-st integral is a short writing to mean selection of an element of *F(x)* by means of a weighted integration. According to [1], Φ is called a *generalized selection function*, iff

$$F(x) \subseteq \mathcal{H} \Leftrightarrow \Phi(x, \omega) \in \mathcal{H}, \quad \forall \omega \in \Omega;$$

ω is a *selection multiplier* (for short, SM).

Proposition 1.3. Let $\overline{x}$ be a minimum point of (1.2), and $\psi_i : C^1(T) \to C(T)$, $i = 0, 1$, be continuous in $N(\overline{x})$. Then, the function

$$\Phi(x) := \{ f(\overline{x}) - f(x), \min_{t \in T} \psi_1(t, x(t), x'(t)), ..., \min_{t \in T} \psi_m(t, x(t), x'(t)) \}$$

is a generalized selection function, which is continuous in $N(\overline{x})$.

Proof. We have to show that the functions:

$$f(\overline{x}) - f(x) \quad \text{and} \quad \min_{t \in T} \psi_i(t, x(t), x'(t)), \quad i \in \mathcal{I},$$

are continuous in $N(\overline{x})$. Let $\tilde{x} \in N(\overline{x})$, and $\varepsilon > 0$; we must prove the inequality:

$$\left| \int_T [\psi_0(t, \tilde{x}(t)) - \psi_0(t, x(t), x'(t))] \right| < \varepsilon, \quad \forall x \in U(\tilde{x}),$$

where $U(\tilde{x})$ is a neighbourhood of $\tilde{x}$. Since ψ_0 is continuous at $\tilde{x}$, $\exists\delta > 0$, such that the inequality

$$\sup_{t\in T} |\tilde{x}(t) - x(t)| < \delta \tag{1.6}$$

implies

$$\sup_{t\in T} |\psi_0(t, \tilde{x}(t), \tilde{x}(t)) - \psi_0(t, x(t), x'(t))| < \frac{\varepsilon}{b-a}.$$

Hence, for each x which fulfils (1.6), we have:

$$\left| \int_T [\psi_0(t, \tilde{x}(t), \tilde{x}\,'(t)) - \psi_0(t, x(t), x'(t))]\, dt \right| < \int_T \frac{\varepsilon}{b-a} dt = \varepsilon.$$

Let $i \in \mathcal{I}$. Consider the inequality:

$$\left| \min_{t\in T} \psi_i(t, \tilde{x}(t), \tilde{x}\,'(t)) - \min_{t\in T} \psi_i(t, x(t), x'(t)) \right| < \varepsilon. \tag{1.7}$$

ψ_0 being continuous at $\tilde{x}$, $\exists\overline{\delta} > 0$, such that the inequality

$$\sup_{t\in T} |\tilde{x}(t) - x(t)| < \overline{\delta} \tag{1.8}$$

implies:

$$\sup_{t\in T} |\psi_i(t, \tilde{x}(t), \tilde{x}\,'(t)) - \psi_i(t, x(t), x'(t))| < \varepsilon.$$

Let

$$\psi_i(\overline{t}, \tilde{x}(\overline{t}), \tilde{x}'(\overline{t})) = \min_{t\in T} \psi_i(t, \tilde{x}(t), \tilde{x}'(t)),$$

$$\psi_i(t^0, x(t^0), x'(t^0)) = \min_{t\in T} \psi_i(t, x(t), x'(t)).$$

We have, $\forall t \in T$, :

$$\psi_i(\overline{t}, \tilde{x}(\overline{t}), \tilde{x}'(\overline{t})) - \psi_i(t, x(t), x'(t)) \le \psi_i(t, \tilde{x}(t), \tilde{x}'(t) - \psi_i(t, x(t), x'(t)) < \varepsilon,$$

so that:

$$\psi_i(\overline{t}, \tilde{x}(\overline{t}), \tilde{x}'(\overline{t})) < \psi_i(t^0, x(t^0), x'(t^0)) + \varepsilon.$$

Similarly, $\forall t \in T$,

$$\psi_i(t^0, x(t^0), x'(t^0)) - \psi_i(t, \tilde{x}(t), \tilde{x}'(t)) \le \psi_i(t, x(t), x'(t)) - \psi_i(t, \tilde{x}(t), \tilde{x}\,'(t)) < \varepsilon,$$

so that:

$$\psi_i(\overline{t}, \tilde{x}(\overline{t}), \tilde{x}\,'(\overline{t})) > \psi_i(t^0, x(t^0), x'(t^0)) - \varepsilon.$$

Hence, for each x which fulfils (1.8), we have that (1.7) is satisfied. □

2. Quasi-multipliers

Here we analyse the case where the selection α belongs to a particular class of functions, namely:

$$\alpha_i(x) = \int_T \omega_i(t,x)\psi_i(t,x,x')\; dt \;, \quad i \in \mathcal{I}, \tag{2.1}$$

where $\omega_i : T \times X \to \mathbb{R} \quad i \in \mathcal{I},$ and $\omega := (\omega_1, ..., \omega_m) \in \Omega, \quad \Omega$ being a given class of parameters.

