ON AUXILIARY PRINCIPLE FOR EQUILIBRIUM PROBLEMS

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Abstract. The auxiliary problem principle introduced by Cohen is extended to a general equilibrium problem. In particular, applications to variational inequalities and to convex optimization problems are analysed.

1 Introduction

The analysis of a general equilibrium problem (for short, EP):

$$\text{find } y^* \in K \ \text{ s.t. } f(x, y^*) \geq 0, \quad \forall x \in K, \tag{EP}$$

where $f : K \times K \longrightarrow \mathbb{R}$, with $f(x, x) = 0$, for all $x \in K$ a closed convex subset of the reflexive Banach space X, has led to a unified approach in the development of the study of different topics in Optimization such as constrained extremum problems and variational inequalities (for short, VI). In particular VI have been extensively studied since they allow to generalize the classical optimality conditions for constrained extremum problems and to formalize equilibrium conditions for problems of different nature as network flow, economic and mechanical engineering equilibrium problems [7, 9].

If we define $f(x, y) := \langle F(y), x - y\rangle$ then EP collapses into the classic VI:

$$\text{find } y^* \in K \ \text{ s.t. } \langle F(y^*), x - y^*\rangle \geq 0, \quad \forall x \in K, \tag{VI}$$

where $F : K \longrightarrow X^*$, $K \subseteq X$ and $\langle \cdot, \cdot \rangle$ is the duality pairing between X and its topological dual X^*. If $f(x, y) := h(x) - h(y)$ then EP is equivalent to the optimization problem

$$\min_{x \in K} \ h(x) \quad \text{ s.t. } x \in K,$$

where $h : K \longrightarrow \mathbb{R}$. We refer to [1] and references therein for a wide analysis concerning the existence results for the problem EP and for further examples of equivalent formulations obtained by means of an EP.

The aim of this paper is to show that, exploiting a suitable fixed point formulation of EP, it is possible to define a class of iterative methods which are a straightforward extension of those proposed by Cohen [3, 4] for solving VI and constrained extremum problems.

In Section 2 we will introduce a suitable regularization of EP which we will refer to as the auxiliary equilibrium problem (for short AEP).
AEP is a further equilibrium problem which is equivalent to the original EP. By means of the fixed point formulation of AEP we will show that it is possible to define a sequence strongly convergent to a solution of AEP and therefore of EP (Section 3). The proposed method collapses into the one developed by Cohen [4] when EP represents a VI. Applications to constrained extremum problems are considered in Section 4.
We recall the main notations and definitions that will be used in the sequel.
A function $f : X \times X \longrightarrow \mathbb{R}$ is said strongly monotone on $K \in X$, with modulus $a > 0$, iff:

$$f(x, y) + f(y, x) \leq \; -a\|y - x\|^2, \quad \forall x, y \in K.$$

A functional $G : X \longrightarrow \mathbb{R}$ is said to be "differentiable" (in the sense of Gateaux) at the point $x^* \in X$ iff there exists finite:

$$\lim_{t \to 0} \frac{G(x^* + ty) - G(x^*)}{t} =: \langle G'(x^*), y \rangle,$$

with $G'(x^*) \in X^*$ (see [5]).

A function $h : X \longrightarrow \mathbb{R}$ is said to be "directionally differentiable" (in the sense of Dini) at the point $x^* \in X$ in the direction y, iff there exists finite:

$$\lim_{t \to 0^+} \frac{h(x^* + ty) - h(x^*)}{t} =: h'(x^*; y).$$

$h : K \longrightarrow \mathbb{R}$ is said strongly convex on K with modulus a ($a \geq 0$) iff $\forall x_1, x_2 \in K, \forall \lambda \in [0, 1]$,

$$h(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda h(x_1) + (1 - \lambda)h(x_2) - a[\lambda(1 - \lambda)/2]\|x_1 - x_2\|^2.$$

If h is differentiable then h is strongly convex on K, with modulus $a > 0$, iff $\forall x_1, x_2 \in K$,

$$h(x_1) - h(x_2) \geq \langle \nabla h(x_2), x_1 - x_2 \rangle + a/2\|x_1 - x_2\|^2.$$

We will say that the mapping $F : X \longrightarrow X^*$ is monotone on $K \subseteq X$ iff:

$$\langle F(y) - F(x), y - x \rangle \geq 0, \quad \forall x, y \in K;$$

it is strictly monotone if strict inequality holds $\forall x \neq y$.
We will say that the mapping F is strongly monotone, with modulus $\mu > 0$, on K iff:

$$\langle F(y) - F(x), y - x \rangle \geq \ \mu\|y - x\|^2, \quad \forall x, y \in K;$$

F is Lipschitz continuous with modulus $L > 0$ over K iff

$$\|F(x) - F(y)\| \leq \ L\|x - y\|, \quad \forall x, y \in K.$$

2 The auxiliary equilibrium problem

Most of the algorithms developed for solving EP can be derived from equivalent formulations of the equilibrium problem. We will focus our attention on fixed-point formulations of EP: we will show that such formulations lead to a generalization of the methods developed by Cohen for variational inequalities and optimization problems.

