Mauriana Pesaresi Seminar
June 25, 2019

Simple mathematical proofs that you may not have seen in school

Fabrizio Luccio

The most famous theorem

countless proofs

Irrational numbers

reductio ad absurdum

The golden equation: Φ and F_i

power series

Faces, edges, and vertices

invariant-preserving algorithms

Archangel Gabriel's horn

integral calculus

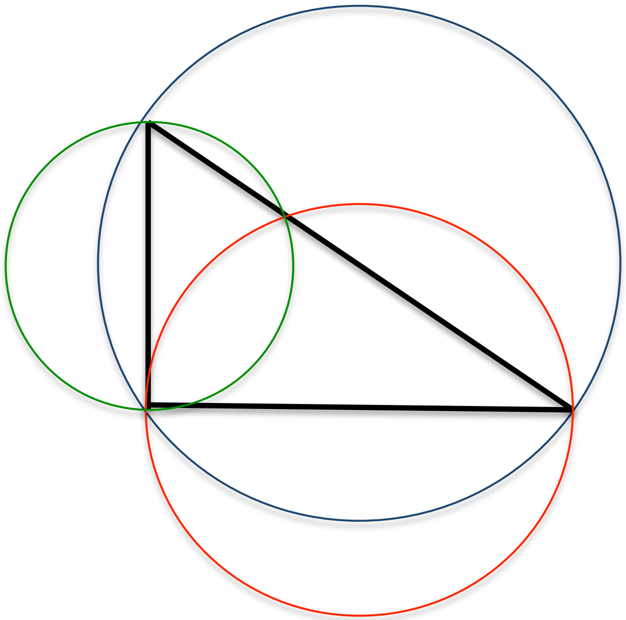
Intersecting a cone

symmetry and metrics

Two amazing sequences

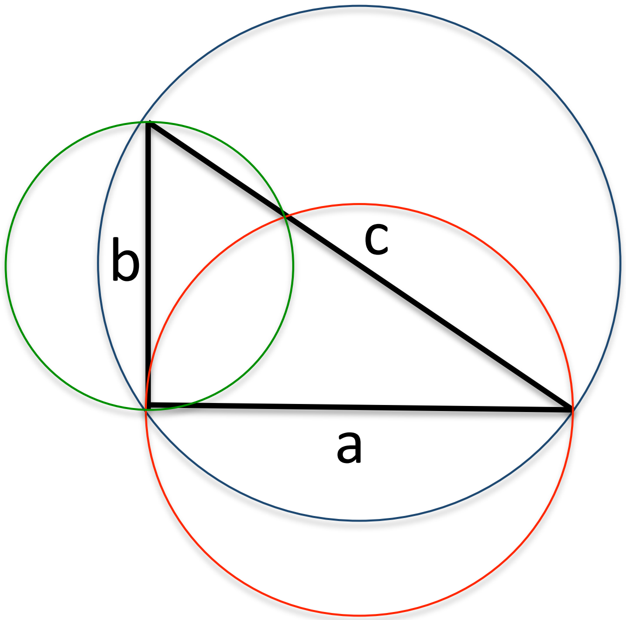
potential functions

Pythagora's Theorem



The circle built on the hypotenuse of a right triangle is equal to the sum of the circles built on the other two sides.

