On Minty Vector Variational Inequality

Giandomenico Mastroeni

Department of Mathematics, University of Pisa, Via Buonarroti 2, 56127 Pisa, Italy

Abstract

The analysis of the connections among generalized systems, Vector Optimization Problems and Variational Inequalities allows to deepen the study of the properties of the Minty Vector Variational Inequality. In particular, under suitable regularity assumptions,the equivalence between Minty Vector Variational Inequality and Stampacchia Vector Variational Inequality is shown.

Key Words. Generalized systems, Vector Optimization, Vector Variational Inequalities, gap functions.

1 Introduction

It has been recently shown [8, 6] that a Vector Optimization Problem (for short, VOP) or a Vector Variational Inequality (for short, VVI) [4] can be formulated under the form of the impossibility of a suitable parametric system $S(y)$ (in the unknown x):

$$f(x,y) \in C \setminus \{0\}, \quad g(x) \in D, \quad x \in X, \tag{1.1}$$

where $f : X \times X \longrightarrow \mathbb{R}^\ell$, $X \subseteq \mathbb{R}^n$, C is a convex cone in $\mathbb{R}^\ell$, $g : X \longrightarrow \mathbb{R}^m$, D is a closed and convex cone in $\mathbb{R}^m$ (the cones are always considered with the apex at the origin). This leads to a unified approach of the analysis of these topics, allowing to point out the connections between VOP and VVI and also between different formulations of VVI. In this paper, we aim to clarify the relationships between the classic VVI, introduced in [4], and the Minty Vector Variational Inequality (for short, $MVVI$) [3].

In Sect. 2 we will recall the relationships among VOP, VVI and generalized systems. In Sect. 3 we will consider the applications to the $MVVI$ while in Sect. 4, following the approach introduced in [8], we will define a gap function [2, 11, 5] associated to $MVVI$.

We recall the main notations and definitions that will be used in the sequel. Let $M \subseteq \mathbb{R}^n$; $intM$ and clM, will denote the interior and the closure of M, respectively. Let $y \in \mathbb{R}^n$,$y := (y_1, \cdots, y_n)$; $y_{(1^-)} := (y_2, \cdots, y_n)$,
$y_{(i^-)} := (y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n)$ $i = 2, \cdots, n-1$, $y_{(n^-)} := (y_1, \cdots, y_{n-1})$.
$\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$. Let $D \subseteq \mathbb{R}^m$ be a convex cone, the positive polar of D is the set $D^* := \{x^* \in \mathbb{R}^m \ : \ \langle x^*, x\rangle \geq 0 \quad \forall x \in D\}$. The closed and convex cone D is called *pointed* iff $D \cap (-D) = \{0\}$. Let $a, b \in \mathbb{R}^n$, $a \geq_D b$ iff $a - b \in D$.
$g : K \longrightarrow \mathbb{R}^m$ is called D–*function on the convex set* $K \in \mathbb{R}^n$ iff:

$$g(\lambda x_1 + (1-\lambda)x_2) - \lambda g(x_1) - (1-\lambda)g(x_2) \ \in D, \quad \forall x_1, x_2 \in K, \forall \lambda \in [0,1].$$

A differentiable function $f : K \longrightarrow \mathbb{R}$ is called *quasi–convex on the convex set* $K \subseteq \mathbb{R}^n$ iff:

$$f(x) \leq f(y) \quad \Longrightarrow \quad \langle \nabla f(y), x - y\rangle \leq 0 \ , \quad \forall x, y \in K;$$

We will say that the mapping $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is *monotone on* K iff:

$$\langle F(y) - F(x), y - x\rangle \geq 0, \quad \forall x, y \in K;$$

We will say that the mapping F is *pseudomonotone on* K iff:

$$\langle F(y), x - y\rangle \geq 0 \Longrightarrow \langle F(x), x - y\rangle \geq 0, \quad \forall x, y \in K.$$

It is called *strictly pseudomonotone on* K iff:

$$\langle F(y), x - y\rangle \geq 0 \Longrightarrow \langle F(x), x - y\rangle > 0, \quad \forall x, y \in K, \ x \neq y.$$

2 Generalized systems, Vector Optimization Problems and Variational Inequalities

Consider the functions $f : X \times X \longrightarrow \mathbb{R}^\ell$, $X \subseteq \mathbb{R}^n$, $g : X \longrightarrow \mathbb{R}^m$, the convex cone $C \in \mathbb{R}^\ell$, the closed and convex cone $D \in \mathbb{R}^m$ and the problem which consists in finding $y \in K := \{x \in X : g(x) \in D\}$ such that the system $S(y)$, defined by (1.1), be impossible.

