The fix point THEORY

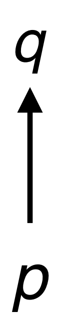
POSET (Partially ordered set, PO)

$(P, \sqsubseteq)$ $\qquad \sqsubseteq \subseteq P \times P$

reflexive $\quad \forall p \in P. \quad p \sqsubseteq p$

antisymmetry $\quad \forall p, q \in P. \quad p \sqsubseteq q \ \wedge \ q \sqsubseteq p \ \Rightarrow \ p = q$

transitive $\quad \forall p, q, r \in P. \quad p \sqsubseteq q \ \wedge \ q \sqsubseteq r \ \Rightarrow \ p \sqsubseteq r$



$p \sqsubseteq q$ means that p is less than (or equal to) q

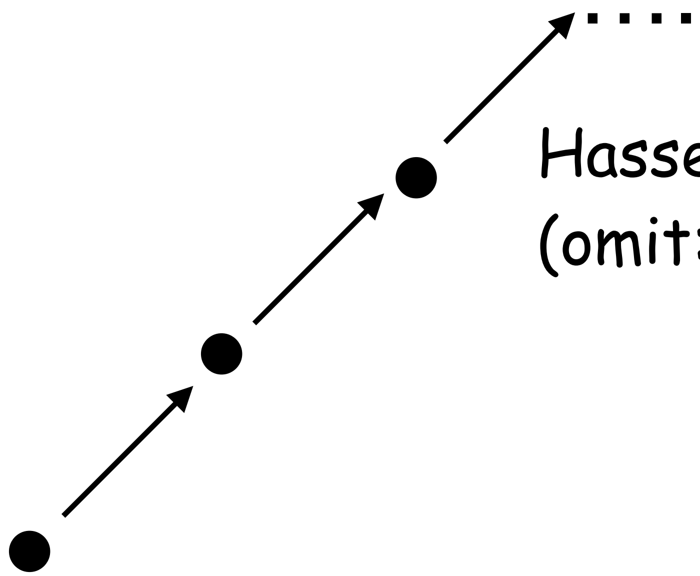
$p \sqsubset q$ means $p \sqsubseteq q \ \wedge \ p \neq q$

Total Order

A PO $(P, \sqsubseteq)$ is total iff

$$\forall p, q \in P. \quad p \sqsubseteq q \ \vee \ q \sqsubseteq p$$

A PO where every two elements are comparable



Hasse diagram notation
(omit: reflexive arcs,
transitive arcs)

Discrete orders

$(P, \sqsubseteq)$ PO

discrete $\forall p, q \in P. \quad p \sqsubseteq q \Leftrightarrow p = q$

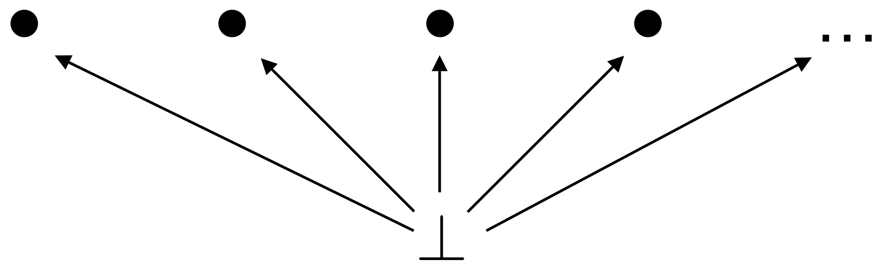
any element is **comparable** only to itself

Flat orders

$(P, \sqsubseteq)$ PO

flat $\forall p, q \in P. \quad p \sqsubseteq q \ \Leftrightarrow \ p = q \ \vee \ p = \bot$

any element is **comparable** only to itself
and with a distinguished (smaller) element $\bot$



Exercise

$(\mathbb{N}, \leq)$

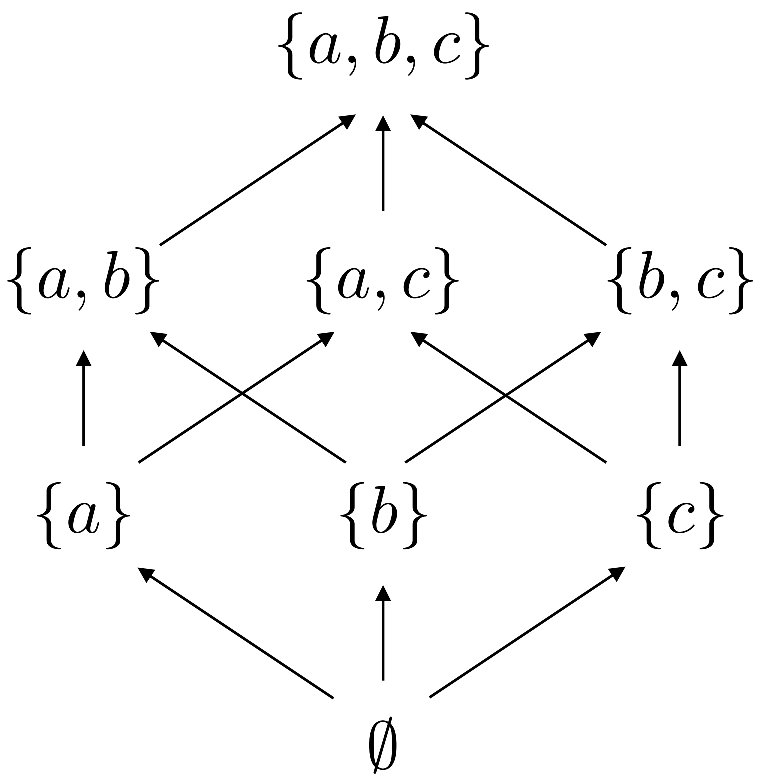
PO?	Total?	Discrete?	Flat?
✓	✓	✗	✗

Exercise

$(\wp(S), \subseteq)$

PO?	Total?	Discrete?	Flat?
✓	$\lvert S\rvert < 2$	$S = \emptyset$	$\lvert S\rvert < 2$

example: $S = \{a, b, c\}$



$\{a, b\} \not\subseteq \{b, c\}$

$\{a, b\} \not\supseteq \{b, c\}$

$\{a\} \not\subseteq \{b\}$

$\{a\} \not\supseteq \{b\}$

Exercise

$(\mathbb{N}, =)$

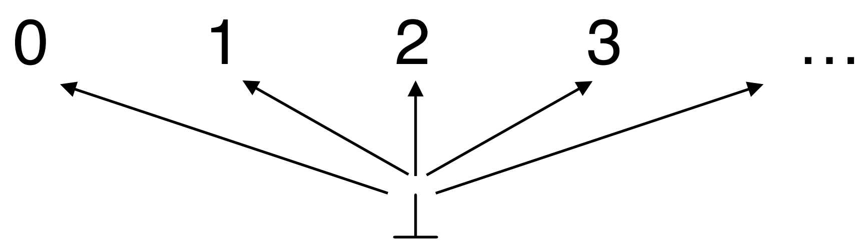
PO?	Total?	Discrete?	Flat?
✓	✗	✓	✗

0 1 2 3 …

Exercise

$$(\mathbb{N} \cup \{\bot\}, \{(\bot, n) \mid n \in \mathbb{N}\} \;\;)$$

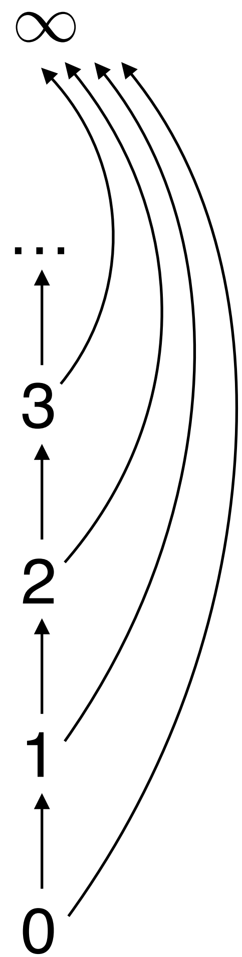
PO?	Total?	Discrete?	Flat?
✓	✗	✗	✓



Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

PO?	Total?	Discrete?	Flat?
✓	✓	✗	✗



Exercise

	PO?	Total?	Discrete?	Flat?
$(\mathbb{N}, <)$	✗	✗	✗	✗
$(\mathbb{Z}, \leq)$	✓	✓	✗	✗
$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$	✓	✓	✗	✗
$(\mathbb{N}, \neq)$	✗	✗	✗	✗

Element properties
(least, minimal, ...)