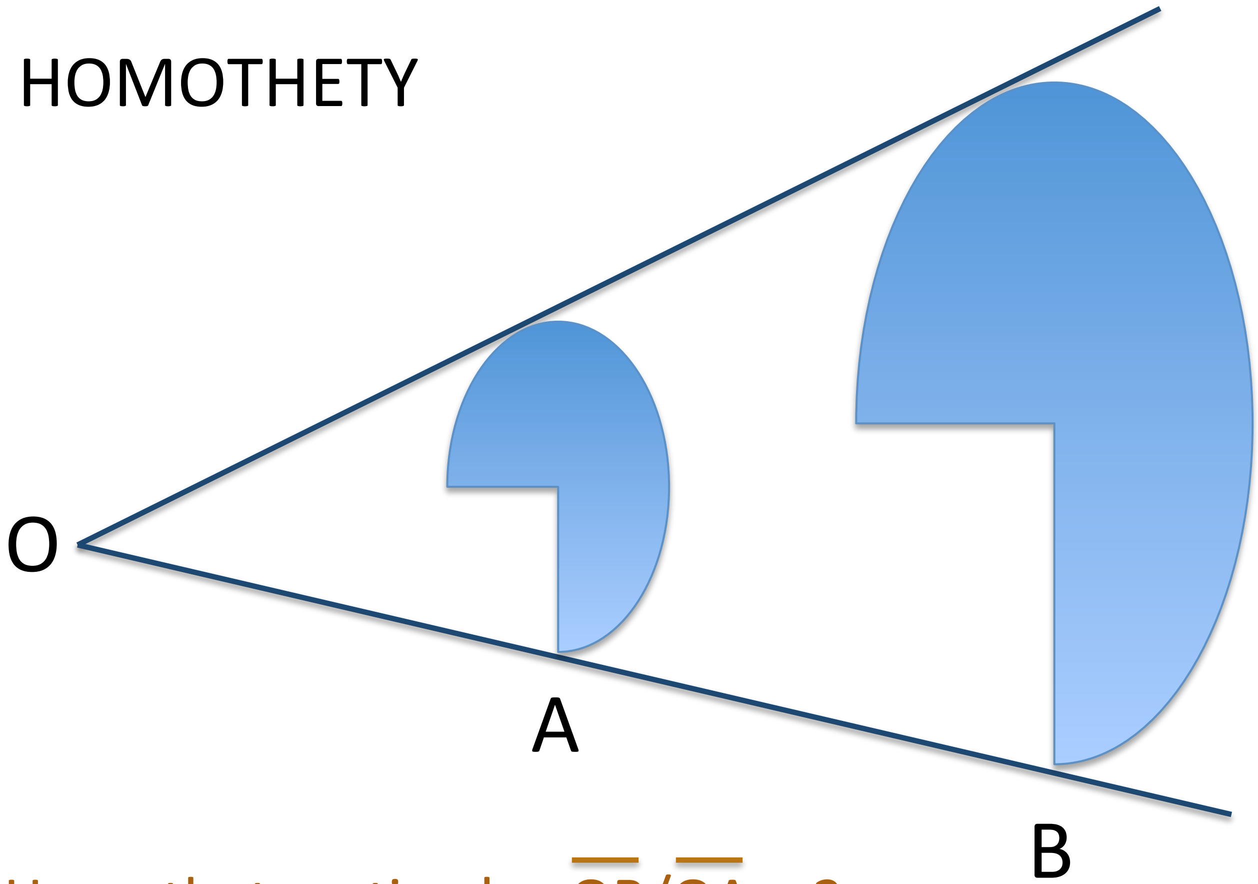
Pythagora's Theorem



$$a^2 + b^2 = c^2$$

$$(\pi/4)\ (a^2 + b^2) = (\pi/4)\ c^2$$

HOMOTHETY

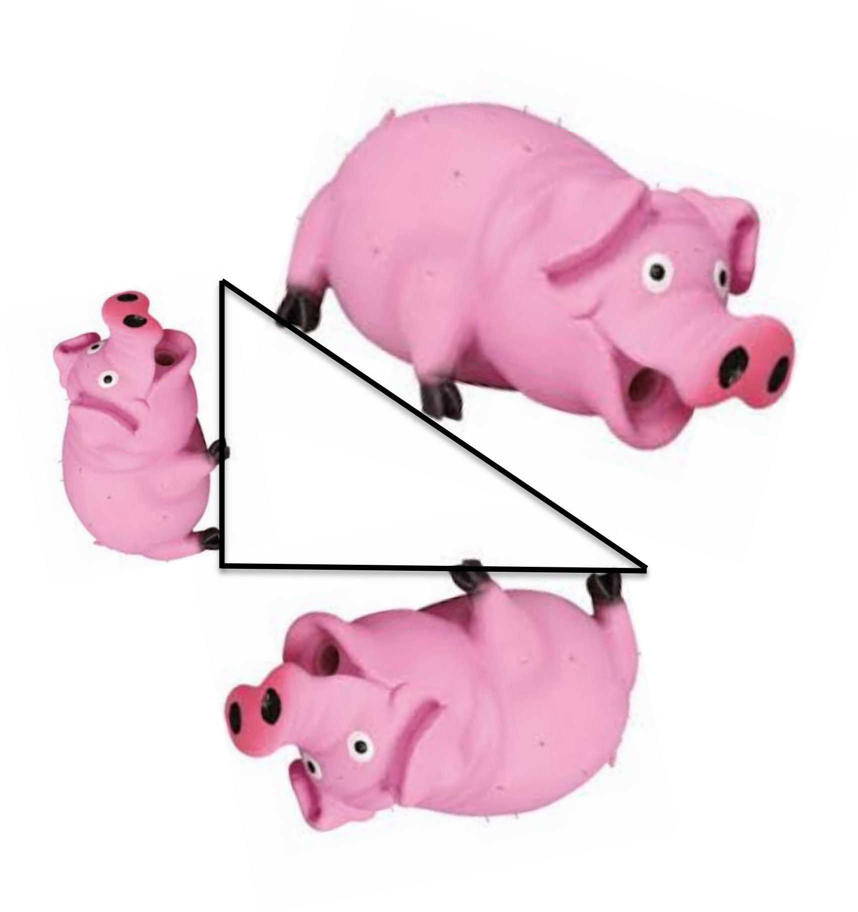


Homothety ratio $k = \overline{OB}/\overline{OA} = 2$

Ratio of the lengths $k = 2$

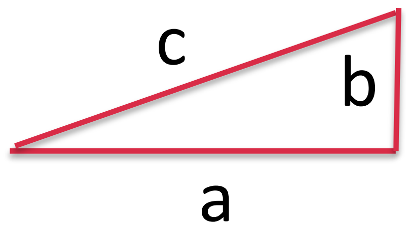
Ratio of the areas $k^2 = 4$

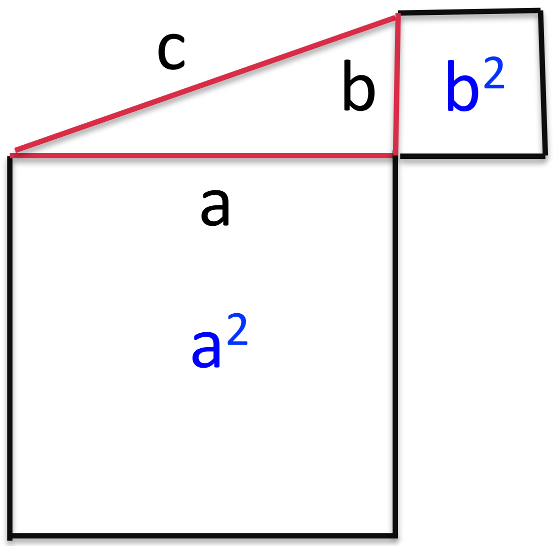
Homothetic Pythagora's Theorem

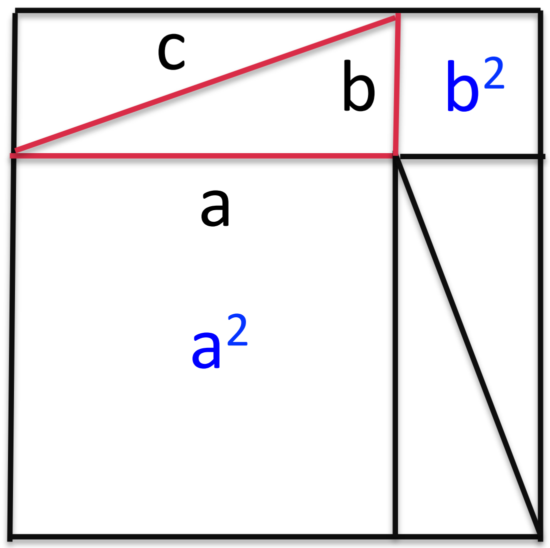


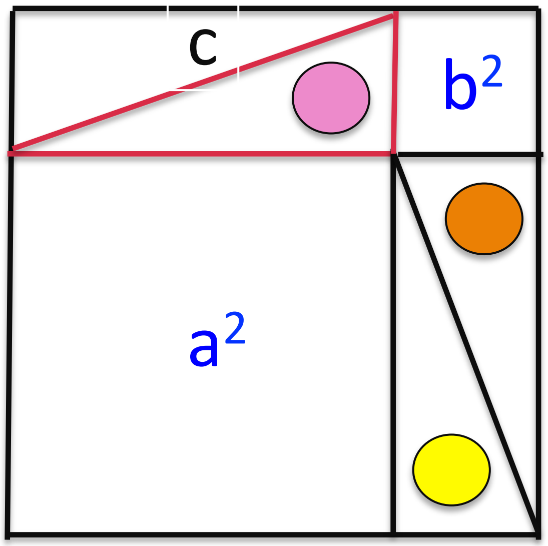
The pig built on the hypotenuse is equal to the sum of the pigs built on the other two sides

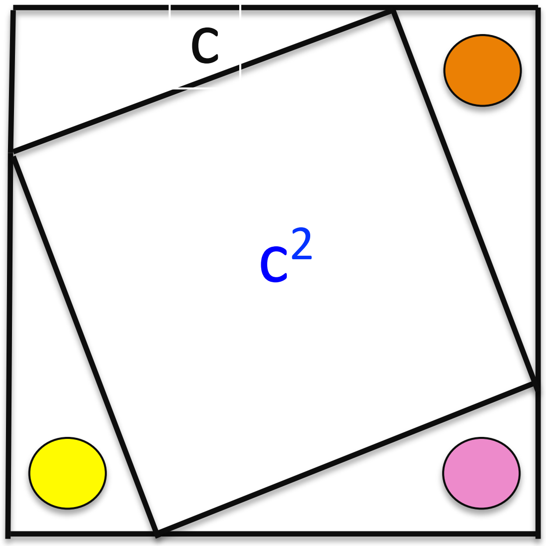
Possibly the simplest proof
is based on translations


c
b
a


c
b
b2
a
a2

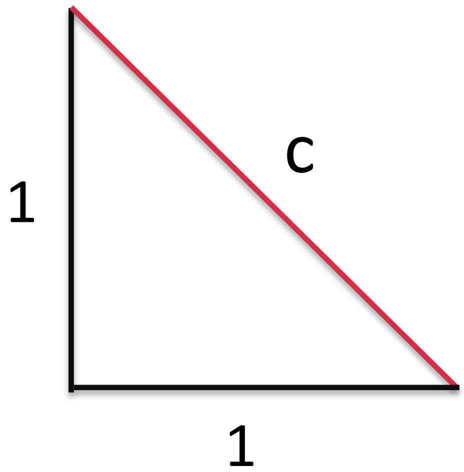

c
b
b2
a
a2


c
b^2
a^2


c
c^2

Reductio ad absurdum

Híppasos of Metapontum is credited for discovering irrationality: he may have speculated on a right triangle