We observe that both a VOP and a VVI can be formulated in terms of the impossibility of the system (1.1) choosing a suitable function $f(x, y)$.

Consider the following VOP:

$$min_C \; \phi(x) \quad s.t. \quad x \in K, \tag{2.1}$$

where $\phi : X \longrightarrow \mathbb{R}^\ell$. We recall that $y^* \in K$ is said a *C–minimum point* for (2.1) iff the following system:

$$\phi(y^*) - \phi(x) \in C \setminus \{0\}, \quad x \in K,$$

is impossible.
The following result is an immediate consequence of the definition of an optimal solution of a VOP.

Proposition 1 Let $f(x, y) := \phi(y) - \phi(x)$, then y^* is a *C–minimum point* for (P) iff $S(y^*)$ is impossible.

We recall the definitions of the Stampacchia and Minty Vector Variational Inequalities introduced in [4] and [3], respectively. The Stampacchia VVI is defined by the following inequality which consists in finding $y \in K$, such that

$$F(y)(x - y) \not\leq_{C\setminus\{0\}} 0, \quad \forall x \in K$$

where $F : X \longrightarrow \mathbb{R}^{\ell\times n}$, while the $MVVI$ consists in finding $y \in K$ such that

$$F(x)(y - x) \not\geq_{C\setminus\{0\}} 0, \quad \forall x \in K.$$

From now on, the notation VVI will be referred to the Stampacchia VVI.

Proposition 2.

1. Let $f(x,y) := F(y)(y-x)$. Then y^* is a solution of VVI iff the system $S(y^*)$ is impossible.
2. Let $f(x,y) := F(x)(y-x)$. Then y^* is a solution of $MVVI$ iff the system $S(y^*)$ is impossible.

Lemma 1. If $f(y^*, y^*) = 0$, then $S(y^*)$ is impossible iff y^* is a solution of the following VOP:

$$min_C(-f(x,y^*)) \quad s.t. \quad x \in K. \tag{2.2}$$

Proof. It follows from the definition of C–minimum point applied to y^*.□

We observe that the condition $f(y,y) = 0$ is fulfilled either when the impossibility of $S(y)$ is equivalent to a VVI or a VOP so that the classic optimality conditions stated for Vector Optimization can be considered for a generalized system and then applied to a VVI.

We recall the Kuhn-Tucker conditions for a differentiable VOP.
Let $g : X \longrightarrow \mathbb{R}^m$, be differentiable at y^* and X be an open set in $\mathbb{R}^n$.

Consider the Lagrangian function associated to the problem (2.1),
$\mathcal{L} : C \times D \times X \longrightarrow \mathbb{R}$,

$$\mathcal{L}(\mu, \lambda, x) := \langle \mu, \phi(x) \rangle - \langle \lambda, g(x) \rangle.$$

The Kuhn–Tucker conditions for (2.1) are given by the following system:

$$\begin{cases} \nabla \mathcal{L}(\mu, \lambda, x) = 0 \\ \langle \lambda, g(x) \rangle = 0 \\ g(x) \in D, \ \mu \in C^*, \ \lambda \in D^*, \quad x \in X. \end{cases}$$

Under suitable regularity assumptions on the involved functions, the previous system is a necessary optimality condition for (2.1). We recall the most important regularity conditions stated in [7, 8].
Let $I(y^*) := \{i \in [1, \dots, m] : g_i(y^*) = 0\}$.

R1. $C := \mathbb{R}^\ell_+$, $D := \mathbb{R}^m_+$, $\forall i = 1, \dots, \ell$ the following system is possible:

$$\nabla \phi_{i^-}(y^*) z < 0, \quad \langle \nabla g_j(y^*), z \rangle > 0, \ \ j \in I(y^*), \quad z \in X.$$

R2. $C := \mathbb{R}^\ell_+$, ϕ is convex, g is a D–function and, for every $i = 1, \dots, \ell$ the following system is possible:

$$S_i(y^*) \qquad \phi_{i^-}(y^*) - \phi_{i^-}(y) > 0, \quad g(y) \in intD, \quad y \in X.$$

R3. C is an open convex cone, ϕ is a (clC)–function, g is a D–function and there exists $\bar{y} \in \mathbb{R}^n$ such that $g(\bar{y}) \in intD$.