Least element

$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad \ell \in Q$

ℓ is a least element of Q if $\forall q \in Q. \;\; \ell \sqsubseteq q$

TH. (uniqueness of least element)

$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad \ell_1, \ell_2$ least elements of Q implies $\ell_1 = \ell_2$

Bottom

$(P, \sqsubseteq)$ PO

the least element of P
(if it exists) is called bottom and denoted $\bot$

sometimes written $\bot_P$

Examples

PO	bottom?
$(\mathbb{N} \cup \{\infty\}, \leq)$	0
$(\wp(S), \subseteq)$	$\varnothing$
$(\mathbb{Z}, \leq)$	✗

Minimal element

$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad m \in Q$

m is a minimal element of Q if $\forall q \in Q. \quad q \sqsubseteq m \Rightarrow q = m$

(no smaller element can be found in Q)

least $\forall q \in Q. \quad \ell \sqsubseteq q$	minimal $\forall q \in Q. \quad q \sqsubseteq m \Rightarrow q = m$
unique	not necessarily unique
minimal	not necessarily least can be least

Reverse order

TH. $(P, \sqsubseteq)$ PO implies $(P, \sqsupseteq)$ PO

$(P, \sqsubseteq)$ PO $Q \subseteq P$

greatest element: least element of Q w.r.t. $(P, \sqsupseteq)$

top element: $\top$ greatest element of P (if it exists) becomes $\bot$

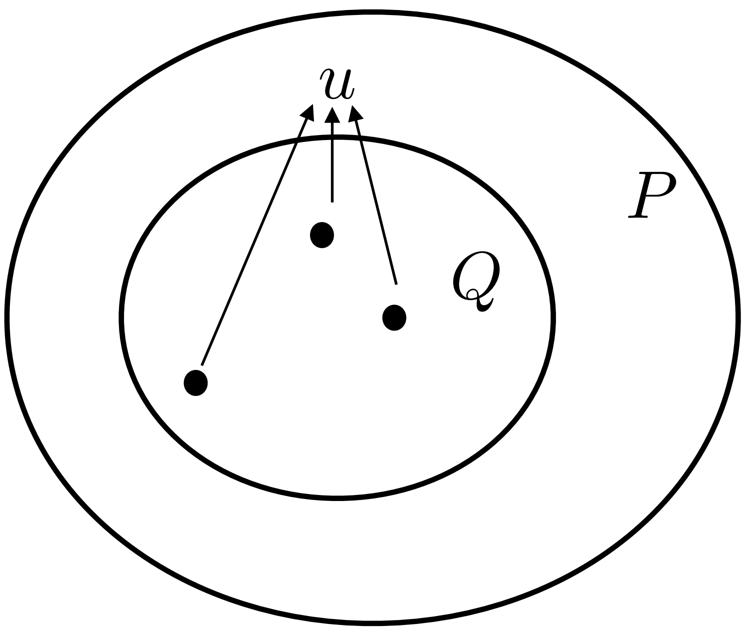
maximal element: minimal element of Q w.r.t. $(P, \sqsupseteq)$

Upper bound

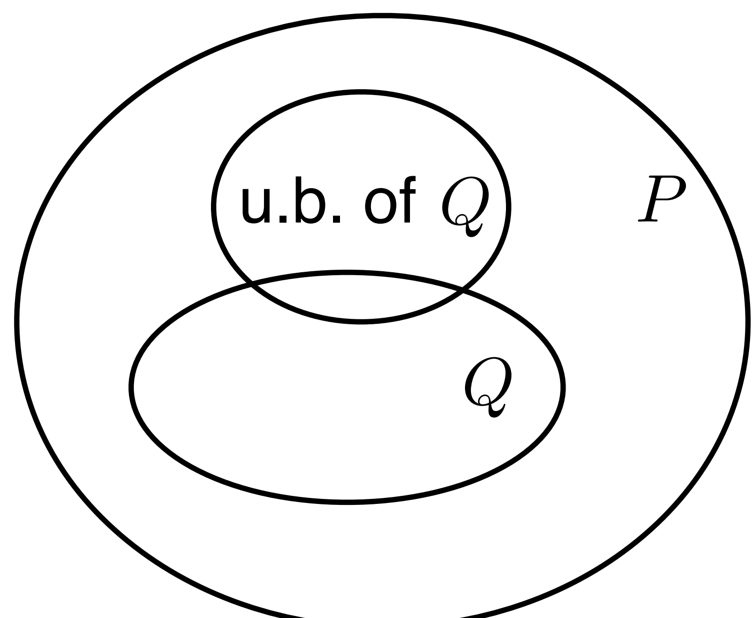
$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad u \in P$

u is an upper bound of Q if $\forall q \in Q.\ q \sqsubseteq u$

(all the elements of Q are smaller than u)



Q may have many upper bounds



Least upper bound

$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad p \in P$

p is the least upper bound (lub) of Q if

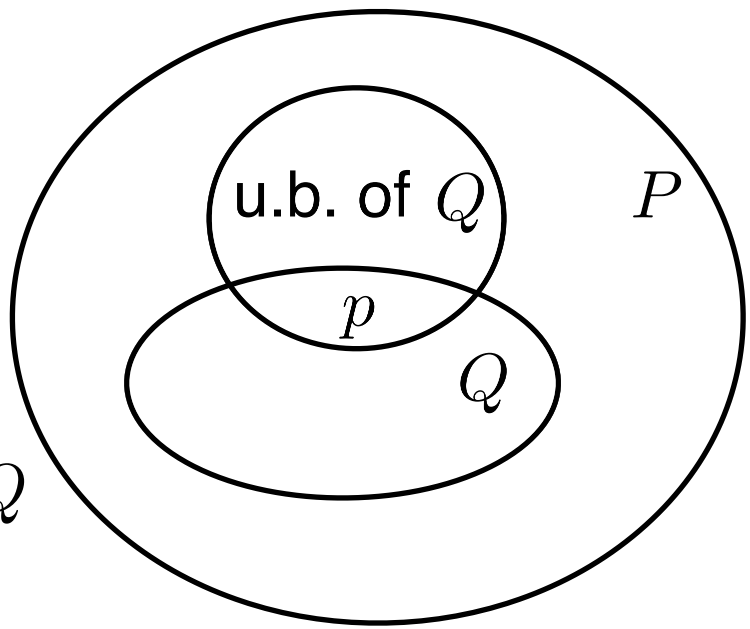
1. it is an upper bound of Q $\quad \forall q \in Q. \quad q \sqsubseteq p$
2. it is smaller than any other upper bound of Q

$$\forall u \in P. \quad (\forall q \in Q. q \sqsubseteq u) \Rightarrow p \sqsubseteq u$$