Let us recall the following preliminary result which states the above mentioned equivalent formulation of EP.

Lemma 2.1 *Suppose that $f(x, x) = 0, \ \forall x \in K$. Then, the following statements are equivalent:*
i) there exists $y^* \in K$ s.t. $f(x, y^*) \geq 0, \quad \forall x \in K$.
ii) $y^* \in K$ is a solution of the problem

$$\min_{x \in K} \ f(x, y^*). \tag{1}$$

If we assume that, for any $y^* \in K$, (1) has a unique solution we can define the following general iterative method:

General Algorithm.

(i) Let $k = 0$, $y^0 \in K$;
(ii) let y^{k+1} be the solution of the problem:

$$\min_{x \in K} f(x, y^k). \tag{2}$$

(iii) if $\|y^{k+1} - y^k\| < \mu$, for some fixed $\mu > 0$, then STOP, otherwise put $k = k + 1$ and go to step (ii).

Unfortunately, in most of the cases, it is not possible (or not convenient) to apply the previous algorithm directly to the problem EP. It is necessary to introduce an auxiliary equilibrium problem, equivalent to the given one, for which the above procedure leads to a solution of EP.

Proposition 2.1 *Let $f(x, y)$ be a convex differentiable function with respect to x at $y = y^* \in K$ and $\epsilon > 0$. Let $H(x, y) : K \times K \longrightarrow \mathbb{R}$ be non negative, differentiable on the convex set K with respect to x and such that*
i) $H(y, y) = 0, \quad \forall y \in K$;
ii) $H'_x(y, y) = 0, \quad \forall y \in K$.
Then y^* is a solution of EP iff it is a solution of the auxiliary equilibrium problem (AEP):

$$\textit{find } y^* \in K \quad s.t. \ \epsilon f(x, y^*) + H(x, y^*) \geq 0 \quad \forall x \in K.$$

Proof. It is obvious that if y^* is a solution of EP then it is also a solution of AEP.
Vice versa let y^* be a solution of AEP. Then y^* is a minimum point of the problem

$$\min_{x \in K} [\epsilon f(x, y^*) \ + \ H(x, y^*)]. \tag{3}$$

Since K is convex then y^* is an optimal solution for (3) iff

$$\langle \epsilon f'_x(y^*, y^*) + H'_x(y^*, y^*), x - y^* \rangle \geq 0, \ \forall x \in K,$$

so that

$$\langle \epsilon f'_x(y^*, y^*), x - y^* \rangle \geq 0, \ \forall x \in K. \tag{4}$$

Dividing by ϵ, we obtain that (4) implies, by the convexity of $f(\cdot, y^*)$, that

$$f(x, y^*) \geq f(y^*, y^*) = 0, \quad \forall x \in K.$$

$\Box$

Corollary 2.1 *y^* is a solution of EP iff y^* is an optimal solution of the extremum problem*

$$\min_{x \in K} [\epsilon f(x, y^*) \ + H(x, y^*)].$$

In the next section we will see that applying the General Algorithm to the auxiliary equilibrium problem AEP, for a suitable choice of the function H, it will be possible to define a sequence $\{y_k\}$ convergent to a solution of EP.

3 The auxiliary problem principle

Following the approach of Cohen [3, 4] we will extend the auxiliary problem principle to the equilibrium problem EP. Given $G : K \longrightarrow \mathbb{R}$, a strongly convex differentiable functional, $\epsilon > 0$, let us introduce the auxiliary problem AEP obtained putting $H(x, y) := G(x) - G(y) - \langle G'(y), x - y \rangle$:

find $y^* \in K$ such that

$$\epsilon f(x, y^*) \ - \ \langle G'(y^*), x - y^* \rangle + G(x) - G(y^*) \geq 0, \quad \forall x \in K.$$

Lemma 3.1 *Let $f(x, y)$ be a convex differentiable function with respect to x at $y = y^* \in K$. Then y^* is a solution of EP iff it is a solution of AEP.*

Proof. It follows from the Proposition 2.1 noticing that the function $H(x, y) := G(x) - G(y) - \langle G'(y), x - y \rangle$ fulfils the hypotheses i) and ii) of the Proposition. $\Box$

Taking into account Corollary 2.1 we have that y^* is a solution of EP iff y^* is an optimal solution of the extremum problem

$$\min_{x \in K} [\epsilon f(x, y^*) \ - \ \langle G'(y^*), x \rangle + G(x)].$$

Applying the General Algorithm to the problem AEP we obtain the following iterative method:

Algorithm 3.1.