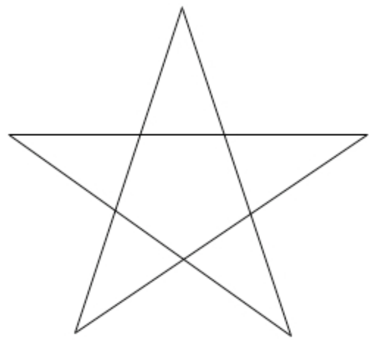
$c^2 = 1^2 + 1^2 = 2$

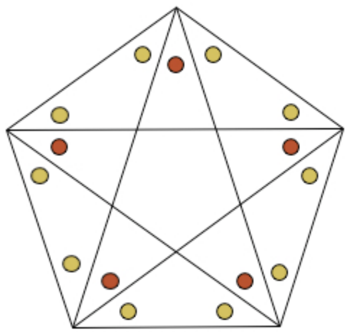
By contradiction assume that c is rational, that is $c = a/b$ with a and b mutually prime integers

$a/b = c \rightarrow a^2/b^2 = 2$ and $a^2 = 2b^2 \rightarrow$ a even hence b odd

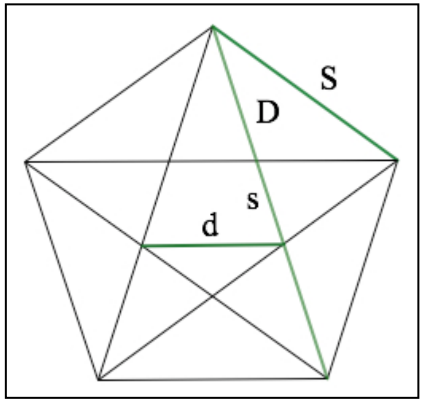
$a = 2q \rightarrow 4q^2 = 2b^2 \rightarrow 2q^2 = b^2 \rightarrow$ b even

It is generally believed that Híppasos discovered irrationality from the *hugieia*, the Pythagoric symbol of perfection from which the pentagon was generated

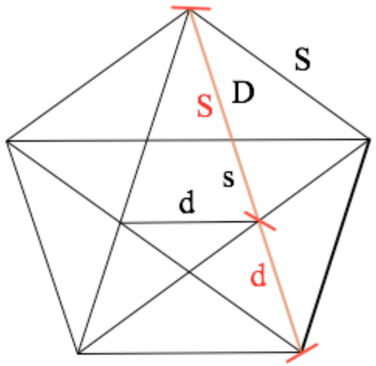




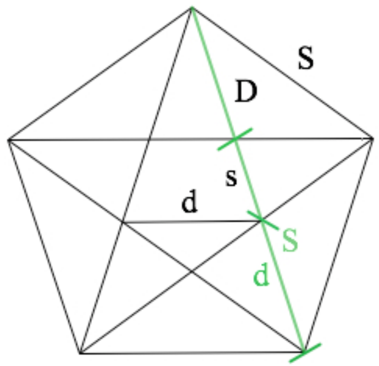
The overall figure contains many similar isosceles triangles



Consider side S
and diagonal D



D - S = d   (1)

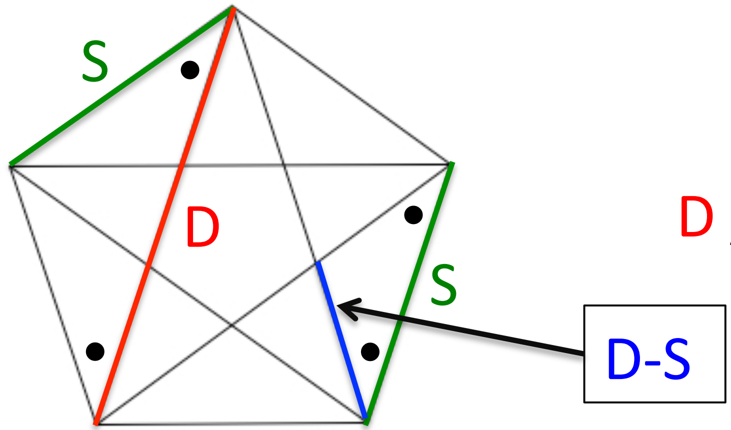


S - d = s   (2)

Proof by reductio ad absurdum that D and S are incommensurable, hence their ratio is an irrational number.

Let k be an arbitrary unit of measure (a segment), such that D, S, are multiples of k. Then by (1) also d is a multiple of k, and by (2) also s is a multiple of k.

Since the partition of the pentagon can go on indefinitely, for any given k we reach a pentagon side s’ with s’ < k.



$D / S = S / D\text{-}S$ (3)

from (3) we have $D^2/S^2 - D/S - 1 = 0$

Letting $D/S = x$ we have the Golden Equation:

$x^2 - x - 1 = 0$ with solutions $x_1 = (1 + \sqrt{5}) / 2$

$x_2 = (1 - \sqrt{5}) / 2$

$(1 + \sqrt{5}) / 2 = \Phi$ is the Golden Ratio

$(1 - \sqrt{5}) / 2 = \Phi'$

$\Phi = 1.618033988 \ldots$ $\Phi' = -0.618033988 \ldots$

$\Phi^2 = 2.618033988 \ldots$

$1/\Phi = 0.618033988 \ldots$

Power series

In the middle of 1800 Jacques Binet discovered a surprising formula based on Φ and Φ' to express the Fibonacci numbers

$$F_n = (1/\sqrt{5})\,(\Phi^n - \Phi'^n)$$

Since $\Phi'^n \rightarrow 0$ for $n \rightarrow \infty$ we have

$$F_{n+1} / F_n \rightarrow \Phi \quad \text{for} \quad n \rightarrow \infty$$

Let us prove Binet's formula (not as Binet did)