A necessary optimality condition for (1.2) will be reached by extending the approach of [2,3]. Consider the set:

$$\mathcal{K}(\omega) := \bigcup_{x \in X} \{\Phi(F(x);\omega)\} =$$

$$= \{(u, v_1, ..., v_m) \in \mathbb{R}^{1+m} : u = f(\overline{x}) - f(x),\; v_i = g_i(x;\omega_i), \quad x \in X, \quad i \in \mathcal{I}\},$$

where

$$g_i(x;\omega_i) := \int_T \omega_i(t,x)\psi_i(t,x,x')\, dt.$$

$\mathcal{K}(\omega)$ will be called the *selected image*, and will play the same role as $\mathcal{K}$ in [2,3]. More precisely, F is now a multifunction and *F(x)* is a set, not necessarily a singleton. Thus the optimality cannot be expressed by a disjunction of $\mathcal{H}$ and $\mathcal{K} = F(X)$ as in [2,3]. However, by selecting an element from *F(x)* or from its convex hull, say conv *F(x)* (this means to select $\mathcal{K}(\omega)$ from $\mathcal{K}$), we may hope to reduce ourselves to the scheme of [2,3]. Sect.4 of [1] shows a way of doing this. The infinite dimensionality of the image is overcome by the selection: instead of considering the image of (1.2), which would lead us to an infinite-dimensional image space, we introduce the multifunction F, so that we have a finite-dimensional image space, where the scheme of [2,3] can be adopted by replacing $\mathcal{K}$ with $\mathcal{K}(\omega)$. The selected element from *F(x)* is $F(x;\omega) := (f(\overline{x}) - f(x), \quad g_i(x;\omega_i), \quad i \in \mathcal{I})$; hence $\overline{h} := F(\overline{x};\omega)$ is the selected image of $\overline{x}$ and will play a role quite analogous to that played by $\overline{h}$ in [2,3] where *F(x)* was a singleton.

As in [1,2] the analysis will be carried out within the class of $\mathcal{C}$-differentiable functions; in the sequel the $\mathcal{C}$-derivative will be always assumed to be bounded (with respect to the 2-nd argument). Here there is a further difficulty: the $\mathcal{C}$-differentiability must be enjoyed by *f(x)* and $g_i(x;\omega_i), \quad i \in \mathcal{I}$, and should be unsuitable to assume it; it is more appropriate that any assumption is made on the given data ψ_i and on the selection multiplier ω_i. To this end, we need the following Proposition 2.1, where the general case of SM depending on the unknown x is considered. In fact, next examples show the need of enlarging the class of SM from $\omega_i(t)$ to $\omega_i(t;x)$; X is the set of continuous functions. The example shows that this holds also when X is made by C^1 functions.

Example 2.1 In (1.2) let us set $T = [0,3]$, $m = 1$; $\psi_0(t,x,x') = x$; $\psi_1(t,x,x') =$

$t^2 - 3t + 2 + x(t)$; $\overline{x}(t) = 3t - t^2 - 2, \quad \forall t \in T; \quad X = C^1(T)$. Now consider the functions:

$$\hat{x}(t) = \begin{cases} 0 & , \text{if } 0 \leq t \leq 1 \\ (t-1)^2 & , \text{if } 1 < t \leq \frac{3}{2} \\ -\frac{7}{9}t^2 + \frac{10}{3}t - 3 & , \text{if } \frac{3}{2} < t < 3 \end{cases} \quad \tilde{x}(t) = \begin{cases} -\frac{7}{9}t^2 + \frac{4}{3}t & , \text{if } 0 \leq t \leq \frac{3}{2} \\ (t-2)^2 & , \text{if } 3/2 < t \leq 2 \\ 0 & , \text{if } 2 < t \leq 3. \end{cases}$$

Neither $\hat{x}$ nor $\tilde{x}$ fulfil (1.2b), so that $(\omega \equiv \omega_1, \ \psi \equiv \psi_1)$:

$$F(\hat{x}) \not\subseteq \mathcal{H} = \mathbb{R}_+ \backslash \{O\} \times \mathbb{R}_+; \quad F(\tilde{x}) \not\subseteq \mathcal{H}.$$

Hence, $\Phi(F(\hat{x}); \omega) \notin \mathcal{H}$ and $\Phi(F(\tilde{x}); \omega) \notin \mathcal{H}$ must be true for some SM ω. Assume that ω does not depend on *x(t)*. Since $f(\overline{x}) - f(\hat{x})$ and $f(\overline{x}) - f(\tilde{x})$ are obviously positive, $\Phi(F(\hat{x}); \omega) \notin \mathcal{H}$ and $\Phi(F(\tilde{x}); \omega) \notin \mathcal{H}$ are equivalent to $(\psi(t, x, x') \equiv \psi(t, x))$:

$$\int_T \omega(t)\psi(t, \hat{x})dt = \int_0^1 \omega(t)(t^2 - 3t + 2)dt + \int_1^{3/2} \omega(t)(2t^2 - 5t + 3)dt +$$
$$+ \int_{\frac{3}{2}}^3 \omega(t)(\frac{2}{9}t^2 + \frac{1}{3}t - 1)dt < 0 \tag{2.3a}$$

and

$$\int_T \omega(t)\psi(t, \tilde{x})dt = \int_0^{3/2} \omega(t)(\frac{2}{9}t^2 - \frac{5}{3}t + 2)dt$$
$$+ \int_{\frac{3}{2}}^2 \omega(t)(2t^2 - 7t + 6)dt + \int_2^3 \omega(t)(t^2 - 3t + 2)dt < 0, \tag{2.3b}$$

respectively. Let $S_1 \subseteq [1, \frac{3}{2}]$, $S_2 \subseteq [\frac{3}{2}, 2]$ be such that $\omega(t) = 0$ on $[1, \frac{3}{2}] \backslash S_1$ and on $[\frac{3}{2}, 2] \backslash S_2$. Since $\psi(t, \hat{x}(t))$ and $\psi(t, \tilde{x}(t))$ are continuous and are negative, respectively, only on $]1, \frac{3}{2}[$ and on $]\frac{3}{2}, 2[$, it follows that (2.3) hold only if:

$$\int_{S_1} \omega(t)(2t^2 - 5t + 3)dt + \int_{S_2} \omega(t)(\frac{2}{9}t^2 + \frac{1}{3}t - 1)dt < 0, \tag{2.4a}$$

$$\int_{S_1} \omega(t)(\frac{2}{9}t^2 - \frac{5}{3}t + 2)dt + \int_{S_2} \omega(t)(2t^2 - 7t + 6)dt < 0. \tag{2.4b}$$