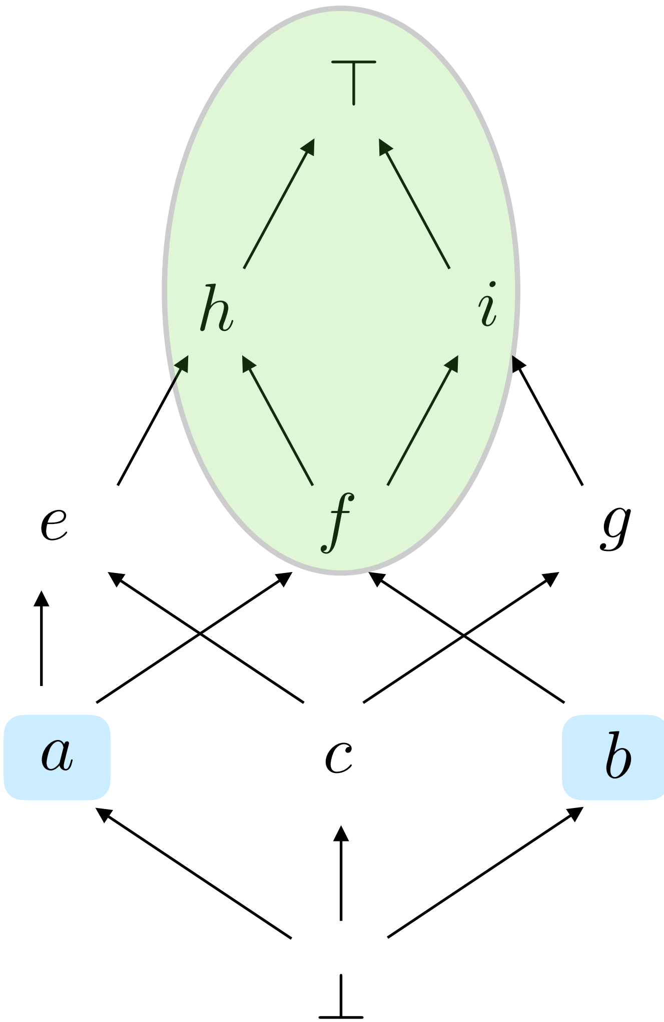
we write $p = lub\ Q$

intuitively, it is the least element that represents all of Q

p not necessarily an element of Q



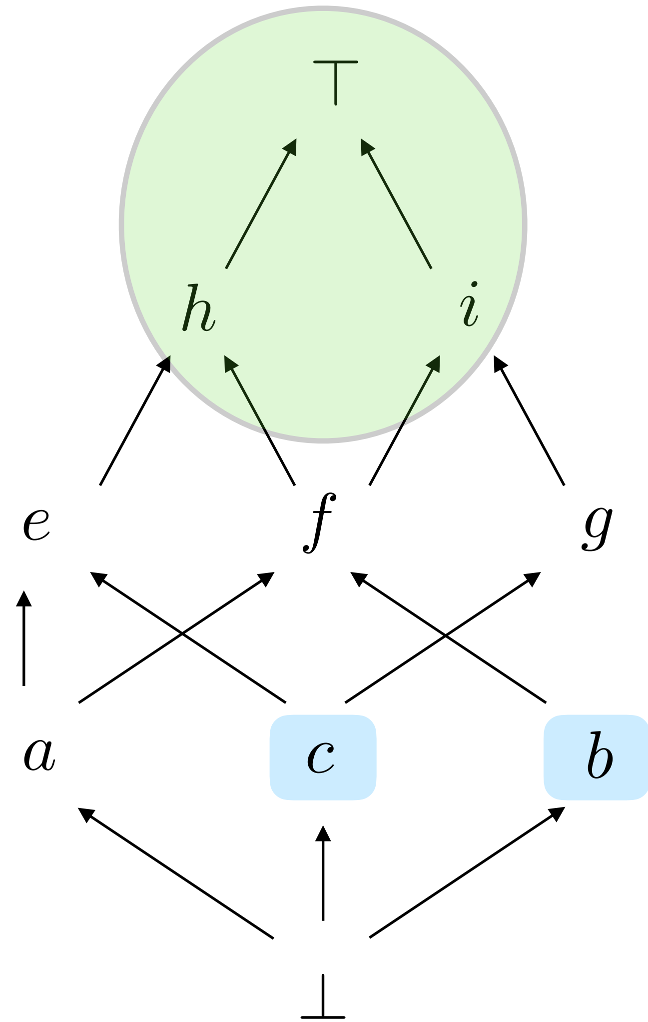
Exercise



Upper bounds of $\{a, b\}$? $\{f, h, i, \top\}$

lub? f

Exercise



Upper bounds of $\{b, c\}$? $\{h, i, \top\}$

lub? no lub!

Exercise

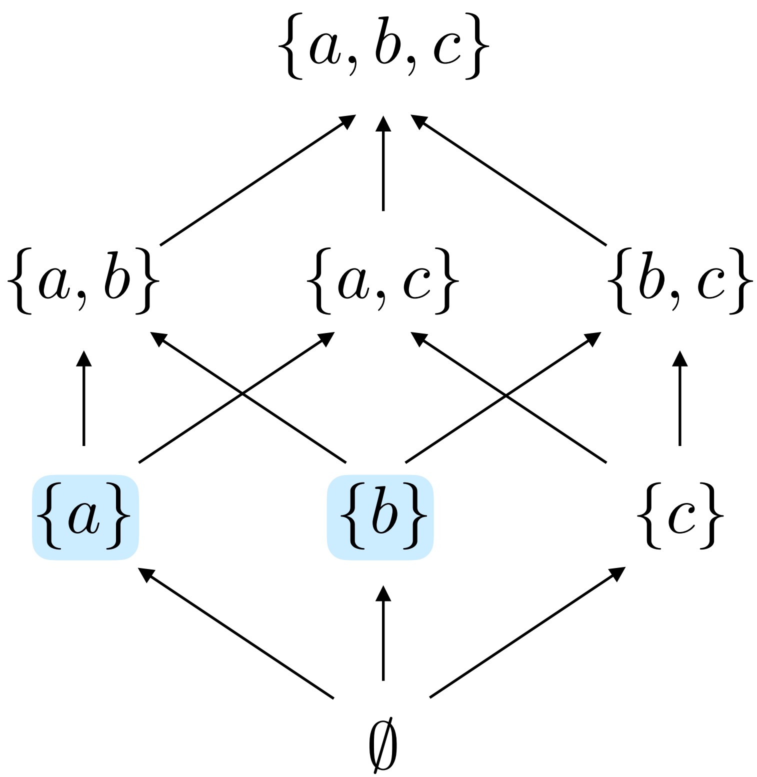
$(\mathbb{N}, \leq)$ $Q \subseteq \mathbb{N}$ lub? if Q finite $lub\ Q = \max Q$

otherwise no lub

Exercise

$(\wp(S), \subseteq)$ $Q \subseteq \wp(S)$ lub?

$$lub\ Q = \bigcup_{T \in Q} T$$



$$lub\ \{\{a\}, \{b\}\} = \{a, b\}$$

Complete partial orders (CPO)

Completeness: the idea

D a domain

$\sqsubseteq$ a way to compare elements

$x \sqsubseteq y$ — x is a (less precise) approximation of y

x and y are consistent, but y is more accurate than x

} PO

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$$

- x_0: first approximation
- x_1: second approximation
- x_2: third approximation
- x_n: (n+1)th approximation

does any sequence of approximations tend to some limit?

Chain

$(P, \sqsubseteq)$ PO

$\{d_i\}_{i\in\mathbb{N}}$ is a chain if $\forall i \in \mathbb{N}.\ d_i \sqsubseteq d_{i+1}$