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}$$

$F_n = c z^n$ with constant c and z

$c z^n = c z^{n-1} + c z^{n-2} \Rightarrow c z^{n-2}(z^2 - z - 1) = 0$

then, either $c = 0$, $z = 0$, implying $F_n = 0$,
or $z^2 - z - 1 = 0$ with $z_1 = \Phi$, $z_2 = \Phi'$

we have two solutions: $F_n = c_1 \Phi^n$, $F_n = c_2 \Phi'^n$

$2 F_n = c_1 \Phi^n + c_2 \Phi'^n$ i.e. $F_n = b_1 \Phi^n + b_2 \Phi'^n$

Imposing initial conditions to $F_n = b_1 \Phi^n + b_2 \Phi'^n$

$F_0 = 0, \; F_1 = 1$

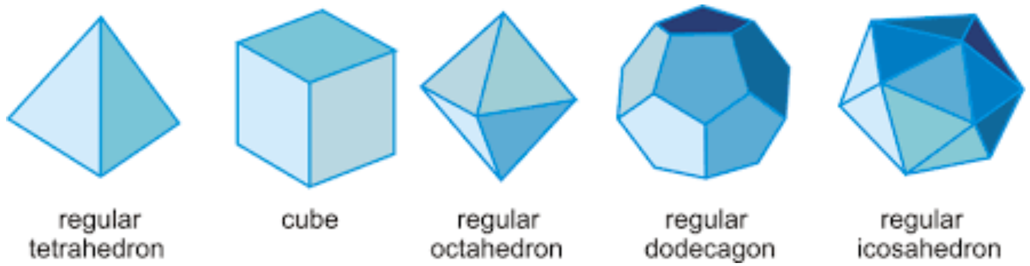
$$0 = b_1 + b_2 \qquad 1 = b_1 \Phi + b_2 \Phi'$$

The solution is $b_1 = 1/\sqrt{5}$ $b_2 = -1/\sqrt{5}$

hence Binet's formula $F_n = (1/\sqrt{5})(\Phi^n - \Phi'^n)$

Invariant-preserving algorithms

In the dialogue Timaeus, Plato explains the nature of the physical world referring to regular polyhedra.



"I have now to speak of their several kinds, and show out of what combinations of numbers each of them was formed".

Then a discussion follows, showing that exactly five regular polyhedra can exist

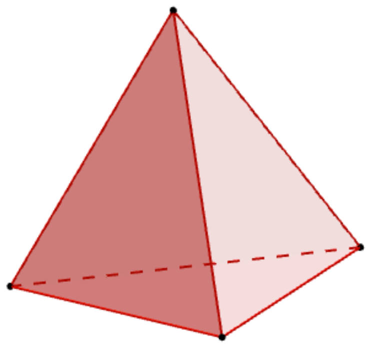
Descartes in the 1600's, Euler in the 1700's, and Cauchy in the 1800's studied polyhedra again. We consider "Euler's formula" and its prove given by Cauchy.

$$V - E + F = k$$

where k is the Euler's characteristic

for polyhedra without holes $k = 2$

e.g.: $V = 4$, $E = 6$, $F = 4$



Let us see Cauchy's proof for $k = 2$.

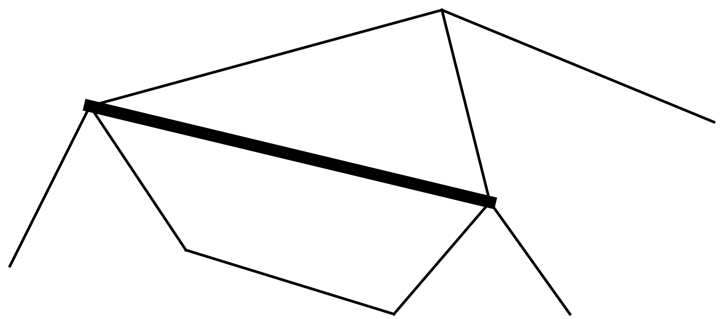
Let f be a face, p be the plane of f, and O be the point at infinity on a straight line orthogonal to p

1. pull the edges of f outwards and project all the other vertices and edges from O onto p so that all their projections are contained into f: we have now a planar graph in p with border b

   ➔ one face is lost, hence we have to prove that $V - E + F = 1$

2. triangulate the graph: for each new edge one new face arises

   $E = E+1$, $F = F+1$ ➔ $V - E + F = 1$ is unchanged



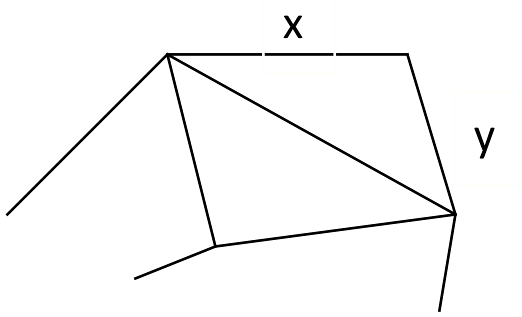
3. while (E > 3)

{ if (there is a face sharing two edges x, y with b)

remove x, y updating b ;

// two edges, one vertex, and one face are lost:

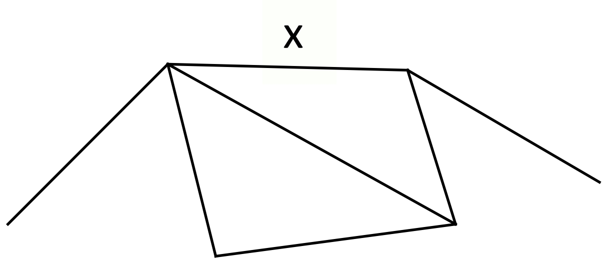
$V - E + F = 1$ is unchanged //



if (there is a face sharing one edge x with b)

remove x updating b ;

// one edge and one face are lost: $V - E + F = 1$ is unchanged // }



The output is one triangle for which

$$V = 3, \quad E = 3, \quad F = 1, \qquad \text{where} \quad V - E + F = 1$$

therefore $V - E + F = 2$ in the original polyhedron.

Euler's formula can be used to rigorously prove that the regular polyhedra are exactly five.

Reaching infinity with integral calculus

Archangel Gabriel's horn

$Y = 1/x$ between 1 and ∞

Evangelista Torricelli built this surface in 1641 by rotating a section of hyperbola around the x axis, with $x \geq 1$

using Cavalieri's techniques, he proved that the corresponding solid has volume π

The radius: $y = 1/x$

$$V = \int_1^\infty \pi(1/x^2)dx = \pi(\lim_{x\to\infty}(-1/x)+1) = \pi$$

The fact that an infinite body had finite volume was considered a paradox.

Worse than that, Torricelli proved that the surface of the horn is infinite !

$$S = \int_1^\infty 2\pi(1/x)dx = 2\pi(\lim_{x\to\infty}(\ln x) + 0) = \infty$$

The paradox caused a strong mathematical and philosophical debate in the XVII century

Cavalieri apparently doubted that his method of computing volumes could contain a bug