(i) Let $k = 0$, $y^0 \in K$;
(ii) let y^{k+1} be the solution of the problem $P(k)$:

$$\min_{x \in K} \, [\epsilon f(x, y^k) \; - \; \langle G'(y^k), x\rangle + G(x)];$$

(iii) if $\|y^{k+1} - y^k\| < \mu$, for some fixed $\mu > 0$, then STOP, otherwise put $k = k + 1$ and go to step (ii).

Remark 3.1 We observe that $P(k)$ has a unique solution since G is a strongly convex function.

Theorem 3.1 *Suppose that the following conditions hold:*
(i) $f(x, y)$ is a convex l.s.c. function with respect to x, $\forall y \in K$;
(ii) $f(x, y)$ is continuous with respect to y, on every finite dimensional subspace of X, $\forall x \in K$;
(iii) f is strongly monotone on K with modulus a;
(iv) G is strongly convex on K with modulus b;
(v) there exist constants α, $\beta > 0$ such that $\forall x, y, z \in K$:

$$f(x, y) + f(y, z) \geq f(x, z) - \alpha\|x - y\|^2 - \beta\|y - z\|^2;$$

Then, if $\epsilon \leq \frac{b}{2\alpha}$ and $\beta < a$, the sequence $\{y^k\}$, defined in the Algorithm 3.1,strongly converges towards the solution y^* of EP.

Proof. In our hypotheses the solution y^* of EP exists and it is unique (see e.g. [1] Theorem 1), as well as the solution of $P(k)$. Consider the functional

$$\Lambda(x) := G(y^*) - G(x) - \langle G'(x), y^* - x\rangle \geq \frac{b}{2}\|x - y^*\|^2, \qquad (5)$$

and the difference
$\Lambda(y^k) - \Lambda(y^{k+1}) = G(y^{k+1}) - G(y^k) - \langle G'(y^k), y^* - y^k\rangle + \langle G'(y^{k+1}), y^* - y^{k+1}\rangle - \langle G'(y^k), y^{k+1}\rangle + \langle G'(y^k), y^{k+1}\rangle = G(y^{k+1}) - G(y^k) - \langle G'(y^k), y^{k+1} -$

$y^k\rangle + \langle G'(y^{k+1}) - G'(y^k), y^* - y^{k+1}\rangle \geq \frac{b}{2}\|y^{k+1} - y^k\|^2 + \epsilon(f(y^{k+1}, y^k) - f(y^*, y^k))$.

The previous inequality is due to the strong convexity of G and from the fact that, since y^{k+1} solves $P(k)$, we have

$$\langle \epsilon f_x(y^{k+1}, y^k) - G'(y^k) + G'(y^{k+1}), x - y^{k+1}\rangle \geq 0 \quad \forall x \in K.$$

Computing the previous inequality for $x = y^*$, we obtain

$$\langle G'(y^{k+1}) - G'(y^k), y^* - y^{k+1}\rangle \geq \langle \epsilon f_x(y^{k+1}, y^k), y^{k+1} - y^*\rangle \geq$$

$$\epsilon(f(y^{k+1}, y^k) - f(y^*, y^k)), \quad \text{recalling that } f(\cdot, y^k) \text{ is convex.}$$

Therefore

$$\Lambda(y^k) - \Lambda(y^{k+1}) \geq \frac{b}{2}\|y^{k+1} - y^k\|^2 + \epsilon(f(y^{k+1}, y^k) + f(y^k, y^*)) - \epsilon(f(y^*, y^k) + f(y^k, y^*)).$$

Exploiting *(iv)* and *(v)*, we obtain

$$\Lambda(y^k) - \Lambda(y^{k+1}) \geq \frac{b}{2}\|y^{k+1} - y^k\|^2 + \epsilon(f(y^{k+1}, y^*) - \alpha\|y^k - y^{k+1}\|^2 - \beta\|y^k - y^*\|^2) +$$

$$\epsilon a\|y^k - y^*\|^2 \geq (\frac{b}{2} - \epsilon\alpha)\|y^k - y^{k+1}\|^2 + \epsilon(a - \beta)\|y^k - y^*\|^2.$$

Therefore the sequence $\Lambda(y^k)$ is strictly decreasing and bounded from below,by (5), so that $\Lambda(y^k) \longrightarrow \bar{\Lambda}$ and $y^k \longrightarrow y^*$. □

In the next section we will see that the Algorithm 3.1, if applied to a variational inequality problem, collapses into the one proposed by Cohen in [4].