$$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$$

any chain is an infinite list

finite chain: there are only finitely many distinct elements

$$\exists k \in \mathbb{N}.\ \forall i \geq k.\ d_i = d_{i+1}$$

or equivalently

$$\exists k \in \mathbb{N}.\ \forall i \geq k.\ d_i = d_k$$

Example

$(\mathbb{N}, \leq)$

$0 \leq 2 \leq 4 \leq \cdots \leq 2n \leq \cdots$ is an infinite chain

$0 \leq 1 \leq 3 \leq 3 \leq 5 \leq \cdots \leq 5 \leq \cdots$ is a finite chain

any chain has infinite length

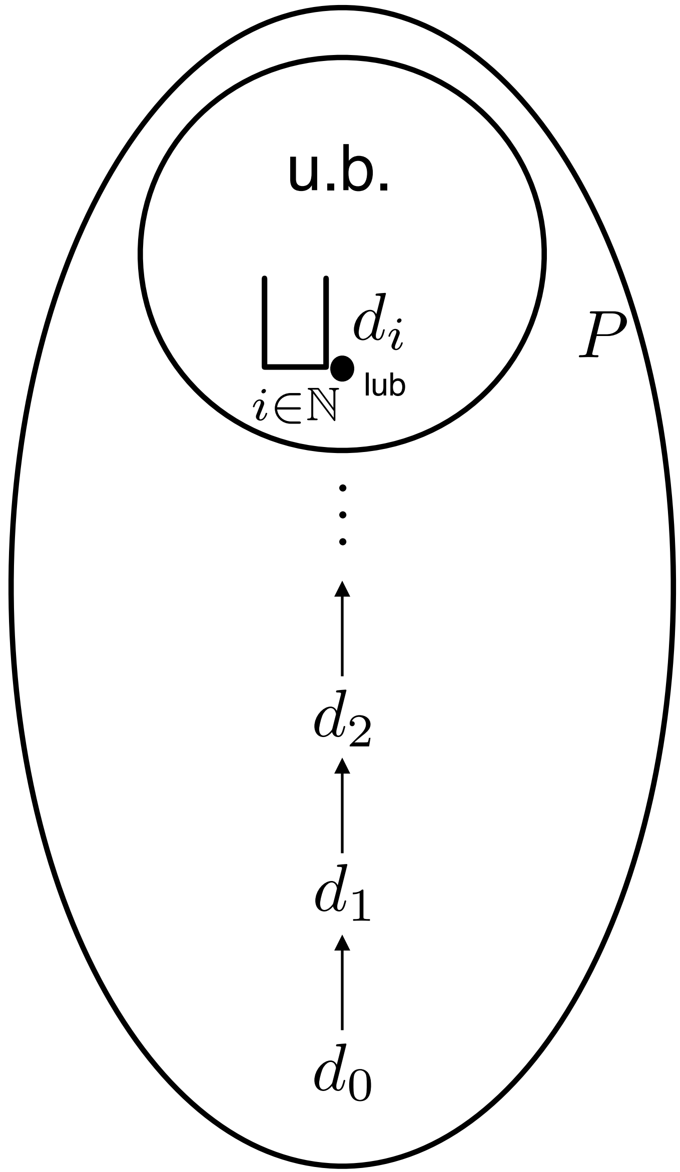
Limit of a chain

$(P, \sqsubseteq)$ PO $\qquad \{d_i\}_{i\in\mathbb{N}}$ a chain

we denote by $\bigsqcup_{i\in\mathbb{N}} d_i$ the lub of $\{d_i\}_{i\in\mathbb{N}}$ if it exists

and call it the limit of the chain

Limit illustrated



Example

$(\mathbb{N}, \leq)$

$0 \leq 2 \leq 4 \leq \cdots \leq 2n \leq \cdots$

has no lub
(empty set of upper bounds)

$0 \leq 1 \leq 3 \leq 3 \leq 5 \leq \cdots \leq 5 \leq \cdots$

has lub 5
(which upper bounds?)

Lemma on finite chains

Lemma (any finite chain has a limit)

$(P, \sqsubseteq)$ PO $\{d_i\}_{i \in \mathbb{N}}$ a finite chain $\Rightarrow$ $\bigsqcup_{i \in \mathbb{N}} d_i$ exists

Complete partial order

$(P, \sqsubseteq)$ PO $\quad P$ is complete if each chain has a limit (lub)

TH. Any finite chain has a limit
(the last element in the sequence)

If P has only finite chains it is complete

If P is finite it is complete

Any discrete order is complete

Any flat order is complete

Example

$(\mathbb{N}, \leq)$ is not complete
(it is enough to exhibit a chain with no limit)

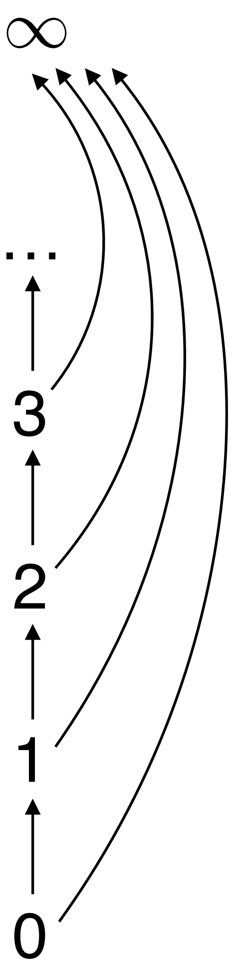
$$0 \leq 2 \leq 4 \leq \cdots \leq 2n \leq \cdots$$

has no lub
(empty set of u.b.)

Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

complete? ✓



any infinite chain has limit ∞
(set of u.b. $\{\infty\}$)

Exercise

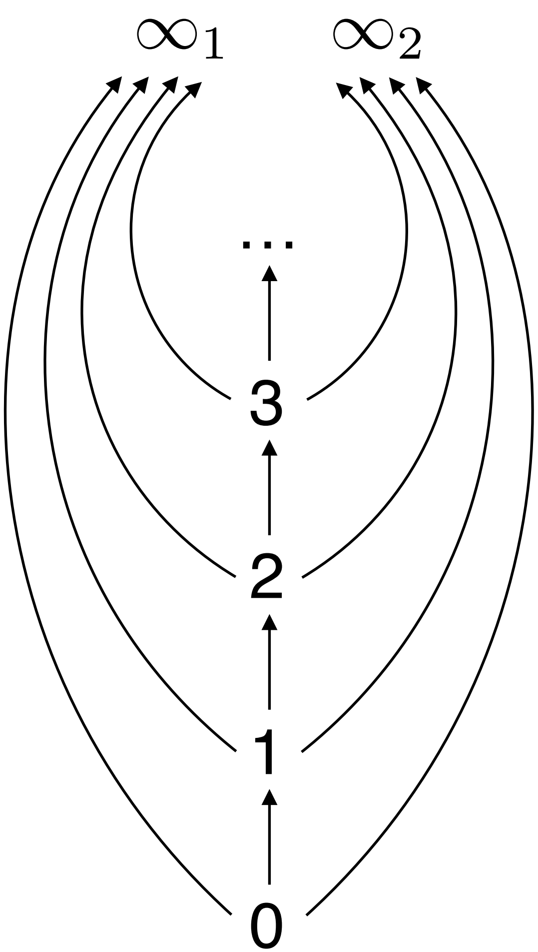
$(\wp(S), \subseteq)$

complete? ✔

$$\{S_i\}_{i \in \mathbb{N}} \qquad \bigsqcup_{i \in \mathbb{N}} S_i = \bigcup_{i \in \mathbb{N}} S_i = \{x \mid \exists k \in \mathbb{N}. x \in S_k\}$$

Exercise

$(\mathbb{N} \cup \{\infty_1, \infty_2\}, \leq)$ complete? ✖



any infinite chain has no limit
(set of u.b. $\{\infty_1, \infty_2\}$)

Partial functions

Partial functions

$D = (A \rightharpoonup B) = \mathbf{Pf}(A, B) = \{f : A \rightharpoonup B\}$ partial functions

$f \sqsubseteq g$ if $f(a)$ is defined, $g(a)$ is defined and $g(a) = f(a)$

but $g(a)$ can be defined when $f(a)$ is not

if we see partial functions as relations

$$\{(x, f(x)) \mid f(x) \neq \bot\} \subseteq A \times B$$

$f \sqsubseteq g$ means essentially $f \subseteq g$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ \perp & \text{otherwise} \end{cases}$$

$$\begin{array}{ll} & \quad n \quad f(n) \\ f = \{ & (0,0), \\ & (2,1), \\ & (4,2), \\ & (6,3), \\ & \ldots \\ & (2k, k), \\ & \ldots \qquad\qquad \} \end{array}$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ 2 \cdot n & \text{otherwise} \end{cases}$$

$$\begin{aligned} g = \{ \quad & (0,0),(1,2), \\ & (2,1),(3,6), \\ & (4,2),(5,10), \\ & (6,3),(7,14), \\ & \ldots \\ & (2k,k),(1+2k,2+4k), \\ & \ldots \qquad \} \end{aligned}$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ 2 \cdot n & \text{otherwise} \end{cases}$$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ \bot & \text{otherwise} \end{cases}$$

$$g = \{ \begin{array}{l} (0,0), (1,2), \\ (2,1), (3,6), \\ (4,2), (5,10), \\ (6,3), (7,14), \\ \ldots \\ (2k,k), (1+2k, 2+4k), \\ \ldots \end{array} \}$$