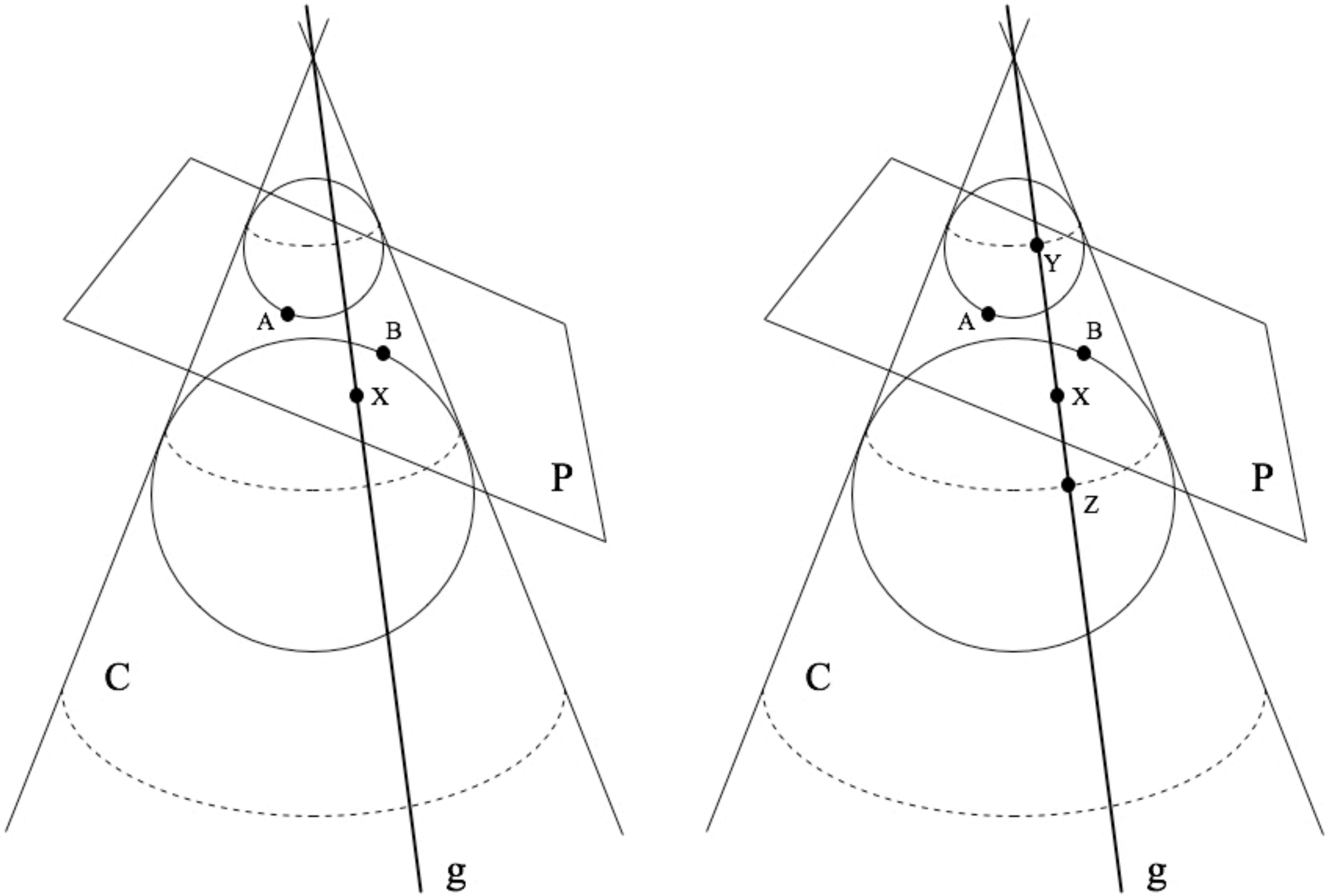
Thomas Hobbes is reported to have said:

“To understand this for sense it is not required that a man should be a geometrician or a logician, but that he should be mad”

Symmetry and metrics

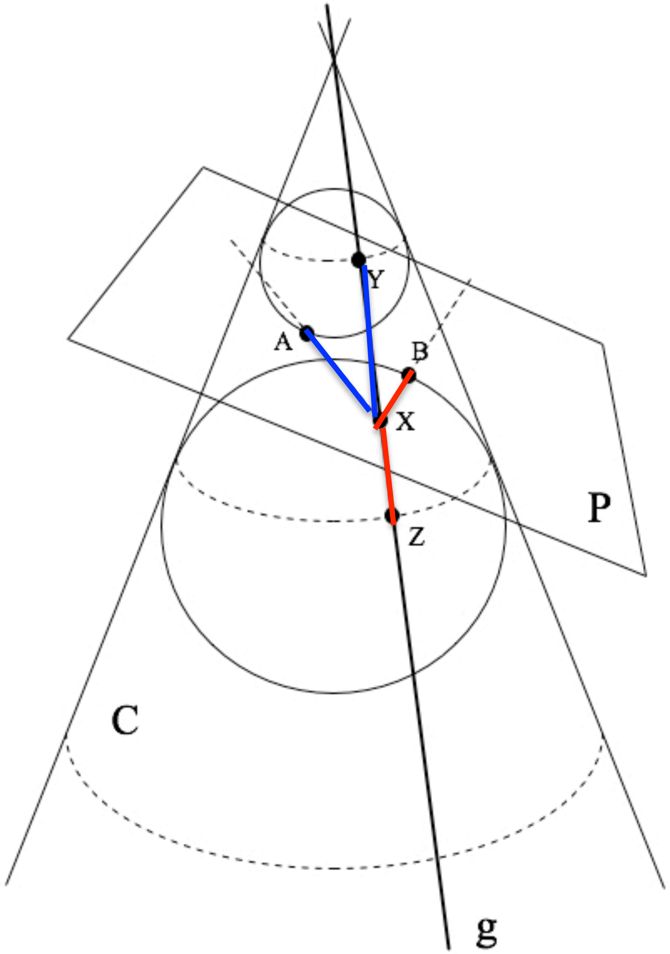
Two centuries B.C., Apollonius studied the *conic sections* obtained intersecting a right circular cone with a plane

In the late 1800, Dandelin gave a metric proof of the construction of the ellipse


A
B
X
P
C
g
Y
A
B
X
Z
P
C
g

$\overline{XY} = \overline{XA}$

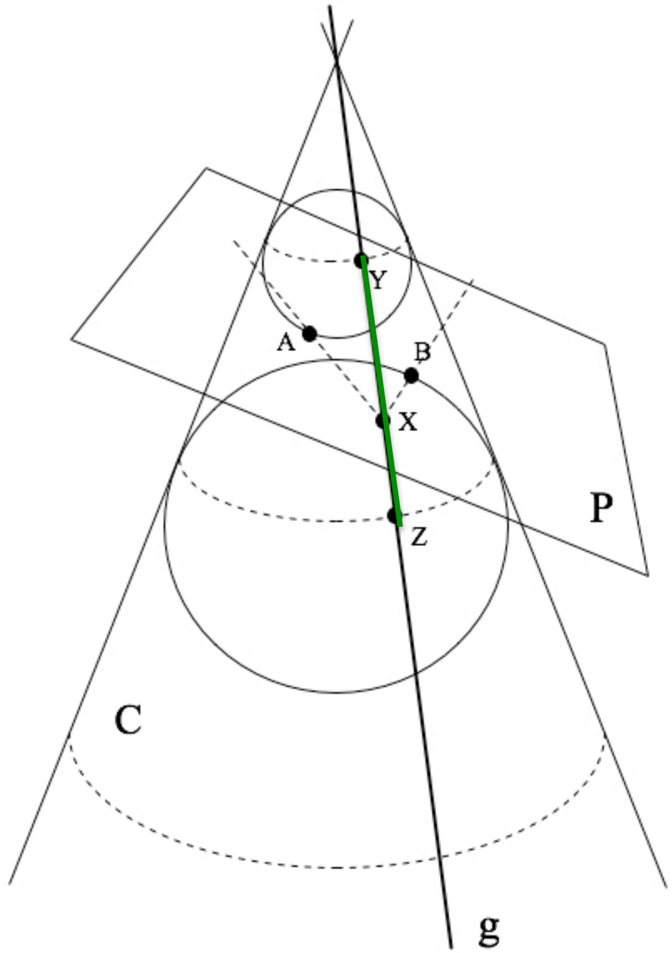
$\overline{XZ} = \overline{XB}$



$\overline{XY} = \overline{XA}$

$\overline{XZ} = \overline{XB}$

$\overline{YZ} = \overline{XA} + \overline{XB}$ **constant**



X describes the locus of points in a plane with a constant sum of distances from two points A, B which in turn are the foci of an ellipse

