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June 25, 2019

Simple mathematical proofs that you may not have seen in school

Fabrizio Luccio
The most famous theorem
countless proofs
Irrational numbers
reductio ad absurdum
The golden equation: $\Phi$ and $F_i$
power series
Faces, edges, and vertices
invariant-preserving algorithms
Archangel Gabriel’s horn
integral calculus
Intersecting a cone
symmetry and metrics
Two amazing sequences
potential functions
Pythagora’s Theorem

The circle built on the hypotenuse of a right triangle is equal to the sum of the circles built on the other two sides.
Pythagora’s Theorem

\[ a^2 + b^2 = c^2 \]

\[ (\pi/4) \left( a^2 + b^2 \right) = (\pi/4) c^2 \]
Homothety ratio \( k = \frac{OB}{OA} = 2 \)

Ratio of the lengths \( k = 2 \)

Ratio of the areas \( k^2 = 4 \)
Homothetic Pythagora’s Theorem

The pig built on the hypotenuse is equal to the sum of the pigs built on the other two sides
Possibly the simplest proof is based on translations.
Híppasos of Metapontum is credited for discovering irrationality: he may have speculated on a right triangle

\[ c^2 = 1^2 + 1^2 = 2 \]

By contradiction assume that \( c \) is rational, that is \( c = \frac{a}{b} \) with \( a \) and \( b \) mutually prime integers

\[
\frac{a}{b} = c \quad \Rightarrow \quad \frac{a^2}{b^2} = 2 \quad \text{and} \quad a^2 = 2b^2 \quad \Rightarrow \quad a \text{ even hence } b \text{ odd}
\]

\[
a = 2q \quad \Rightarrow \quad 4q^2 = 2b^2 \quad \Rightarrow \quad 2q^2 = b^2 \quad \Rightarrow \quad b \text{ even}
\]
It is generally believed that Híppasos discovered irrationality from the *hugieia*, the Pythagoric symbol of perfection from which the pentagon was generated.

The overall figure contains many similar isosceles triangles.
Consider side $S$ and diagonal $D$

$$D - S = d \quad (1)$$

$$S - d = s \quad (2)$$

Proof by *reductio ad absurdum* that $D$ and $S$ are *incommensurable*, hence their ratio is an irrational number.

Let $k$ be an arbitrary unit of measure (a segment), such that $D, S$ are multiples of $k$. Then by (1) also $d$ is a multiple of $k$, and by (2) also $s$ is a multiple of $k$.

Since the partition of the pentagon can go on indefinitely, for any given $k$ we reach a pentagon side $s'$ with $s' < k$. 

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<table>
<thead>
<tr>
<th>Consider side S</th>
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from (3) we have \[ D^2/S^2 - D/S - 1 = 0 \]

Letting \( D/S = x \) we have the **Golden Equation:**

\[ x^2 - x - 1 = 0 \]

with solutions

\[ x_1 = \frac{1 + \sqrt{5}}{2} \]
\[ x_2 = \frac{1 - \sqrt{5}}{2} \]
(1 + \sqrt{5}) / 2 = \Phi  \text{ is the Golden Ratio}
(1 - \sqrt{5}) / 2 = \Phi' \\
\Phi = 1.618033988 \ldots \quad \Phi' = -0.618033988 \ldots \\
\Phi^2 = 2.618033988 \ldots \\
1/\Phi = 0.618033988 \ldots
Power series

In the middle of 1800 Jacques Binet discovered a surprising formula based on $\Phi$ and $\Phi'$ to express the Fibonacci numbers

$$F_n = \left(\frac{1}{\sqrt{5}}\right)(\Phi^n - \Phi'^n)$$

Since $\Phi'^n \to 0$ for $n \to \infty$ we have

$$\frac{F_{n+1}}{F_n} \to \Phi \text{ for } n \to \infty$$

Let us prove Binet’s formula (not as Binet did)
\[ F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \]

\[ F_n = cz^n \quad \text{with constant } c \text{ and } z \]

\[ cz^n = cz^{n-1} + cz^{n-2} \implies cz^{n-2}(z^2 - z - 1) = 0 \]

then, either \( c = 0, \ z = 0 \), implying \( F_n = 0 \),

or \( z^2 - z - 1 = 0 \) with \( z_1 = \Phi, \ z_2 = \Phi' \)

we have two solutions: \( F_n = c_1 \Phi^n, \ F_n = c_2 \Phi'^n \)

\[ 2F_n = c_1 \Phi^n + c_2 \Phi'^n \quad \text{i.e.} \quad F_n = b_1 \Phi^n + b_2 \Phi'^n \]
Imposing initial conditions to \( F_n = b_1 \Phi^n + b_2 \Phi'^n \)

\( F_0 = 0, \quad F_1 = 1 \)

\[
0 = b_1 + b_2 \quad 1 = b_1 \Phi + b_2 \Phi'
\]

The solution is \( b_1 = 1/\sqrt{5} \quad b_2 = -1/\sqrt{5} \)

hence Binet’s formula \( F_n = (1/\sqrt{5})(\Phi^n - \Phi'^n) \)
Invariant-preserving algorithms

In the dialogue Timaeus, Plato explains the nature of the physical world referring to regular polyhedra.

“\(I\) have now to speak of their several kinds, and show out of what combinations of numbers each of them was formed”.

Then a discussion follows, showing that exactly five regular polyhedra can exist.
Descartes in the 1600’s, Euler in the 1700’s, and Cauchy in the 1800’s studied polyhedra again. We consider “Euler’s formula” and its prove given by Cauchy.

\[ V - E + F = k \]

where \( k \) is the Euler’s characteristic

for polyhedra without holes \( k = 2 \)

e.g.: \( V = 4, E = 6, F = 4 \)

Let us see Cauchy’s proof for \( k = 2 \).
Let $f$ be a face, $p$ be the plane of $f$, and $O$ be the point at infinity on a straight line orthogonal to $p$.