$f \sqsubseteq g?$ ✓

$g \sqsubseteq f?$ ✗

$$f = \{ \begin{array}{l} (0,0), \\ (2,1), \\ (4,2), \\ (6,3), \\ \ldots \\ (2k,k), \\ \ldots \end{array} \}$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{\ (0,0)\ \} \sqsubseteq \{\ (0,0),\ (1,1)\ \} \sqsubseteq \ \cdots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,0) \} \sqsubseteq \{ (0,0), (1,1) \} \sqsubseteq \{ (0,0), (1,1), (2,2) \} \sqsubseteq \cdots$$

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Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

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which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \{ (0,1), (1,1) \} \sqsubseteq \{ (0,1), (1,1), (2,2) \} \sqsubseteq \cdots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \{ (0,1), (1,1) \} \sqsubseteq \{ (0,1), (1,1), (2,2) \} \sqsubseteq \{ (0,1), (1,1), (2,2), (3,6) \} \sqsubseteq \cdots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$\emptyset \sqsubseteq$ { (0,1) } $\sqsubseteq$ { (0,1), (1,1) } $\sqsubseteq$ { (0,1), (1,1), (2,2) } $\sqsubseteq$ { (0,1), (1,1), (2,2), (3,6) } $\sqsubseteq$ { (0,1), (1,1), (2,2), (3,6), (4,24) } $\sqsubseteq \cdots$

which function(s) are we approximating?

Pf is complete?

$\mathbf{Pf}(A, B) = \{f : A \rightharpoonup B\}$ partial functions

$(\mathbf{Pf}(A, B), \sqsubseteq)$ is a PO with bottom

what is bottom?

the empty relation
(the function always undefined)

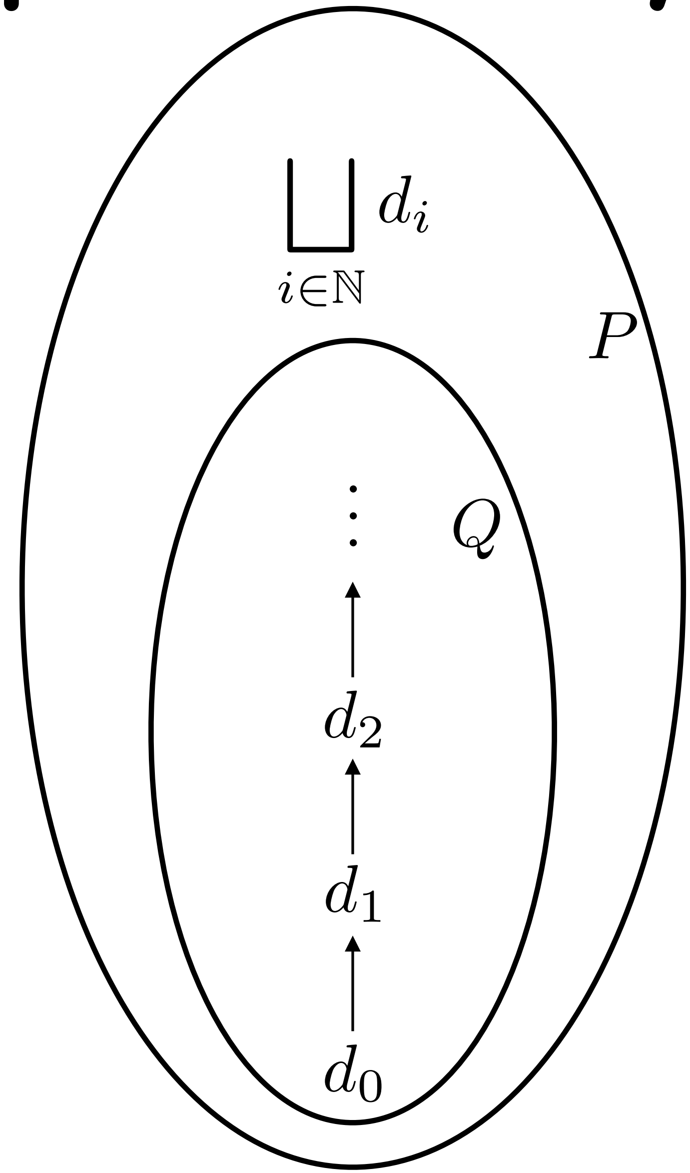
is it complete?

yes!

Given a chain $\{f_i\}_{i \in \mathbb{N}}$ let us consider $\bigcup_{i \in \mathbb{N}} f_i \subseteq A \times B$

$$\bigcup_{i \in \mathbb{N}} f_i \in \mathbf{Pf}(A, B)$$

pictorially



is the limit in Q?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$
\begin{array}{rll}
f_0\ \emptyset & \subseteq & \{(0,1)\}\ f_1 \\
 & \subseteq & \{(0,1),(1,1)\}\ f_2 \\
 & \subseteq & \{(0,1),(1,1),(2,2)\}\ f_3 \\
 & \subseteq & \{(0,1),(1,1),(2,2),(3,6)\}\ f_4 \\
 & \subseteq & \{(0,1),(1,1),(2,2),(3,6),(4,24)\}\ f_5 \\
 & \subseteq & \ldots
\end{array}
$$

$\bigcup_{i \in \mathbb{N}} f_i$ is (maybe) the factorial function

note: the limit of partial functions can be a total function

Monotone functions

Monotone function

$(D, \sqsubseteq_D)$ PO $\quad (E, \sqsubseteq_E)$ PO $\quad f : D \to E$

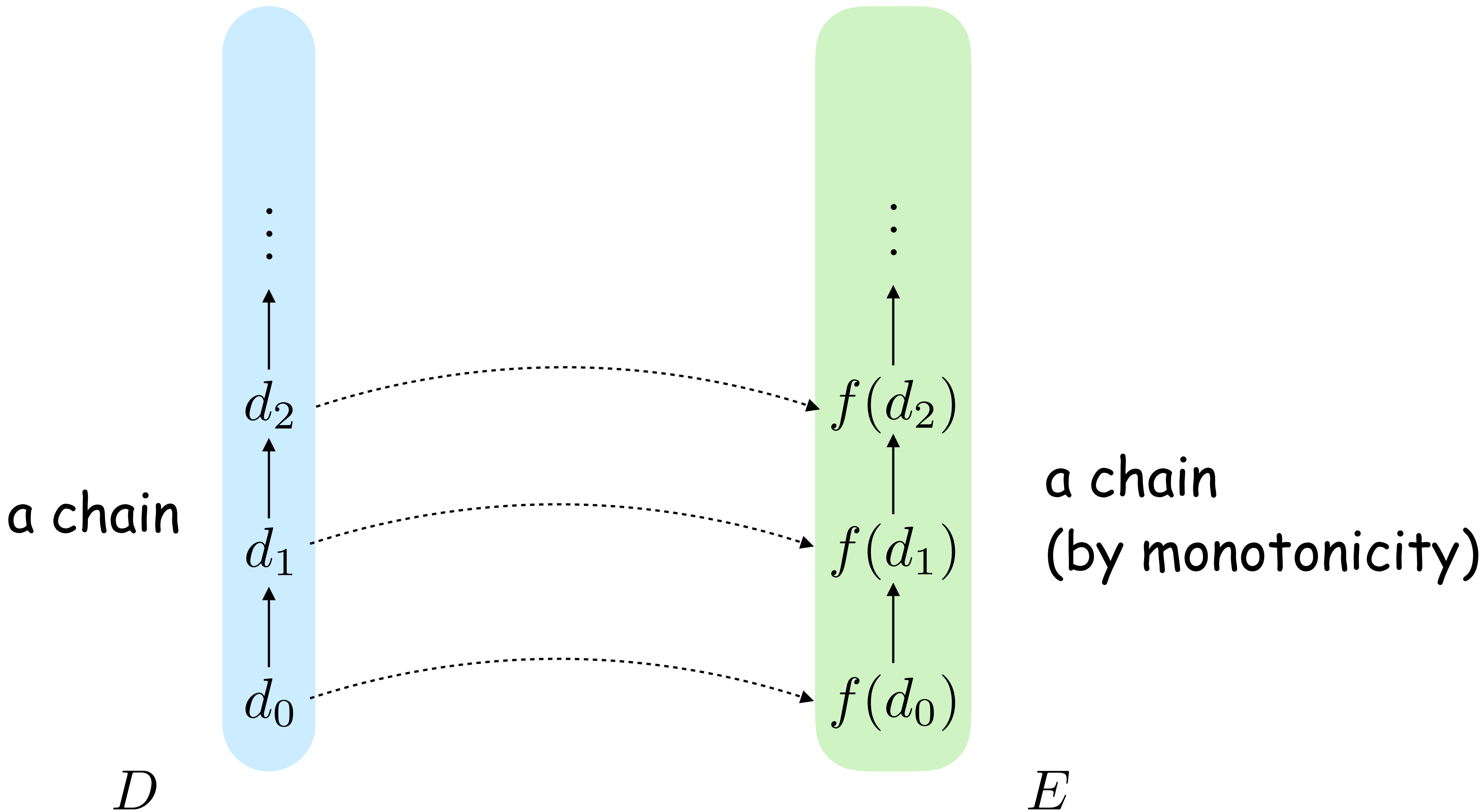
f is monotone if $\forall d_1, d_2 \in D.\ d_1 \sqsubseteq_D d_2 \Rightarrow f(d_1) \sqsubseteq_E f(d_2)$