Remark 1. In [9] the previous regularity conditions have been generalized to the nondifferentiable case.

Remark 2. As observed in the Lemma 1, in the hypothesis that $f(y^*, y^*) = 0$, putting $\phi(x) := -f(x, y^*)$ in (2.1), we can relate to the generalized system $S(y^*)$ the optimality conditions stated for (2.2).

Consider the following function $L : C \times D \times X \longrightarrow R$, that we will call the *Lagrangian function* associated to the system $S(y^*)$:

$$L(y^*; \mu, \lambda, x) := -[\langle \mu, f(x, y^*) \rangle + \langle \lambda, g(x) \rangle].$$

The following result holds:

Proposition 3. Suppose that $f(y^*, y^*) = 0$.

1. Assume that condition R1 holds, with $\phi(x) := -f(x, y^*)$.

   If $S(y^*)$ is impossible, then $\exists (\mu, \lambda) \in (\mathbb{R}^\ell \times \mathbb{R}^m)$ such that (μ, λ, y^*) is a solution of the following system (S)

   $$\begin{cases} \nabla L(y^*; \mu, \lambda, x) = 0 \\ \langle \lambda, g(x) \rangle = 0 \\ g(x) \in D, \ \mu \in C^*, \ \lambda \in D^*, \quad x \in X. \end{cases}$$

2. Assume that one of the conditions R2, R3 holds with $\phi(x) := -f(x, y^*)$. Then $S(y^*)$ is impossible if and only if $\exists (\mu, \lambda) \in (\mathbb{R}^\ell \times \mathbb{R}^m) \setminus \{0\}$ such that (μ, λ, y^*) is a solution of the system (S).

Proof. It is sufficient to observe that the system (S) represent the Kuhn–Tucker conditions for the problem (2.2), that, under the conditions R2 or R3, are also sufficient for y^* to be a solution of (2.2). □

Corollary 1.

1. Assume that condition R1 holds, with $\phi(x) := F(y^*)(x - y^*)$.

   If y^* is a solution of VVI then $\exists(\mu, \lambda) \in (\mathbb{R}^\ell \times \mathbb{R}^m)$ such that (y^*, μ, λ) is a solution of the following system (VS)

   $$\begin{cases} \mu F(y) - \lambda \nabla g(y) = 0 \\ \langle \lambda, g(y) \rangle = 0 \\ g(y) \in D, \ \mu \in C^*, \ \lambda \in D^*, \quad y \in X. \end{cases}$$

2. Assume that one of the conditions R2,R3 holds with $\phi(x) := F(y^*)(x - y^*)$. Then x^* is a solution of VVI if and only if $\exists(\mu, \lambda) \in (C^* \times D^*) \setminus \{0\}$ such that (y^*, μ, λ) is a solution of the system (VS).

3 The Minty Vector Variational Inequality

In this section we will deepen the analysis of the $MVVI$. The results reported in the previous section will allow to derive Lagrangian–type optimality conditions for $MVVI$ by which it will be possible to prove, under suitable regularity assumptions and convexity properties of the constraint function g, that VVI and $MVVI$ are equivalent.

Let $f : X \times X \longrightarrow \mathbb{R}^\ell$, $f(x, y) = F(x)(y - x)$ and $F_i(x)$ be the i–th row of the matrix $F(x)$.

Proposition 4. Assume that X is an open set in $\mathbb{R}^n$ and that F is differentiable at y^*.