1. pull the edges of $f$ outwards and project all the other vertices and edges from $O$ onto $p$ so that all their projections are contained into $f$: we have now a planar graph in $p$ with border $b$

$\Rightarrow$ one face is lost, hence we have to prove that $V - E + F = 1$

2. triangulate the graph: for each new edge one new face arises

$E = E+1, \ F = F+1 \ \Rightarrow \ V - E + F = 1 \ \text{is unchanged}$

\[\begin{array}{c}
\end{array}\]
3. while (E > 3) 
   { if (there is a face sharing two edges x, y with b) 
       remove x, y updating b; 
       // two edges, one vertex, and one face are lost: 
       \( V - E + F = 1 \) is unchanged // 
   
   if (there is a face sharing one edge x with b) 
       remove x updating b; 
       // one edge and one face are lost: \( V - E + F = 1 \) is unchanged // 
   }
The output is one triangle for which

\[ V = 3, \ E = 3, \ F = 1, \quad \text{where} \quad V - E + F = 1 \]

therefore \[ V - E + F = 2 \] in the original polyhedron.

Euler’s formula can be used to rigorously prove that the regular polyhedra are exactly five.
Reaching infinity with integral calculus

Archangel Gabriel’s horn

\[ Y = \frac{1}{x} \text{ between } 1 \text{ and } \infty \]
Evangelista Torricelli built this surface in 1641 by rotating a section of hyperbola around the $x$ axis, with $x \geq 1$

using Cavalieri’s techniques, he proved that the corresponding solid has volume $\pi$

The radius: $y = 1/x$

$$V = \int_1^\infty \pi (1/x^2) dx = \pi (\lim_{x \to \infty} (-1/x) + 1) = \pi$$
The fact that an infinite body had finite volume was considered a paradox.

Worse than that, Torricelli proved that the surface of the horn is infinite!

\[ S = \int_1^\infty 2\pi \left(\frac{1}{x}\right) dx = 2\pi \left(\lim_{x \to \infty} (\ln x + 0)\right) = \infty \]
The paradox caused a strong mathematical and philosophical debate in the XVII century

Cavalieri apparently doubted that his method of computing volumes could contain a bug

Thomas Hobbes is reported to have said:
  “To understand this for sense it is not required that a man should be a geometrician or a logician, but that he should be mad”
Symmetry and metrics

Two centuries B.C., Apollonius studied the **conic sections** obtained intersecting a right circular cone with a plane.

In the late 1800, Dandelin gave a metric proof of the construction of the ellipse.
$X$ describes the locus of points in a plane with a constant sum of distances from two points $A$, $B$ which in turn are the foci of an ellipse.
Potential functions

Two amazing sequences

Hercules and the Hydra, 1982

Collatz conjecture, 1937 (still open)

later named after Ulam, Kakutani, Thwaites, Hasse,
or called the Syracuse problem
The battle between Hercules and the Hydra

A Hydra is a finite rooted tree
if Hercules chops off one head  x
2 copies of the neck and heads sprout off the Hydra’s shoulder
Hercules *wins* if, after a finite sequence of head choppings, nothing is left of the Hydra but its root.

Different winning strategies can be found, requiring different (and huge) numbers of steps. More surprisingly:

*every head-chopping order is winning*

i.e., even the most inexperienced Hercules would eventually win.

To prove this, we define a potential for the Hydra as a function of her shape.
To each node $y$ of the tree, assign a nonnegative integer potential $\pi(y)$ relative to the subtree rooted at $y$:

$$\pi(y) = 0, \text{ if } y \text{ is a leaf}$$

$$\pi(y) = 4^{\pi(x_1)} + \ldots + 4^{\pi(x_m)}, \text{ where } x_1, \ldots, x_m \text{ are the children of } y.$$
If the Hydra reduces to the sole root $r$ its potential drops to zero.

It is then sufficient to prove that $\pi(r)$ decreases at each head chopping.
Let $x$ be the leaf to be cut off and $x_1, \ldots, x_k$ be its siblings.

Before the cut:

$\pi(y) = 4^{\pi(x_1)} + \ldots + 4^{\pi(x_k)} + 4^{\pi(x)} = \alpha + 1$

$\pi(z) = 4^{\alpha+1} + \beta = 4 \cdot 4^\alpha + \beta$
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\]
\[
\pi(z) = 4^{\alpha+1} + \beta = 4 \cdot 4^\alpha + \beta
\]

After the cut:

\[
\pi(y) = \alpha
\]
\[
\pi(z) = 3 \cdot 4^\alpha + \beta
\]
The decrease of \( \pi(z) \) propagates down to the root \( r \) thereby decreasing the value of \( \pi(r) \), thus proving our claim.
Collatz rule \quad n \quad \text{positive integer}

\begin{align*}
  n &= n/2, \quad n \quad \text{even} \\
  n &= 3n + 1, \quad n \quad \text{odd}
\end{align*}
Collatz conjecture:
For any n, the series converges to 1

n positive integer

n = n/2, n even
n = 3n + 1, n odd
The sequence for $n = 27$ takes 111 steps, climbing to 9232 before descending to 1

Although it has been extensively studied, Collatz conjecture is still open. It would be interesting finding e. g. an ad hoc potential function for the integers to prove the conjecture.
Sturgeon’s Law (1958):  90% of everything is crud

I hope we stayed in the 10% complementary set