Monotone = Order preserving

$\{d_i\}_{i \in \mathbb{N}}$ a chain in D
f monotone
$\Rightarrow \{f(d_i)\}_{i \in \mathbb{N}}$ a chain in E

When $D = E$ we say $f : D \to D$ is a function on D

Monotonicity illustrated

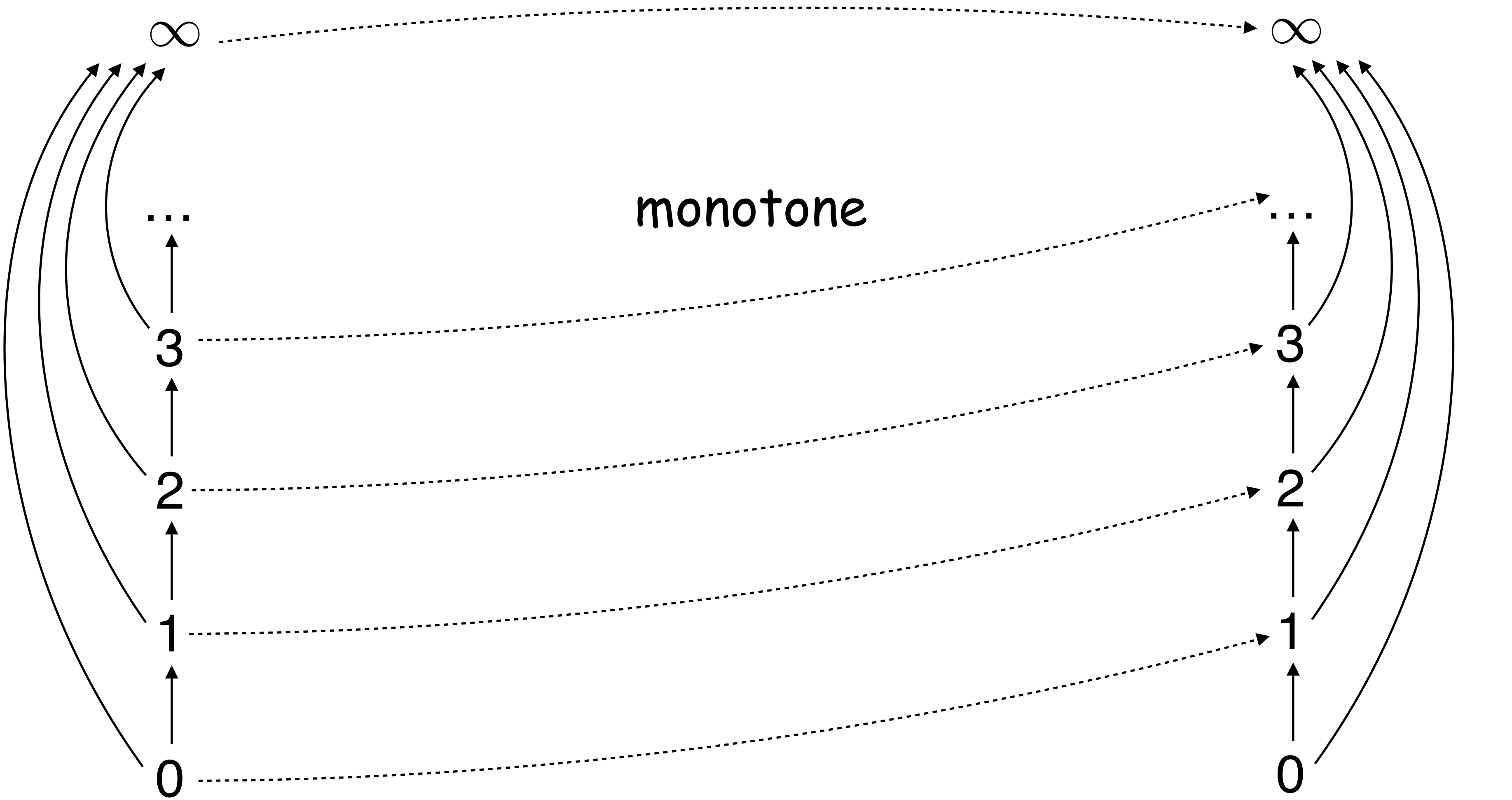


Example

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = n + 1$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$