1. Suppose that $C := \mathbb{R}^\ell_+$, $D := \mathbb{R}^m_+$, g is differentiable at y^* and $\forall i = 1, \dots, \ell$ the following system is possible:

   $$\begin{cases} \langle F_k(y^*), z \rangle > 0, \quad k = 1, \dots, \ell \ \ k \neq i, \\ \langle \nabla g_j(y^*), z \rangle > 0, \quad j \in I(y^*), \quad y \in X. \end{cases} \tag{3.1}$$

   Let $y^* \in K$ be a solution of $MVVI$. Then $\exists \mu \in \mathbb{R}^\ell$ *and* $\lambda \in \mathbb{R}^m$, $(\mu, \lambda) \neq 0$, such that (μ, λ, y^*) is a solution of the system (VS)

   $$\begin{cases} \mu F(x) - \lambda \nabla g(x) = 0 \\ \langle \lambda, g(x) \rangle = 0 \\ g(x) \in D, \ \mu \in C^*, \ \lambda \in D^*, \ x \in X. \end{cases}$$

2. If $C := \mathbb{R}^\ell_+$, $-f(\cdot, y^*)$ is a convex function, g is a D–function and, for every $i := 1, \dots, \ell$ the following system is possible

$$f_{i^-}(x, y^*) > 0, \quad g(x) \in intD, \quad x \in X;$$

Then, $y^* \in K$ is a solution of $MVVI$ if and only if $\exists \mu \in \mathbb{R}^\ell$ *and* $\lambda \in D^*, (\mu, \lambda) \neq 0$, such that (μ, λ, y^*) is a solution of the system (VS).

3. If C is an open convex cone, $f(\cdot, y^*)$ is a cl$C$–function, g is a D–function and there exists $\bar{y} \in \mathbb{R}^n$ such that $g(\bar{y}) \in intD$.

Then, $y^* \in K$ is a solution of $MVVI$ if and only if $\exists \mu \in C^*$ *and* $\lambda \in D^*, (\mu, \lambda) \neq 0$, such that (μ, λ, y^*) is a solution of the system (VS).

Proof. From Proposition 2 it follows that y^* is a solution of $MVVI$ iff $S(y^*)$ is impossible, where $f(x, y) := F(x)(y - x)$. We observe that $\nabla_x f(x, y) = -F(x) + \nabla F(x)(y - x)$ so that $\nabla_x f(y^*, y^*) = -F(y^*)$; applying Proposition 3, we complete the proof. □

Theorem 1.

1. Let $-g_i$ be a quasiconvex function , $\forall i \in I(y^*)$, and the hypotheses stated in 1. of Proposition 4 hold. Then if y^* is a solution of $MVVI$, y^* is a solution of VVI.

2. Suppose that the hypotheses stated in 3. of Proposition 4 hold. Then y^* is a solution of $MVVI$, if and only if y^* is a solution of VVI.

Proof. 1. Since y^* is a solution of $MVVI$, applying the previous Proposition, we have that $\exists (\lambda, \mu) \in (D^* \times C^*)$ such that (λ, μ, y^*) is a solution of the system (VS). Moreover, taking into account Theorem 4.1 of [7] (applied to the problem (2.2), at the point y^*), we obtain that $\mu \in intR^\ell_+ = intC^*$.

Since $-g_i$ is a quasiconvex function $\forall i \in I(y^*)$, then

$$-g_i(x) \leq -g_i(y^*) \quad \Longrightarrow \quad -\langle \nabla g_i(y^*), x - y^* \rangle \leq 0, \quad \forall x \in X.$$

Adding the previous inequalities, multiplied by $\lambda_i \geq 0$, we have

$$g_i(x) \geq 0, \ i \in I(y^*), \quad \Longrightarrow \quad \langle \sum_{i \in I(y^*)} \lambda_i \nabla g_i(y^*), x - y^* \rangle \geq 0, \quad \forall x \in X.$$

Therefore

$$\langle \mu F(y^*), x - y^* \rangle \geq 0, \quad \forall x \in K. \tag{3.2}$$

We will show that , in the considered hypotheses, (3.2) implies that y^* is a solution of VVI.
Ab absurdo, suppose that $\exists \bar{x} \in K$ such that

$$-F(y^*)(\bar{x} - y^*) \ = \ z \in C \setminus \{0\}. \tag{3.3}$$

Since $z \in C \setminus \{0\}$ and $\mu \in \text{int } C^*$, then $\langle \mu, z \rangle > 0$, and therefore

$$-\langle \mu F(y^*), \bar{x} - y^* \rangle > 0,$$

which is against (3.2).
2. It follows from Proposition 4 and Corollary 1. □

Corollary 2. Suppose that the hypotheses stated in 1. of Theorem 1 hold and that F is a (componentwise) monotone operator on K. Then y^* is a solution of $MVVI$ if and only if y^* is a solution of VVI.