4 Applications to variational inequalities and optimization problems

Consider the generalized variational inequality (for short GVI):

$$\text{find } y^* \in K \text{ s.t. } \langle F(y^*), x - y^*\rangle \geq \phi(y^*) - \phi(x) \quad \forall x \in K, \qquad (GVI)$$

where $F : K \longrightarrow X^*$, $\phi : K \longrightarrow \mathbb{R}$ is a l.s.c. proper convex function. Put

$$f(x,y) := \langle F(y), x-y\rangle - \phi(y) + \phi(x),$$

then GVI is equivalent to EP so that we can apply the Algorithm 3.1 to the problem GVI.

Algorithm 4.1.

(i) Let $k = 0$, $y^0 \in K$;
(ii) let y^{k+1} be the solution of the problem :

$$\min_{x\in K} [\langle \epsilon F(y^k) \ - \ G'(y^k), x\rangle + G(x) + \epsilon\phi(x)].$$

(iii) if $\|y^{k+1} - y^k\| < \mu$, for some fixed $\mu > 0$, then STOP, otherwise put $k = k+1$ and go to step (ii).

Algorithm 4.1 coincides with the one proposed by Cohen [4].

Theorem 4.1 *Under the hypotheses*
(i) F is a strongly monotone operator on K (with modulus a) which is weakly continuous over every finite-dimensional subspace of X;
(ii) G is differentiable and strongly convex on K with modulus b;
(iii) there exist constants α, $\beta > 0$ such that $\forall x, y, z \in K$:

$$\langle F(y) - F(z), x-y\rangle \geq -\alpha\|x-y\|^2 - \beta\|y-z\|^2.$$

Then, if $\epsilon \leq \frac{b}{2\alpha}$ and $\beta < a$, the sequence $\{y^k\}$, defined in the Algorithm 4.1, strongly converges towards the solution y^* of GVI.

Proof. It follows from Theorem 3.1, putting

$$f(x,y) := \langle F(y), x-y\rangle - \phi(y) + \phi(x).$$

□

Theorem 4.1 is a slight generalization of Theorem 2.2 of [4] since it is possible to show that, if F is Lipschitz continuous over K, then iii) holds for a suitable choice of the costants α and β.

Proposition 4.1 *Assume that F is Lipschitz continuous with modulus L over K, then condition iii) of Theorem 4.1 holds provided that $\sqrt{\alpha\beta} \geq L\backslash 2$.*

Proof. Since F is Lipschitz continuous, we have

$$\langle F(y) - F(z), x - y\rangle \geq -\|F(y) - F(z)\|\|x - y\| \geq -L\|y - z\|\|x - y\| \geq -2\sqrt{\alpha\beta}\|y - z\|\|x - y\| \geq -\alpha\|x - y\|^2 - \beta\|y - z\|^2, \quad \forall x, y, z \in K.$$

□

Remark 4.1 In order to apply Theorem 4.1 it is sufficient to choose α and β that fulfil the following system:

$$\sqrt{\alpha\beta} \geq L\backslash 2,\ \beta < a, \alpha > 0,\ \beta > 0.$$

The analysis of the variational inequality GVI allows to define a further algorithm for a constrained extremum problem of the form

$$\min_{x\in K}\ [\psi(x) + \phi(x)] \tag{P}$$

where $\psi : K \longrightarrow \mathbb{R}$ is a (Gateaux) differentiable strongly convex functional, $\phi : K \longrightarrow \mathbb{R}$ is convex with finite directional derivative $\phi'(x; z)$, $\forall x, z \in K$. As already mentioned, (P) is equivalent to the equilibrium problem

$$f(x, y^*) \geq 0, \quad \forall x \in K,$$

provided that $f(x, y) = \psi(x)+\phi(x)-\psi(y)-\phi(y)$. We observe that condition iii) of Theorem 3.1 cannot be fulfilled whatever ψ and ϕ may be. An equivalent equilibrium problem can be stated as a first order optimality condition for (P).