Potential functions

Two amazing sequences

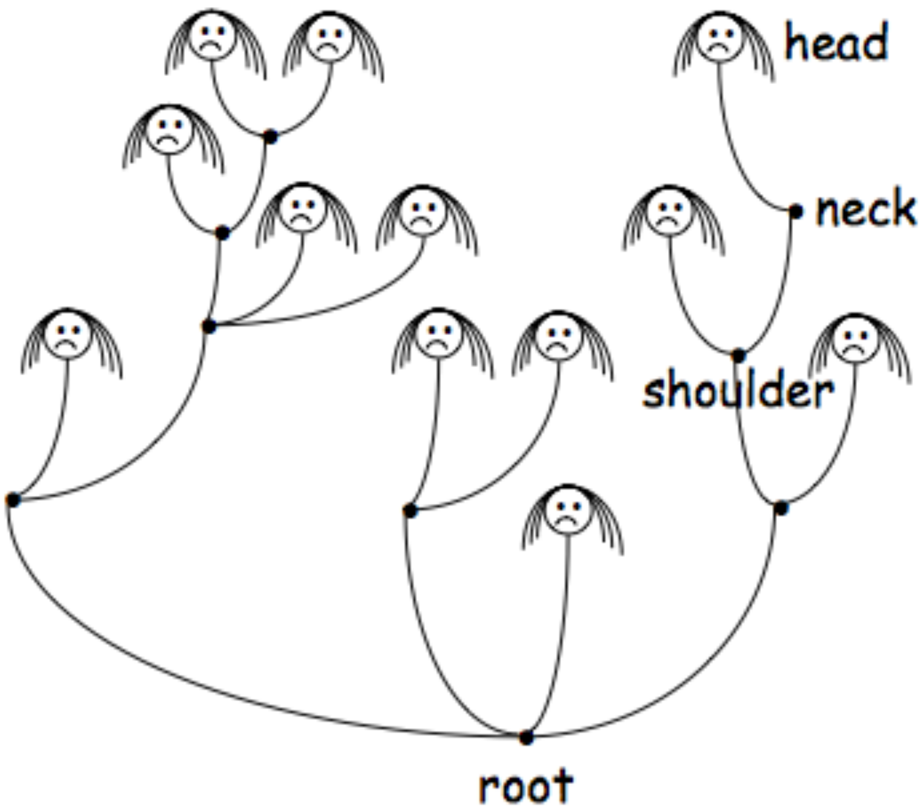
Hercules and the Hydra, 1982

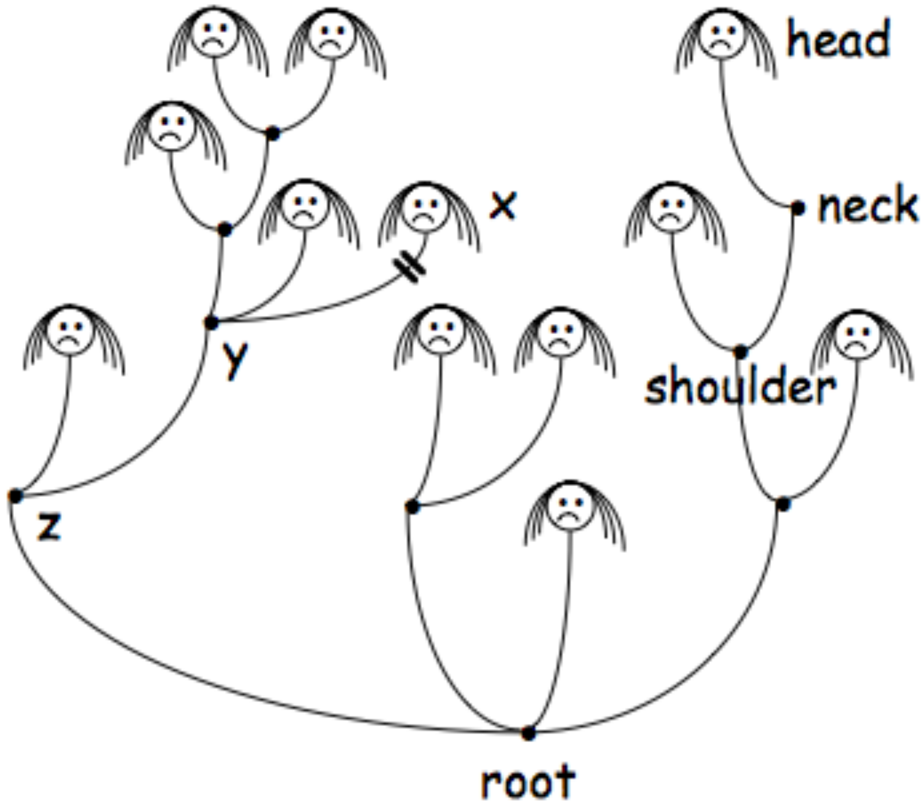
Collatz conjecture, 1937 (still open)

later named after Ulam, Kakutani, Thwaites, Hasse,
or called the Syracuse problem