Proof. The necessity part follows from Theorem 1.
Sufficiency [3]. Since F is a (componentwise) monotone operator then, letting y^* be a solution of VVI, we have that, $\forall x \in K$,

$$F(x)(y^* - x) \leq \quad F(y^*)(y^* - x) \not\geq_{C \setminus \{0\}} 0.$$

□

Remark 3. The hypoyhesis of monotonicity on the operator F can be replaced by the one of strict pseudomonotonicity.

Example 1. Let $g(x) := (x+1, -x)$, $D := R^2_+$, $C := R^2_+$, $F(x) := (1, 2x)^T$, $y^* := 0$. In this example, stated in [3], $y^* = 0$ is a solution of $MVVI$ but not of VVI. It is simple to check that, for $i = 1, 2$, the system (3.1) is impossible so that the hypotheses of Corollary 2 do not hold.

4 A gap function for Minty Vector Variational Inequality

Consider the classic (scalar) Variational Inequality (for short VI) which consists in finding $y^* \in K$, such that

$$\langle F(y^*), x - y^* \rangle \geq 0, \quad \forall x \in K,$$

where $F : K \longrightarrow R^n$, $K \subseteq \mathbb{R}^n$. A gap function $p : K \longrightarrow \mathbb{R}$, associated to VI, is a non-negative function that fulfils the condition $p(y) = 0$ if and only if y is a solution of VI. Therefore VI is equivalent to the minimization of the gap function on the feasible set K. A first example of gap function was given by Auslender [1] who considered the function $p(y) := \sup_{x \in K} \langle F(y), y - x \rangle$. The definition of gap function can be extended to the generalized system $S(y)$.

Definition 1. A function $p : K \longrightarrow \mathbb{R}$ is a gap function for the generalized system $S(y)$ iff
i) $p(y) \geq 0, \quad \forall y \in K$;
ii) $p(y) = 0$ if and only if $S(y)$ is impossible.

Remark 4. If the impossibility of $S(y)$ is equivalent to the fact that y is a solution for $MVVI$ (or VVI), then we will say that p is a gap function for $MVVI$ (or VVI).

Consider the following function $\psi : K \longrightarrow \mathbb{R}$:

$$\psi(y) := \min_{(\mu,\lambda) \in S} \sup_{x \in X} [\langle \mu, f(x, y) \rangle \ + \ \langle \lambda, g(x) \rangle],$$

where $S := \{(\mu, \lambda) \in (C^* \times D^*) : \|(\mu, \lambda)\| = 1\}$.
Let $\Omega := \{x \in K : \psi(x) = 0\}$. We will prove that $\psi(y)$ is a gap function for $S(y)$.

Theorem 2. Suppose that g is a D–function, $f(\cdot, y^*)$ is a (clC)–function on the convex set $X \subseteq \mathbb{R}^n$, $\forall y^* \in \Omega$, and $f(y, y) = 0, \ \forall y \in K$.

1. Assume that $C := R^{\ell}_{+}$ and that, for every $i := 1, \dots, \ell$ and $\forall y^* \in \Omega$, the following system is possible

$$(4.1) \qquad f_{i^-}(x,y^*) > 0, \quad g(x) \in intD, \quad x \in X;$$

then $\psi(y)$ is a gap function for $S(y)$.

2. Assume that C is an open convex cone and that

$$(4.2) \qquad \exists \bar{y} \in X \textit{ such that } g(\bar{y}) \in intD;$$

then $\psi(y)$ is a gap function for $S(y)$.

Proof. *1.* It is easy to prove that $\psi(y) \geq 0, \quad \forall y \in K$; in fact, if $(\mu,\lambda) \in (C^* \times D^*)$, then

$$\langle \mu, f(y,y) \rangle + \langle \lambda, g(y) \rangle = \langle \lambda, g(y) \rangle \geq 0.$$

Suppose that $S(y^*)$ is impossible. Since $f(x,y^*)$ is a (clC)–function in the variable x and g is a D–function, from Proposition 3.1 of [8] , we have that $\exists (\mu^*,\lambda^*) \in (C^* \times D^*)$, $(\mu^*,\lambda^*) \neq 0$, such that (μ^*,λ^*,y^*) is a saddle point for $L(y^*;\mu,\lambda,x) := -[\langle \mu, f(x,y^*) \rangle \ + \ \langle \lambda, g(x) \rangle]$ on $(C^* \times D^*) \times X$.