These inequalities, summing them up side by side, imply:

$$5 \int_{S_1 \cup S_2} \omega(t)(\frac{2}{3}t - 1)^2 dt < 0,$$

which is evidently false, being $\omega(t) \geq 0, \quad \forall t \in T$. $\square$

Example 2.2. In (1.2), let us set $m = 2$, $\psi_0 = -x_1(t)$, $\psi_2 \equiv 1$, and

$$\psi_1 = \begin{cases} x_2(t) & \text{, if } x_1 \in X_1^- := \left\{x_1 : \int_T x_1(t)dt \leq 0\right\}, \\ x_2(t) - x_1(t)^2 & \text{, if } x_1 \in X_1^+ := \left\{x_1 : \int_T x_1\, dt > 0\right\}, \text{ and} \\ & \quad x_2 \in X_2^- := \left\{x_2 : \int_T x_2 dt \leq 0\right\} \\ x_2(t) + x_1(t)^2 & \text{, if } x_1 \in X_1^+, \quad x_2 \in X_2^+ := \left\{x_2 : \int_T x_2 dt > 0\right\} \end{cases}$$

with $x_1, x_2 \in C^0(T)$. The selected problem, namely (1.2) where (1.2b) is replaced by

$$g_i(x; \omega_i) \geq 0, \quad i \in \mathfrak{I}, \tag{2.5}$$

becomes now:

$$\min \int_T -x_1(t)dt \quad , \qquad \text{s.t. } g_2(x; \omega_2) = \int_T \omega_2(t)dt \geq 0,$$

subject to:

$$g_1(x; \omega_1) = \begin{cases} \int_T \omega_1(t)x_2(t)\, dt\,, \; x_1 \in X_1^-, \\ \int_T \omega_1(t)x_2(t)\, dt - \int_T \omega_1(t)x_1(t)^2\, dt\,, \; x_1 \in X_1^+\,, \; x_2 \in X_2^-, \\ \int_T \omega_1(t)x_2(t)\, dt + \int_T \omega_1(t)x_1(t)^2\, dt\,, \; x_1 \in X_1^+\,, \; x_2 \in X_2^+. \end{cases}$$

Note that $\psi_1 \equiv 0 \Leftrightarrow x_1 \in X_1^-$ and $x_2 \equiv 0$. In fact:

$$x_1 \in X_1^+ \text{ and } x_2 - x_1^2 \equiv 0 \Rightarrow x_2 \in X_2^+; \quad x_1 \in X_1^+ \text{ and } x_2 + x_1^2 \equiv 0 \Rightarrow x_2 \in X_2^-.$$

It follows that $(x_1 \in X_1^-, \; x_2 \equiv 0)$ are the only admissible pairs, and therefore $\overline{x} = (\overline{x}_1 \equiv 0, \overline{x}_2 \equiv 0)$ is the unique minimum point of (1.2). □

Example 2.3. Let us identify (1.2) with:

$$\min \int_T \cos x(t)\, dt\,, \quad x(t) = 0, \quad \forall t \in T := [0,1], \quad x(t) \in C(T).$$

Of course, $x(t) \equiv 0$ is the minimum point. The selected problem, namely (1.2) where (1.2b) is replaced by (2.6), is

$$\min \int_T \cos x(t)\, dt\,, \quad \int_T \omega(t)x(t)\, dt = 0, \quad x \in C(T),$$

where $\omega(t) \in C(T)$ is arbitrary. We prove that $x(t) \equiv 0$ is not solution of the selected problem. To this end, it is enough to show that the selected problem admits a not identically zero feasible solution. Note that we have:

$$\int_T \cos x(t)\, dt \leq 1.$$

Therefore, we look for a solution of type

$$x(t) = at + b, \quad a \neq 0, \, b \neq 0,$$

$$\int_T \omega(t)x(t)\,dt = W(t)x(t)\Big|_0^1 - \int_0^1 W(t)x'(t)\,dt,$$

where *W(t)* is an antiderivative of $\omega(t)$ (*W(t)* exists, since $\omega \in C(T)$). Hence, we must have:

$$0 = \int_T \omega(t)x(t)\,dt = W(1)(a+b) - W(0)b - a\int_T W(t)\,dt.$$

If a and b are chosen in such a way to have:

$$a\left[W(1) - \int_T W(t)\,dt\right] + b[W(1) - W(0)] = 0, \qquad a, b \neq 0,$$

then we obtain that $x(t) = at + b \not\equiv 0$. Now, let us evaluate the objective function in the found point:

$$\int_T \cos(at+b)\,dt = \frac{2}{a}\sin\frac{a}{2}\cos\frac{a+2b}{2}.$$

If the minimum were 1, then we should have:

$$\sin\frac{a}{2}\cos\frac{a+2b}{2} = \frac{a}{2}.$$

By choosing $\frac{a}{2} \notin [-1, 1]$, the previous equality is false. □

Example 2.4. Let us identify (1.2) with:

$$\min \int_T x(t)\,dt, \quad x^2(t) \le 0, \quad \forall t \in T := [0,1], \quad x \in C(T).$$

Of course, $x(t) \equiv 0$ is the unique feasible (and hence optimal) solution. The selected problem, namely (1.2) where (1.2b) is replaced by (2.6), is

$$\min \int_T x(t)\,dt, \quad \int_T \omega(t)x^2(t)\,dt \le 0, \quad x \in C(T),$$

where $\omega(t) \in C(T)$. If we choose $\omega(t) > 0$, $\forall t \in T$, then necessarily $x(t) \equiv 0$ to have $x(t)$ admissible for the selected problem; this, therefore, turns out to be equivalent to the given problem.