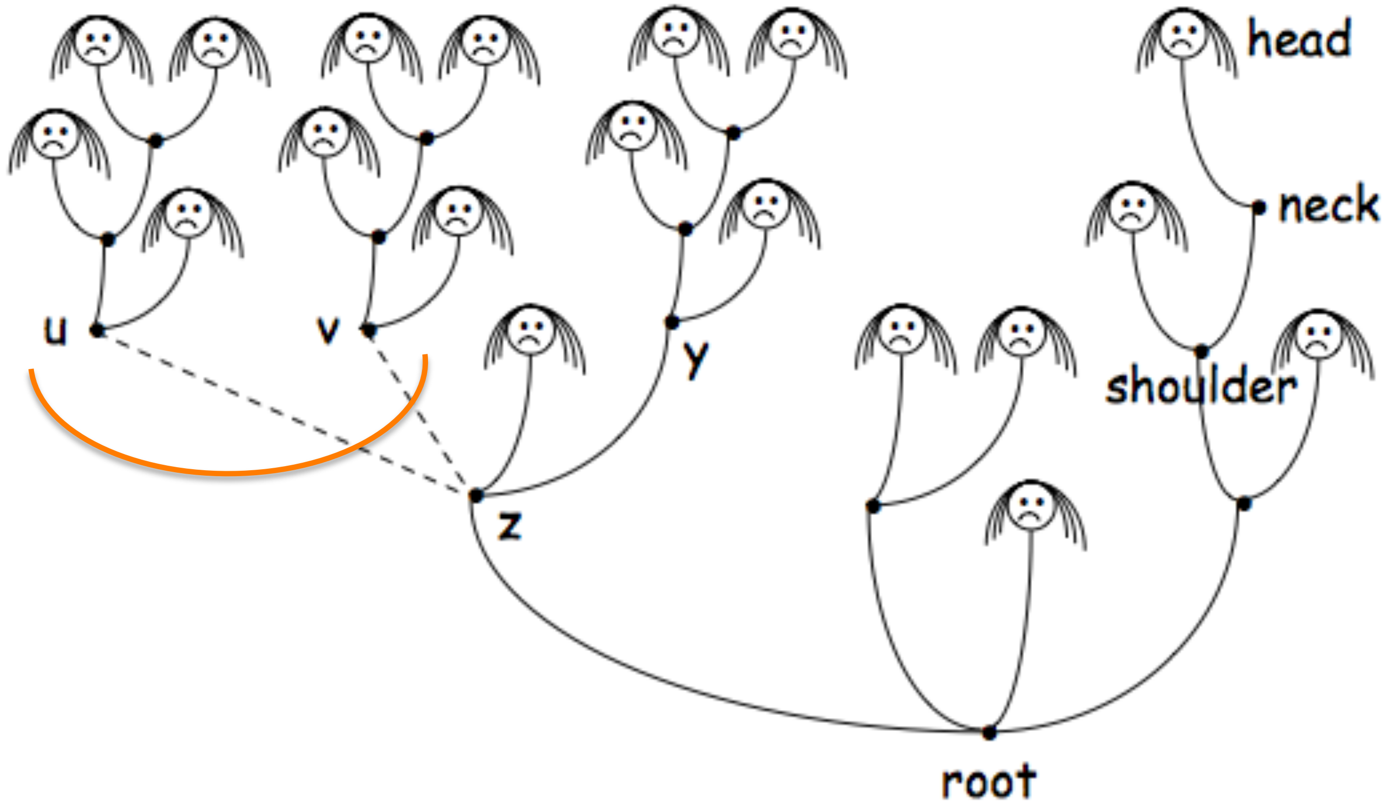
The battle between Hercules and the Hydra

A Hydra is a finite rooted tree





if Hercules chops off one head x



2 copies of the neck and heads sprout off the Hydra's shoulder

Hercules *wins* if, after a finite sequence of head choppings, nothing is left of the Hydra but its root.

Different winning strategies can be found, requiring different (and huge) numbers of steps. More surprisingly:

every head-chopping order is winning

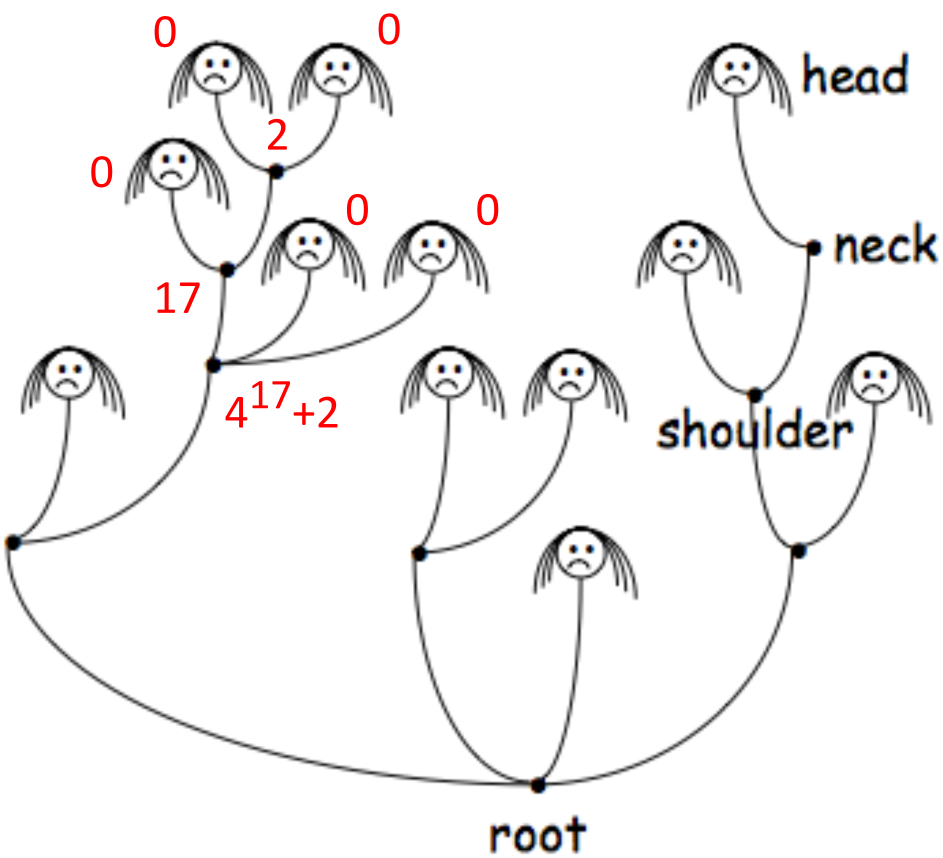
i.e., even the most inexperienced Hercules would eventually win.

To prove this, we define a potential for the Hydra as a function of her shape.

To each node y of the tree, assign a nonnegative integer potential $\pi(y)$ relative to the subtree rooted at y:

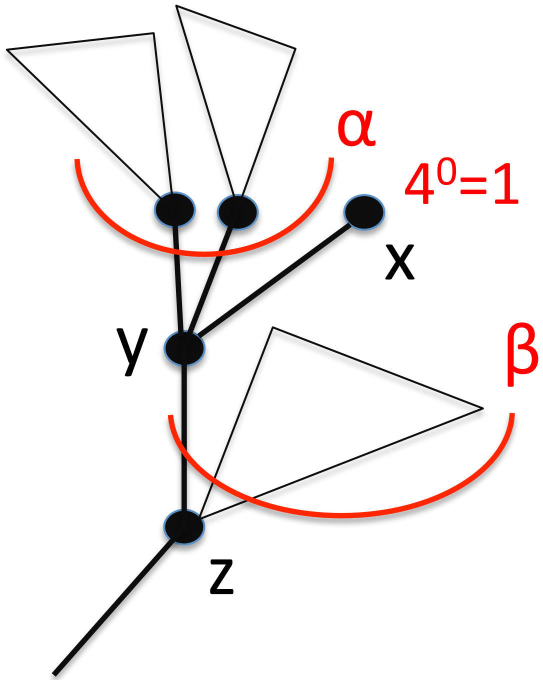
$\pi(y) = 0$, if y is a leaf

$\pi(y) = 4^{\pi(x1)} + \ldots + 4^{\pi(xm)}$, where $x_1, \ldots, x_m$ are the children of y.



If the Hydra reduces to the sole root r
its potential drops to zero.