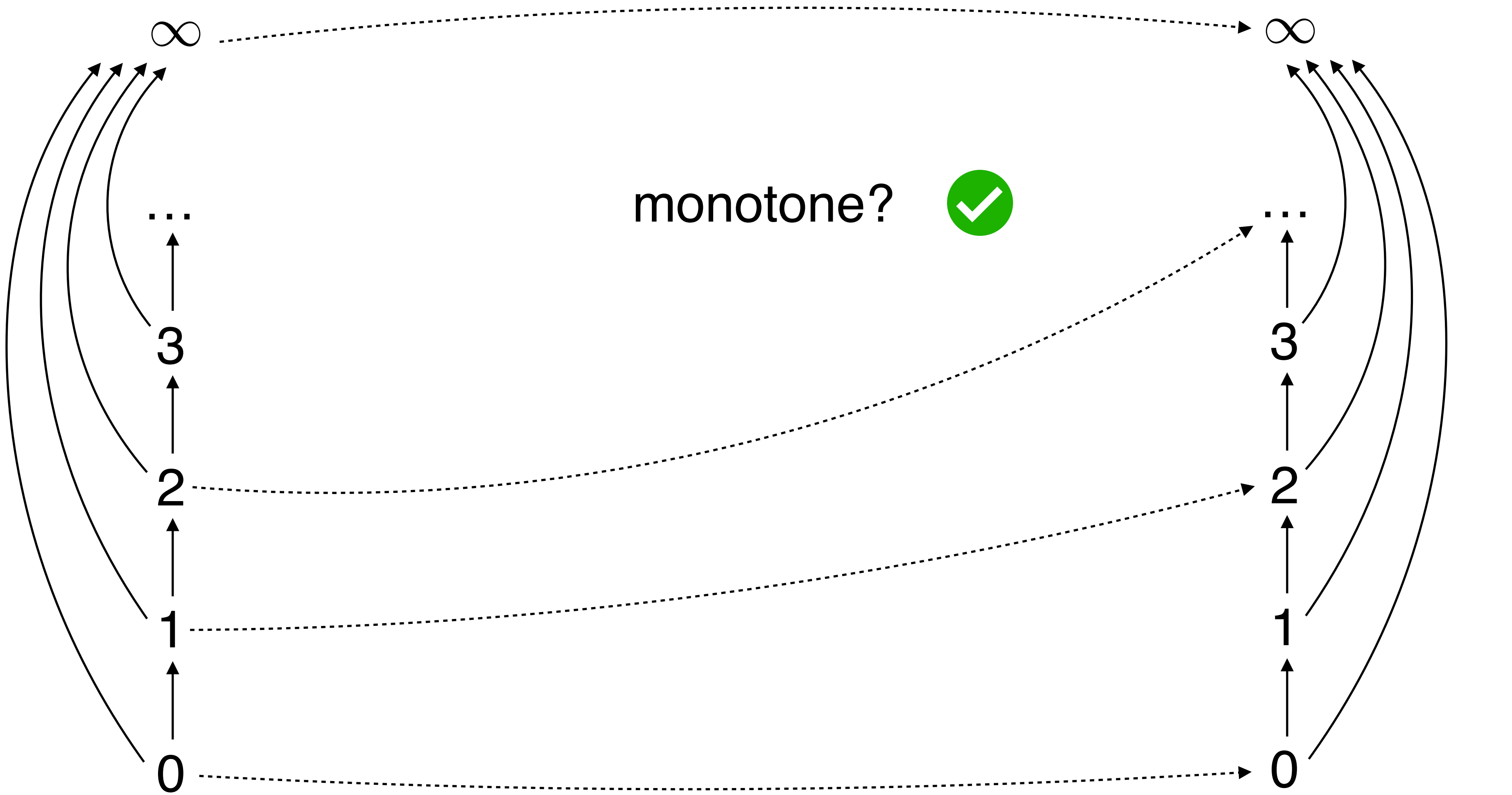


Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$f(n) = 2 \cdot n$
$f(\infty) = \infty$

$(\mathbb{N} \cup \{\infty\}, \leq)$

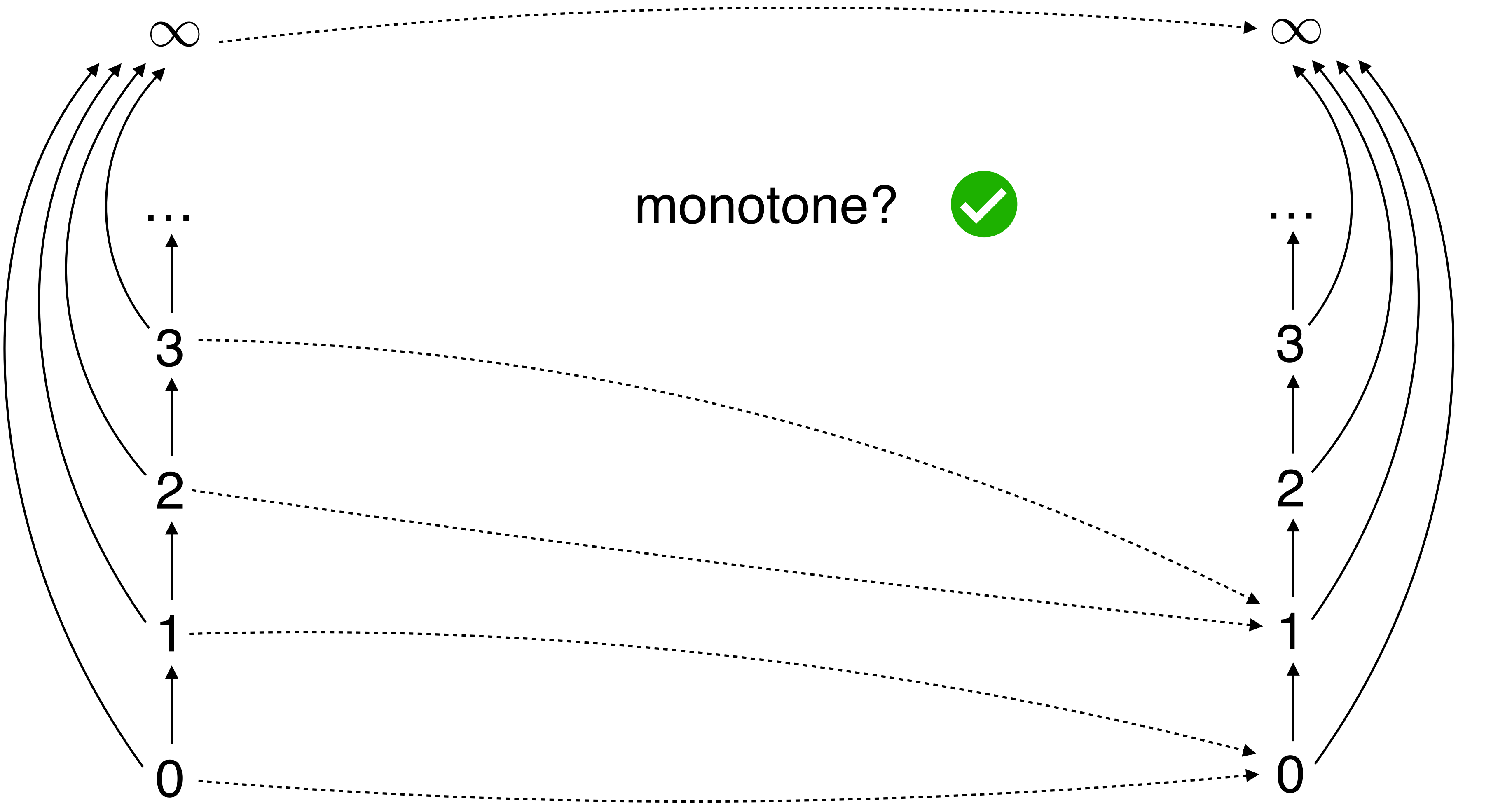


Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

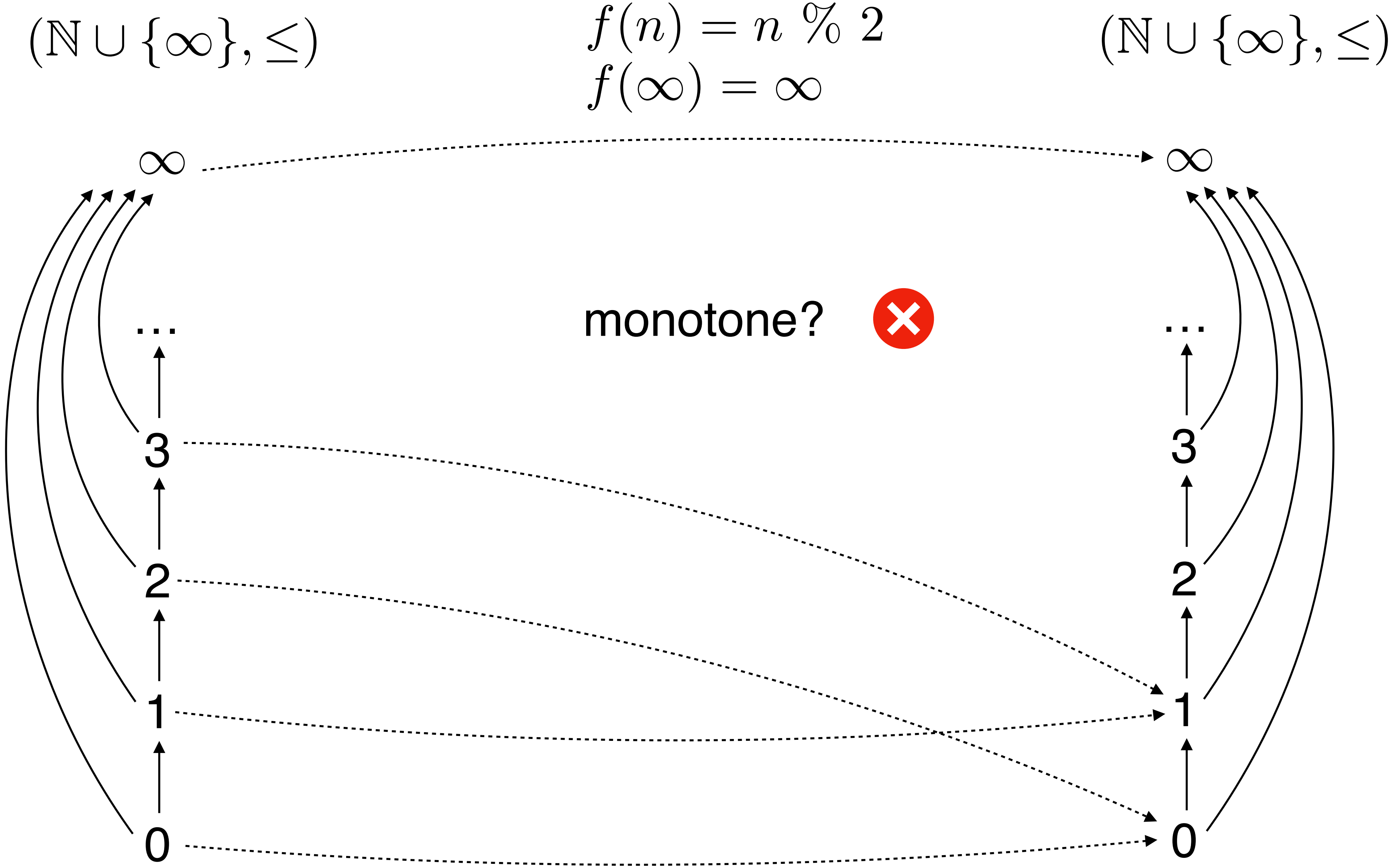
$$f(n) = n/2$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$