The preceding examples shows that, in the general case, we cannot make true the proposition:

$$F(x) \not\subseteq \mathcal{H} \quad \Leftrightarrow \quad \exists \omega \in \Omega, \text{ s.t. } \Phi(F(x);\omega) \notin \mathcal{H},$$

if the elements of Ω are independent of x.

As a consequence of the above example, we have that the approach of [3] cannot be extended, if the SM does not depend on x. Hence, we will introduce a SM depending on x, which will be called *selection quasi-multiplier* (in short, SQM); in this case we will obtain a necessary condition, like that of [3].

Proposition 2.1. Let $f_i : X \to \mathbb{R},\ i = 1, 2$ be $\mathcal{C}$-differentiable at $x = \overline{x}$ and let

$$f_i(x) = f_i(\overline{x}) + \mathcal{D}_{\mathcal{C}} f_i(\overline{x}; z) + \epsilon_i(\overline{x}; z),\ \ i = 1, 2$$

be their expansions, where $\mathcal{D}_{\mathcal{C}} f_i,\ i = 1, 2$ are the $\mathcal{C}$-derivatives. Set $\overline{f} := f_1 \cdot f_2$ and assume that:

$$\mathcal{D}_{\mathcal{C}} \tilde{f}(\overline{x}; z) := \mathcal{D}_{\mathcal{C}} f_1(\overline{x}; z) \cdot f_2(\overline{x}) + f_1(\overline{x}) \cdot \mathcal{D}_{\mathcal{C}} f_2(\overline{x}; z) \in \mathcal{C}. \tag{2.6}$$

Then $\tilde{f}$ is $\mathcal{C}$-differentiable at $\overline{x}$ in the direction z and its expansion is given by

$$\tilde{f}(x) = \tilde{f}(x) + \mathcal{D}_{\mathcal{C}} \tilde{f}(\overline{x}; z) + \tilde{\epsilon}(\overline{x}; z), \tag{2.7}$$

where

$$\tilde{\epsilon}(\overline{x}; z) := \epsilon_1 \cdot \epsilon_2 + \epsilon_1 \cdot [f_2(\overline{x}) + \mathcal{D}_{\mathcal{C}} f_2] + \epsilon_2 [f_1(\overline{x}) + \mathcal{D}_{\mathcal{C}} f_1] + \mathcal{D}_{\mathcal{C}} f_1 \cdot \mathcal{D}_{\mathcal{C}} f_2. \tag{2.8}$$

Proof. The expansion of $\tilde{f}$ is trivially obtained from the product of the expansions of f_1 and f_2. Because of assumption (2.6) $\mathcal{D}_{\mathcal{C}} \tilde{f}$ is sublinear; hence we have to prove only that $\lim_{z \to 0} \tilde{\epsilon} / ||z|| = 0$. As $z \to 0$, obviously $\epsilon_1 \cdot \epsilon_2 / ||z|| \to 0$; the same happens to the 2-nd and 3-rd terms in the RHS of (2.8), since the forms in square brackets are bounded. The boundedness of $\mathcal{D}_{\mathcal{C}} f_1 / ||z||$ and $\lim\limits_{z \to 0} \mathcal{D}_{\mathcal{C}} f_2 = 0$ imply that $\mathcal{D}_{\mathcal{C}} f_1 \cdot \mathcal{D}_{\mathcal{C}} f_2 / ||z|| \to 0$ as $z \to 0$. This completes the proof. □

Assumption (2.6) is fulfilled, when f_1 and f_2 are differentiable, since $\mathcal{D}_{\mathcal{C}} f_i,\ i = 1, 2$ are linear (in this case $\mathcal{D}_{\mathcal{C}} \tilde{f} = \langle f_1'(\overline{x}) f_2(\overline{x}) + f_1(\overline{x}) f_2'(\overline{x}), z \rangle$, which is the classic formula), or when $\mathcal{D}_{\mathcal{C}}\, f_i,\ \ i = 1, 2$ are not linear and $f_i(\overline{x}) \geq 0,\ \ i = 1, 2$. When $f_i(\overline{x}) < 0$, then $\tilde{f}$ may not be $\mathcal{C}$-differentiable; see for instance the case where $f_1(x) = |x|,\ \ f_2(x) = |x| - 1,\ \ x \in \mathbb{R}$.

We will assume the $\mathcal{C}$-differentiability of $\psi_0,\ -\psi_1,\ \ i \in \mathcal{I}$ with respect to the set of 2-nd and 3-rd arguments, of ω_i with respect to the 2-nd argument and that all the hypothesis of Theorem 3.1 and 4.1 in [1] are satisfied. As a consequence we will have the following expansion (for the sake of simplicity, in the sequel $\overline{x}$ will be replaced merely by x):

$$f(x + \delta x) = f(x) + \int_T \mathcal{D}_{\mathcal{C}} \psi_0(t, x, x'; \delta x, \delta x')\ dt + \int_T \epsilon_{\psi_0}(t, x, x'; \delta x, \delta x')\ dt, \tag{2.9a}$$

$$g_i(x + \delta x; \omega_i) = g_i(x; \omega_i) + \int_T \mathcal{D}_{\mathcal{C}} \pi_i(t, x, x'; \delta x, \delta x')\ dt + \\ + \int_T \epsilon_i^{\pi}(t, x, x'; \delta x, \delta x')\ dt,\ \ i \in \mathcal{I}, \tag{2.9b}$$