Without loss of generality we can suppose that $(\mu^*,\lambda^*) \in S$. We observe that the saddle value $L(y^*;\mu^*,\lambda^*,y^*) = 0$. Recalling that the saddle point condition can be characterized by a suitable minimax problem [10], we have

$$(4.3) \qquad \min_{(\mu,\lambda)\in C^*\times D^*} \sup_{x\in X}[\langle \mu, f(x,y) \rangle \ + \ \langle \lambda, g(x) \rangle] = L(y^*;\mu^*,\lambda^*,y^*) = 0.$$

Since $(\mu^*,\lambda^*) \in S$, taking into account (4.3), we obtain that $\psi(y^*) = 0$. Vice–versa, suppose that $\psi(y^*) = 0$. Then $\exists (\mu^*,\lambda^*) \in S$, such that

$$\langle \mu^*, f(x,y^*) \rangle + \langle \lambda^*, g(x) \rangle \leq 0, \quad \forall x \in X.$$

Applying Theorem 2.1 and Proposition 2.4 of [8] we obtain that $S(y^*)$ is impossible.

2. The proof is analogous to the one of *1.* using condition (4.2) instead of the hypothesis that the system (4.1) is possible for $i = 1,\ldots,\ell$. □

Corollary 3. In the hypotheses of Theorem 2 with $f(x,y) := F(x)(y-x)$,

$$\psi(y) := \min_{(\mu,\lambda)\in S} \sup_{x\in X}[\langle \mu, F(x)(y-x) \rangle \ + \ \langle \lambda, g(x) \rangle]$$

is a gap function for $MVVI$.

Corollary 4. [8] In the hypotheses of Theorem 2 with $f(x,y) := F(y)(y-x)$,

$$\psi(y) := \min_{(\mu,\lambda)\in S} \sup_{x\in X}[\langle \mu, F(y)(y-x)\rangle \ + \ \langle \lambda, g(x)\rangle]$$

is a gap function for VVI.

References

[1] AUSLENDER A., " Optimization. Methodes Numeriques". Masson, Paris, 1976.

[2] FUKUSHIMA M., "Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems". Mathem. Programming, **Vol.53**, 1992, pp.99-110.

[3] GIANNESSI F., " On Minty Variational Principle". In "New Trends in Mathematical Programming", F.Giannessi, S.Komlosi, T.Rapcsak Eds., Kluwer, 1998, pp. 93-99.

[4] GIANNESSI F., " Theorems of the Alternative, Quadratic Programs and Complementarity Problems". In "Variational Inequalities and Complementarity Problems", R.W. Cottle, F. Giannessi and J–L. Lions Eds., Wiley, New York, 1980, pp.151-186.

[5] GIANNESSI F. "Separation of sets and Gap Functions for Quasi-Variational Inequalities". In "Variational Inequalities and Network Equilibrium Problems", F.Giannessi and A.Maugeri (eds.),Plenum Publishing Co, 1995, pp.101-121;

[6] GIANNESSI F., MASTROENI G. and PELLEGRINI L., "On the Theory of Vector Optimization and Variational Inequalities. Image Space Analysis and Separation". This Volume.

[7] MAEDA T.," Constraint Qualifications in Multiobjective Optimization Problems: Differentiable Case". Jou. of Optim. Theory and Appl.s, Vol.80, 1994, pp. 483-500.

[8] MASTROENI G., " Separation methods for Vector Variational Inequalities. Saddle point and gap function". To appear in "Nonlinear Optimization and Applications 2" , G. Di Pillo, F.Giannessi Eds., Kluwer Academic Publishers B.V..

[9] PREDA V. and CHITESCU I.," On Constraint Qualifications in Multiobjective Optimization Problems: Semidifferentiable Case". Jou. of Optim. Theory and Appl.s, Vol.100, 1999, pp. 417-433.

[10] ROCKAFELLAR R.T., " Convex Analysis". Princeton University Press, Princeton, 1970.

[11] ZHU D.L.,MARCOTTE P., " An extended descent framework for variational inequalities". Jou. of Optim. Theory and Appl., Vol.80, 1994, pp.349-366.