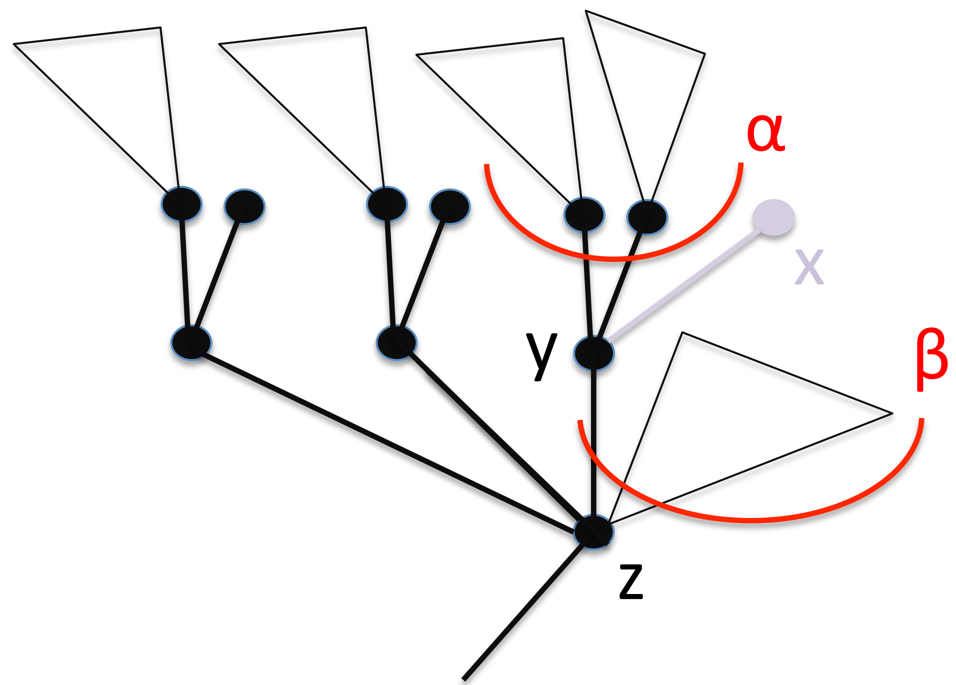
It is then sufficient to prove that π(r)
decreases at each head chopping.



Let x be the leaf to be cut off and $x_1, \ldots, x_k$ be its siblings

Before the cut: $\pi(y) = 4^{\pi(x1)} + \ldots + 4^{\pi(xk)} + 4^{\pi(x)} = \alpha + 1$

$\pi(z) = 4^{\alpha+1} + \beta = 4 \cdot 4^{\alpha} + \beta$



Let x be the leaf to be cut off and $x_1, \ldots, x_k$ be its siblings

Before the cut: $\pi(y) = 4^{\pi(x1)} + \ldots + 4^{\pi(xk)} + 4^{\pi(x)} = \alpha + 1$

$\pi(z) = 4^{\alpha+1} + \beta = 4 \cdot 4^{\alpha} + \beta$

After the cut: $\pi(y) = \alpha$

$\pi(z) = 3 \cdot 4^{\alpha} + \beta$

The decrease of $\pi(z)$ propagates down to the root r thereby decreasing the value of $\pi(r)$, thus proving our claim.

Collatz rule n positive integer

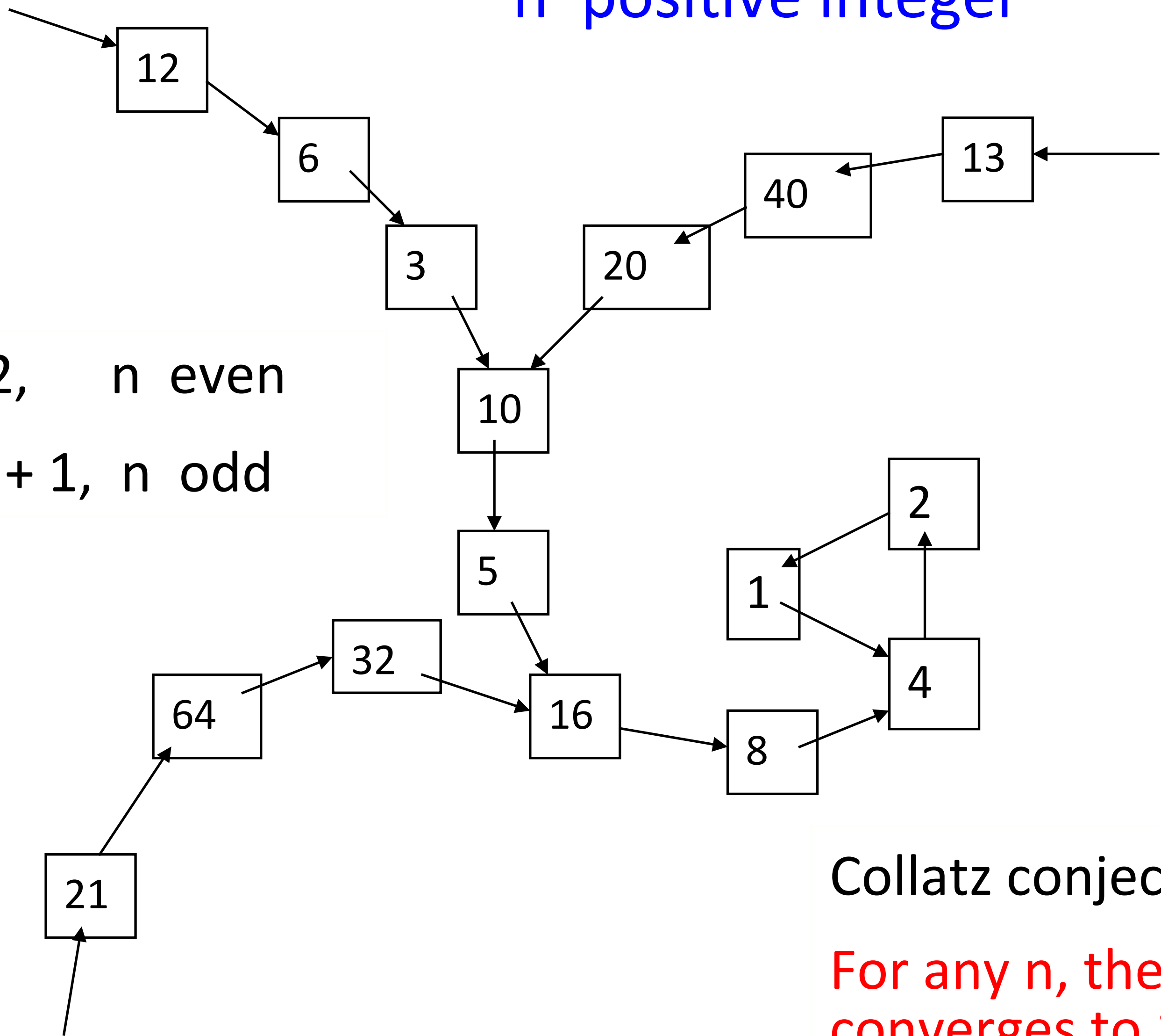
$n = n / 2$, n even

$n = 3n + 1$, n odd

n positive integer

$n = n/2$, n even

$n = 3n + 1$, n odd



Collatz conjecture:

For any n, the series converges to 1

The sequence for $n = 27$ takes 111 steps, climbing to 9232 before descending to 1

27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077, ***9232***, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1

Although it has been extensively studied, Collatz conjecture is still open. It would be interesting finding e. g. an ad hoc potential function for the integers to prove the conjecture.

Sturgeon's Law (1958): 90% of everything is crud

I hope we stayed in the 10% complementary set