where

$$\pi_i := \omega_i \cdot \psi_i;\ \mathcal{D}_{\mathcal{C}} \pi_i := \mathcal{D}_{\mathcal{C}} \omega_i(t, x; \delta x) \cdot \psi_i(t, x, x') + \omega_i(t, x) \cdot \mathcal{D}_{\mathcal{C}} \psi_i(t, x, x'; \delta x, \delta x');$$

$$\epsilon_i^{\pi} := \epsilon_{\omega_i} \cdot \epsilon_{\psi_i} + \epsilon_{\omega_i} \cdot [\psi_i(t, x, x') + \mathcal{D}_{\mathcal{C}} \psi_i] + \epsilon_{\psi_i} \cdot [\omega_i(t, x) + \mathcal{D}_{\mathcal{C}} \omega_i(t, x, \delta x)] + \mathcal{D}_{\mathcal{C}} \omega_i \cdot \mathcal{D}_{\mathcal{C}} \psi_i;$$

and where the pairs $(\mathcal{D}_{\mathcal{C}}\omega_i, \epsilon_{\omega_i})$, $(\mathcal{D}_{\mathcal{C}}\psi_i, \epsilon_{\psi_i})$ give the expansions of ω_i, ψ_i, respectively. Since $\mathcal{D}_{\mathcal{C}}$ is an operator which denotes $\mathcal{C}$-derivative, the use of $\mathcal{D}_{\mathcal{C}}\pi_i$ as a symbol would be improper; this does not happen here since π_i is $\mathcal{C}$-differentiable due to Proposition 2.1. When ω_i and ψ_i are differentiable $\mathcal{D}_{\mathcal{C}}\pi_i$ collapses to the usual derivative of a product. If ω_i is constant with respect to x, so that can be denoted by $\omega_i(t)$, then $\mathcal{D}_{\mathcal{C}}\pi_i = \omega_i(t) \cdot \mathcal{D}_{\mathcal{C}}\psi_i$ and $\epsilon_i^{\pi} = \epsilon_{\psi_i} \cdot [\omega_i(t,x) + \mathcal{D}_{\mathcal{C}}\omega_i(t,x,\delta x)]$.

3. Homogeneization

For the sake of simplicity, in this section we will assume that condition C be fulfilled by a selection function α, where

$$\alpha_i(x) = \int_T \omega_i(t)\psi_i(t,x,x')\, dt, \quad i \in \mathcal{I} \tag{3.1}$$

for $x \in N(\overline{x})$, *a* neighbourhood of $\overline{x}$. We observe that (3.1) coincides with (2.1) except for the fact that the parameters ω_i do not depend on x, but only on t. Next proposition is a consequence of the above assumptions and of the results stated in Sect.1.

Proposition 3.1 Assume that condition C be fulfilled and that $\alpha_i(x)$ be defined by (3.1), $i \in \mathcal{I}$. If the system

$$f(\overline{x}) - f(x) > 0; \quad \psi_i(t, x(t), x'(t)) \geq 0, \; i \in \mathcal{I}, \quad \forall t \in T, \quad x \in X, \tag{3.2a}$$

is impossible (for $x \in X \cap N(\overline{x})$), then the following system is also impossible:

$$f(\overline{x}) - f(x) > 0, \quad g(x,\omega) \geq 0, \quad x \in X \cap N(\overline{x}) \tag{3.2b}$$

Proof. It follows from Proposition 1.1 taking into account (3.1). □

Lemma 3.1 (Homogeneization). Let ψ_0 and $-\psi_i$, $i \in \mathcal{I}$ be $\mathcal{C}$-differentiable with respect to the set of the 2nd and 3rd arguments. If $\overline{x}$ is a minimum point of (1.2), then there exist a non-negative SM $\overline{\omega}(t) = (\overline{\omega}_i(t),\ i \in \mathcal{I}) \in C^0(T)^m$ and a neighbourhood in the sense of closedness of order one, say $N^{(1)}(\overline{x})$, such that the system (in the unknown $\delta\overline{x} = x - \overline{x}$; $\delta\overline{x}' = x' - \overline{x}'$):

$$\int_T \mathcal{D}_{\mathcal{C}}\psi_0(t,\overline{x},x';\delta\overline{x},\delta\overline{x}')\, dt < 0; \; \int_T \overline{\omega}_i(t) \cdot \mathcal{D}_{-\mathcal{C}}\psi_i(t,\overline{x},\overline{x}';\delta\overline{x},\delta\overline{x}')\, dt > 0, \quad i \in \mathcal{I}^0, \tag{3.3}$$

$$g_i(\overline{x};\overline{\omega}_i) + \int_T \overline{\omega}_i(t) \cdot \mathcal{D}_{-\mathcal{C}}\psi_i(t,\overline{x},\overline{x}';\delta\overline{x},\delta\overline{x}')\, dt \geq 0, \quad i \in \mathcal{I}\backslash\mathcal{I}^0; \quad x \in X \cap N^{(1)}(\overline{x}),$$

is impossible, where $\mathcal{I}^0 := \{i \in \mathcal{I} : g_i(\overline{x};\overline{\omega_i}) = 0,\ \int_T \overline{\omega}_i(t) \cdot \epsilon_i(t,\overline{x},\overline{x}';\delta\overline{x},\delta\overline{x}')\, dt \gtreqqless 0\}$.