Proposition 4.2 *y^* is a solution of P iff it a solution of the GVI*

$$\langle\psi'(y^*), x - y^*\rangle + \phi(x) - \phi(y^*) \geq 0, \quad \forall x \in K. \tag{6}$$

Proof. We observe that, since P is a convex problem, y^* is a solution of P if and only if

$$\langle\psi'(y^*), x - y^*\rangle + \phi'(y^*; x - y^*) \geq 0, \quad \forall x \in K. \tag{7}$$

Suppose that (7) holds. Since ϕ is convex we have (see [13])

$$\phi(x) - \phi(y^*) \geq \phi'(y^*; x - y^*), \quad \forall x \in K,$$

so that $\langle \psi'(y^*), x - y^* \rangle \geq -\phi'(y^*; x - y^*) \geq \phi(y^*) - \phi(x), \quad \forall x \in K.$
Vice versa if (6) holds then y^* is minimum point of the problem

$$\min_{x \in K} [\langle \psi'(y^*), x - y^* \rangle + \phi(x) - \phi(y^*)],$$

which first order optimality condition is given by (7) and the proof is complete. □

The variational inequality (6) can be solved by means of the Algorithm 4.1 which, in this setting, turns out to be a particular case of Algorithm 2.1 stated in [3] by Cohen.

5 Concluding remarks

We have considered a fixed point algorithm for solving a general equilibrium problem. We have shown that this method, that could be stated directly for the original problem, must be applied to an equivalent auxiliary equilibrium problem in order to achieve the convergence.

Many are the possible developments of the analysis: t he connections with the proximal methods for equilibrium problems [12]; the applications to decomposition algorithms [3, 6, 2]; the extensions to vector equilibrium problems (see e.g. [8]); the applications of the epiconvergence theory in the analysis of the auxiliary problem principle [14].

References

[1] Blum E. and Oettli W. (1993) From Optimization and Variational Inequalities to Equilibrium Problems, The Mathematics Student, **Vol. 63**, *pp. 1-23*;

[2] Chen G. and Rockafellar R.T. (1997) Convergence Rates in Forward-Backward Splitting, *SIAM Journal of Optimization* , **Vol. 7**, *pp. 421-444*;

[3] Cohen G. (1980) Auxiliary Problem Principle and Decomposition of Optimization Problems, *Journal of Optimization Theory and Applications*, **Vol. 32**, *pp. 277-305*;

[4] Cohen G. (1988) Auxiliary Problem Principle Extended to Variational Inequalities, *Journal of Optimization Theory and Applications*, **Vol. 59**, *pp. 325-333*;

[5] Ekeland I. and Temam R. (1977) Convex Analysis and Variational Problems, North-Holland, Amsterdam;

[6] Farouq E.N. and Cohen G. (1998) Progressive Regularization of Variational Inequalities and Decomposition Algorithms, *Journal of Optimization Theory and Applications*, **Vol. 97**, *pp. 407-433*;

[7] Giannessi F. (1995) Separation of sets and Gap Functions for Quasi-Variational Inequalities, in *"Variational Inequalities and Network Equilibrium Problems"*, F.Giannessi and A.Maugeri (eds), Plenum Publishing Co, *pp.101-121*;

[8] Giannessi F. (2000) Vector Variational Inequalities and Vector Equilibria, Kluwer Acad. Publ., Dordrecht, Boston, London;

[9] Harker P.T.,Pang J.S. (1990) Finite–Dimensional Variational Inequalities and Nonlinear Complementarity Problem: a Survey of Theory, Algorithms and Applications, *Mathematical Programming*, **Vol.48**, *pp.161-220*;

[10] Karamardian S. (1976) An Existence Theorem for the Complementary Problem, *Journal of Optimization Theory and Applications*, **Vol.18**, *pp.445-454*;

[11] Mastroeni G. (1999) Minimax and extremum problems associated to a Variational Inequality, *Supplemento ai Rendiconti del Circolo Matematico di Palermo*, **Suppl. 58** , *pp. 185-196*;

[12] Moudafi A. (1999) Proximal Point Algorithm Extended to Equilibrium Problems, *Journal of Natural Geometry*, **Vol. 15**, *pp. 91-100*;

[13] Rockafellar R.T. (1970) Convex Analysis, Princeton University Press, Princeton.

[14] Salmon G., Nguyen V. H. and Strodiot J. J. (2000) Coupling the Auxiliary Problem Principle and Epiconvergence Theory to Solve General Variational Inequalities, *Journal of Optimization Theory and Applications*, **Vol. 104**, *pp. 629-657*;