Proof. By applying Proposition 3.1 we get the existence of $\overline{\omega}$ such that (3.2) is impossible. Now, ab absurdo, suppose that, at the same $\omega = \overline{\omega}$, (3.3) be possible, and let $\hat{x} \neq \overline{x}$ be a solution. Then $\alpha\hat{x}$ is a solution of (3.3) $\forall \alpha \in]0,1]$, since

$g_i(\overline{x};\overline{\omega}_i) \geq 0$ and $\mathcal{D}_{\mathcal{C}}f$, $\mathcal{D}_{-\mathcal{C}}\psi_i$, $i \in \mathcal{I}$ are positively homogeneous (satisfy (12a) of [2]). The assumption implies that the remainders:

$$\int_T \epsilon_{\psi_0}\, dt, \quad \int_T \overline{\omega}_i \epsilon_i \, dt, \quad i \in \mathcal{I}$$

are infinitesimal of order >1 with respect to $||(\delta\overline{x}, \delta\overline{x}\,')||$, so that, setting $\hat{y} = (\hat{x}, \hat{x}')$, $\overline{y} = (\overline{x}, \overline{x}')$ and $\delta\overline{y} = (\hat{x} - \overline{x}, \hat{x}' - \overline{x}') = (\delta\overline{x}, \delta\overline{x}')$, $\exists\hat{\alpha} \in]0, 1]$ such that:

$$\frac{1}{||\hat{\alpha}\delta\overline{y}||}\int_T \epsilon_{\psi_0}(t, \overline{y}; \hat{\alpha}\delta\overline{y})\, dt < -\frac{1}{||\delta\overline{y}||}\int_T \mathcal{D}_{\mathcal{C}}f(t, \overline{y}; \delta\overline{y})\, dt,$$

$$\frac{1}{||\hat{\alpha}\delta\overline{y}||}\int_T \overline{\omega}_i(t) \cdot \epsilon_i(t, \overline{y}; \hat{\alpha}\delta\overline{y})\, dt > -\frac{1}{||\delta\overline{y}||}\int_T \overline{\omega}_i(t) \cdot \mathcal{D}_{-\mathcal{C}}\psi_i(t, \overline{y}; \delta\overline{y})\, dt, \quad i \in \mathcal{I}^0.$$

From these inequalities, by noting that $g_i(\overline{x};\overline{\omega}_i) = 0$, $\forall i \in \mathcal{I}^0$, we have:

$$\int_T [\mathcal{D}_{\mathcal{C}}\psi_0(t, \overline{y}; \hat{\alpha}\delta\overline{y}) + \epsilon_f(t, \overline{y}; \hat{\alpha}\delta\overline{y})]\, dt < 0, \qquad (3.4a)$$

$$g_i(\overline{x};\overline{\omega}_i) + \int_T \{\overline{\omega}_i(t)[\mathcal{D}_{-\mathcal{C}}\psi_i(t, \overline{y}; \hat{\alpha}\delta\overline{y}) + \epsilon_i(t, \overline{y}_i; \hat{\alpha}\delta\overline{y})]\}\, dt > 0, \quad i \in \mathcal{I}^0. \qquad (3.4b)$$

$\forall i \in \mathcal{I}\backslash\mathcal{I}^0$ either $g_i(\overline{x};\overline{\omega}_i) = 0$ and $\int_T \overline{\omega}_i\epsilon_i\, dt \equiv 0$ or $g_i(\overline{x};\overline{\omega}_i) > 0$. In the former case, with $\hat{\alpha} = 1$, we obviously have:

$$g_i(\overline{x};\overline{\omega}_i) + \int_T \{\overline{\omega}_i(t)[\mathcal{D}_{-\mathcal{C}}\psi_i(t, \overline{y}; \hat{\alpha}\delta\overline{y}) + \epsilon_i(t, \overline{y}; \hat{\alpha}\delta\overline{y})]\}\, dt \geq 0. \qquad (3.4c)$$

In the latter case $\exists\alpha^0 \in]0, 1]$ such that:

$$g_i(\overline{x};\overline{\omega}_i) + \int_T \overline{\omega}_i(t) \cdot \mathcal{D}_{-\mathcal{C}}\psi_i(t, \overline{y}; \alpha\delta\overline{y})\, dt > 0, \quad \forall\alpha \in]0, \alpha^0],$$

and thus $\exists\tilde{\alpha} \in]0, \alpha^0]$ such that:

$$\frac{1}{||\tilde{\alpha}\delta\overline{y}||}\int_T \overline{\omega}_i(t) \cdot \epsilon_i(t, \overline{y}; \tilde{\alpha}\delta\overline{y})\, dt \geq$$

$$\geq -\frac{1}{\alpha^0||\delta\overline{y}||}\left[g_i(\overline{x};\overline{\omega}_i) + \alpha^0\int_T \overline{\omega}_i(t) \cdot \mathcal{D}_{-\mathcal{C}}\psi_i(t, \overline{y}; \delta\overline{y})\, dt\right] \geq$$

$$\geq -\frac{1}{\alpha||\delta\overline{y}||}\left[g_i(\overline{x};\overline{\omega}_i) + \alpha\int_T \overline{\omega}_i(t) \cdot \mathcal{D}_{-\mathcal{C}}\psi_i(t, \overline{y}; \delta\overline{y})\, dt\right], \quad \forall\alpha \in]0, \alpha^0],$$

where the 1st inequality holds since $\int_T \overline{\omega}_i\epsilon_i\, dt$ is infinitesimal of order >1 with respect to $||(\delta\overline{x}, \delta\overline{x}\,')||$ and the 2nd side is fixed and negative, the 2nd inequality holds since the 2nd side is obviously the maximum of the 3rd on $]0, \alpha^0]$. With $\overline{\alpha} := \tilde{\alpha}$ it follows that:

$$g_i(\overline{x};\overline{\omega}_i) + \int_T \{\overline{\omega}_i(t)[\mathcal{D}_{-\mathcal{C}}\psi_i(t, \overline{y}; \hat{\alpha}\delta\overline{y}) + \epsilon_i(t, \overline{y}; \hat{\alpha}\delta\overline{y})]\}\, dt \geq 0. \qquad (3.4d)$$

Collecting all (3.4), recalling that $g_i(\overline{x};\overline{\omega}_i)=0,\ i\in\mathcal{I}^0$, and using the definition of the remainders ϵ_i, we obtain the possibility of system (3.2), and hence the contradiction. This completes the proof. $\square$

The impossibility of system (3.3) can be expressed as disjunction of the two sets of the image space associated to (1.2). To this end introduce the sets:

$$\mathcal{H}_h := \{(u,v)\in\mathrm{I\!R}\times\mathrm{I\!R}^m :\ u>0;\ v_i>0,\ i\in\mathcal{I}^0;\ v_i\geq 0,\ i\in\mathcal{I}\backslash\mathcal{I}^o\};$$

$$\mathcal{K}(\omega) := \{(u,v)\in\mathrm{I\!R}\times\mathrm{I\!R}^m : u=-\int_T \mathcal{D}_{\mathcal{C}} f\,dt;\ v_i=g_i(\overline{x};\omega_i)+$$
$$+\int_T \omega_i\mathcal{D}_{-\mathcal{C}}\psi_i\,dt,\quad i\in\mathcal{I};\quad x\in X\}.$$

It is easily seen that the impossibility of system (3.3) holds iff

$$\mathcal{H}_h\cap\mathcal{K}(\overline{\omega})=\varnothing.$$

Note that system (3.3) is set up with the homogeneous parts of f and the selections g_i and hence $\mathcal{K}_h(\omega)$ represents the homogenization of the selected image $\mathcal{K}(\omega)$; $\mathcal{H}_h$ simply follows the changes in the types of inequalities in going from (3.2a) to (3.3).

When $\psi_0,\psi_i,\ i\in\mathcal{I}$ are differentiable ($\mathcal{C}$ is replaced with its subset $\mathcal{L}$ of linear elements), then (3.3) becomes:

$$\int_T [\langle\nabla_x\psi_0, x-\overline{x}\rangle+\langle\nabla_{x'}\psi_0, x'-\overline{x}\,'\rangle]\,dt<0;$$

$$\int_T \overline{\omega}_i[\langle\nabla_x\psi_i, x-\overline{x}\rangle+\langle\nabla_{x'}\psi_i, x'-\overline{x}\,'\rangle]\,dt>0,\quad i\in\mathcal{I}^0;$$

$$g_i(\overline{x};\omega_i)+\int_T \overline{\omega}_i[\langle\nabla_x\psi_i, x-\overline{x}\rangle+$$
$$+\langle\nabla_{x'}\psi_i, x'-\overline{x}\,'\rangle]\,dt\geq 0,\quad i\in\mathcal{I}\backslash\mathcal{I}^0;\quad x\in X\cap N^{(1)}(\overline{x}),$$

and in this case Lemma 3.1 extends to problem (1.2) a well known Linearization Lemma (see [4]). Note that Lemma 3.1 can be slightly sharpened by requiring differentiability or $\mathcal{C}$-differentiability only for those ψ_i such that $g_i(\overline{x};\overline{\omega}_i)=0$ and continuity for the remaining ones. Lemma 3.1 can be generalized to semidifferentiable functions.

4. Semistationarity

The generalization of the concept of stationary point, which is associated with that of necessary conditions, has received much attention. The crucial point is the kind of convergence that is required. The following definition seems to be quite general, even if it is clear that it is not possible to handle every problem with a single kind of convergence.

Definition 4.1. $\overline{x}\in R\subseteq Y$ will be called a *lower semistationary point* of a problem of type $\min_{x \in R} f(x)$, iff there exists a neighbourhood $N(\overline{x})$ of $\overline{x}$, such that:

$$\liminf_{x \to \overline{x}} \frac{f(x) - f(\overline{x})}{||x - \overline{x}||} \geq 0, \quad x \in N(\overline{x}). \tag{4.1}$$

The following proposition, whose proof is in Sect.3 of [3], is a motivation for adopting the above definition.

Proposition 4.1 (i) If $\overline{x}$ is a minimum point of f on R, then (4.1) holds. (ii) If R and f are convex, then a lower semistationary point of f on R is a global minimum point, and (4.1) becomes:

$$f'(\overline{x}; x - \overline{x}) \geq 0, \quad \forall x \in R,$$

where $f'(\overline{x}; z)$ denotes directional derivative at $\overline{x}$ in the direction z. (iii) If f is differentiable, then (4.1) becomes:

$$\langle f'(\overline{x}), x - \overline{x} \rangle \geq 0, \quad \forall x \in R,$$

which if $\overline{x} \in \operatorname{int} R$, collapses to:

$$f'(\overline{x}) = 0.$$

Note that, in the case of problem (1.2), f' denotes the variation of the functional f. Let us introduce the function:

$$L(x; \theta, \lambda, \omega) := \theta f(x) - \langle \lambda\, g(x; \omega) \rangle, \quad (\theta, \lambda) \in \mathbb{R} \times \mathbb{R}^m, \quad \omega \in \Omega.$$

Note that, if we set $\lambda_i(t) := \lambda_i \cdot \omega_i(t)$, L is the classic Lagrangian function associated to (1.2); hence, here the Lagrangian multiplier is splitted into two parts: a selection part, i.e. $\omega_i(t)$ which in a wider context becomes $\omega_i(t, x)$, and a separation part, i.e. λ_i.

A star as apex of a cone will denote its positive polar. Let $\overline{k}(\omega) := (0, g(\overline{x}; \omega)) := (\overline{u}, \overline{v}(\omega))$ a selection of the image of $\overline{x}$. Unlike before, $y := (x, x')$, $\overline{y} := (\overline{x}, \overline{x}')$, $\delta\overline{y} := y - \overline{y}$.

Lemma 4.1 (Semistationarity). Let ψ_0 be $\mathcal{C}$-differentiable and ψ_i, $i \in \mathcal{I}$ be (-$\mathcal{C}$)-differentiable with respect to the set of 2nd and 3rd arguments at any value of them.

(i) If $\exists \overline{\omega} \in \Omega$ such that:

$$-(\theta, \lambda) \in [\mathcal{K}(\overline{\omega}) - \overline{k}(\overline{\omega})]^*, \tag{4.2}$$

then

$$\liminf_{x \to \overline{x}} \frac{L(x; \theta, \lambda, \overline{\omega}) - L(\overline{x}; \theta, \lambda, \overline{\omega})}{||x - \overline{x}||} \geq 0. \tag{4.3}$$

If $\lim_{||\delta\overline{y}|| \downarrow 0} \mathcal{D}_{\mathcal{C}} \psi_0(t, \overline{y}; \frac{\delta\overline{y}}{||\delta\overline{y}||})$ and $\lim_{||\delta\overline{y}|| \downarrow 0} \mathcal{D}_{-\mathcal{C}} \psi_i(t, \overline{y}; \frac{\delta\overline{y}}{||\delta\overline{y}||})$, $i \in \mathcal{I}$ exist, then the lower limit of (4.1) collapses to the ordinary limit.

(ii) If $\overline{x} \in \operatorname{int} X$ and $\psi_0, \psi_i,\ i \in \mathcal{I}$ are differentiable, then (i) becomes: if

$$-(\theta, \lambda) \in [\mathcal{K}(\overline{\omega}) - \overline{k}(\overline{\omega})]^{\perp},$$

then

$$L'_x(\overline{x}; \theta, \lambda, \overline{\omega}) = 0.$$

Proof. (4.2) is equivalent to:

$$-(\theta\, \lambda) \in \{(u^*, v^*) \in \mathbb{R}\times\mathbb{R}^m : \langle (u^*, v^*), (u-\overline{u}, v(\overline{\omega})-\overline{v}(\overline{\omega}))\rangle \geq 0,\ \ \forall (u, v(\overline{\omega})) \in \mathcal{K}(\omega)\},$$

or, by using Proposition 3.1 of [2],

$$\mathcal{D}_{\mathcal{C}}L(x; \delta\overline{x}; \theta, \lambda, \overline{\omega}) - \mathcal{D}_{\mathcal{C}}L(\overline{x}; \delta\overline{x}; \theta, \lambda, \overline{\omega}) \geq 0,\ \ \forall x \in X, \tag{4.4}$$

where

$$\mathcal{D}_{\mathcal{C}}L = \int_T [\theta \mathcal{D}_{\mathcal{C}}\psi_0 - \sum_{i\in\mathcal{I}} \lambda_i \mathcal{D}_{-\mathcal{C}}\psi_i]\ dt.$$

Divide both sides of (4.4) by $||\delta\overline{x}||$ and add to them:

$$\frac{1}{||\delta\overline{x}||}\overline{\epsilon}(\overline{x}; \delta\overline{x}; \theta, \lambda, \overline{\omega}) := \frac{1}{||\delta\overline{x}||}\int_T \left(\theta\epsilon_{\psi_0} - \sum_{i\in\mathcal{I}} \lambda_i \epsilon_i\right)\ dt;$$

then (4.4) becomes:

$$\frac{1}{||\delta\overline{x}||}[L(x; \theta, \lambda, \overline{\omega}) - L(\overline{x}; \theta, \lambda, \overline{\omega})] \geq \frac{1}{||\delta\overline{x}||}\overline{\epsilon}(\overline{x}; \delta\overline{x}; \theta, \lambda, \overline{\omega}),\ \ \forall x \in X\backslash\{\overline{x}\}.$$

Now (4.3) follows, since $\overline{\epsilon}/||\delta\overline{x}|| \to 0$ as $x \to \overline{x}$. The remaining part is obvious. (ii) Since $\mathcal{K}(\overline{\omega})$ is now affine, the polar becomes the orthogonal complement and hence lim inf collapses to lim and this is zero since both $\geq$ and $\leq$ must hold. □

References

[1] Giannessi F., Uderzo A., "A multifunction approach to extremum problems having infinite-dimensional image. I: general properties." Proceedings of "Seminario Matematico e Fisico", Univ. of Modena, Suppl. Vol. XLVI, 1998, pp. 771-785

[2] Giannessi F., "Semidifferentiable functions and necessary optimality conditions". Journal of Optimization Theory and Applications, Vol. 60, N. 2, Plenum, New York, 1989, pp. 191-241.

[3] Giannessi F., "General optimality conditions via a separation scheme". In "Algorithms for continuous optimization. The state-of-the-art", E. Spedicato (ed.), Kluwer, 1994.

[4] Abadie J., "On the Kuhn-Tucker theorem." In "Nonlinear Programming", North-Holland, Amsterdam, 1967, pp. 19-36.

[5] Uderzo A., "On a generalized differentiability for operators", Rendiconti del Circolo Matematico di Palermo, Serie II, Suppl. Vol. 48, 1997, pp. 205-224.