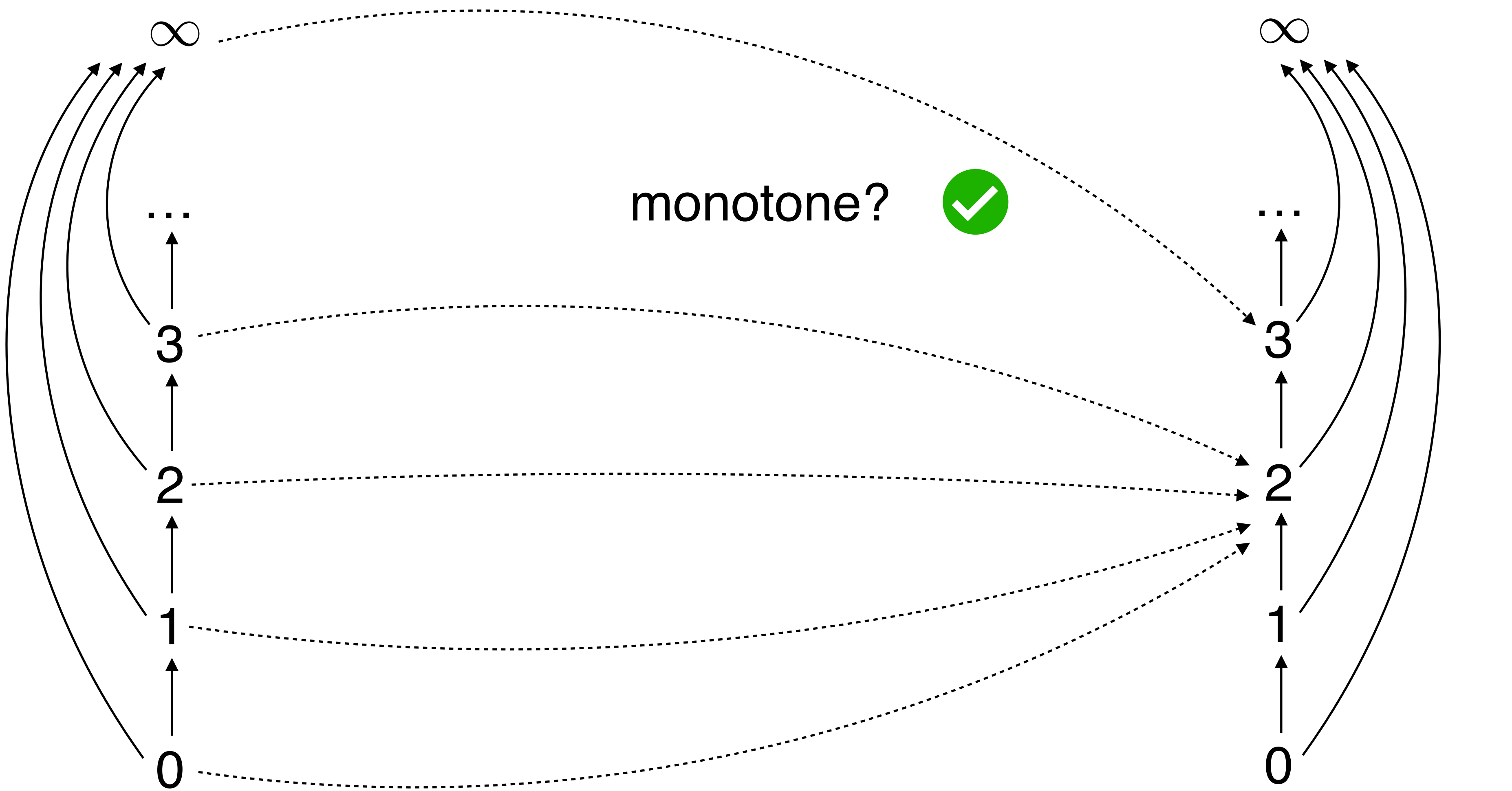
Exercise

$$f(n) = n \,\%\, 2$$
$$f(\infty) = \infty$$

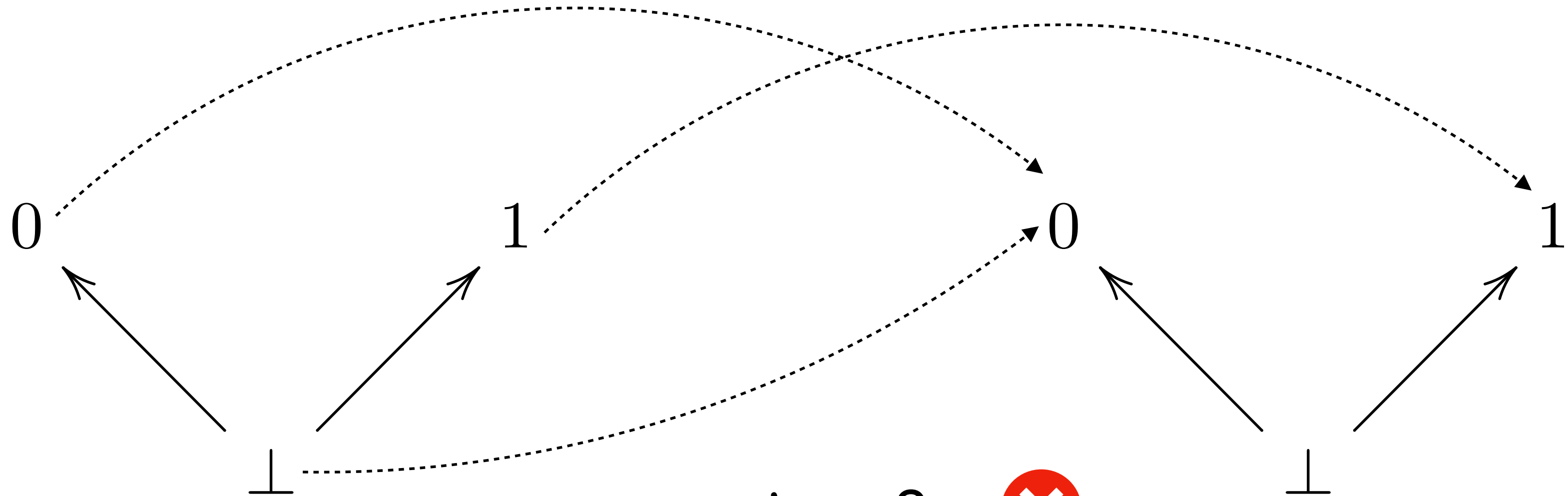


Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$ $\qquad f(n) = 2$, $f(\infty) = 3$ $\qquad (\mathbb{N} \cup \{\infty\}, \leq)$



Exercise



monotone? ✗

D $\qquad f : D \to D$ $\qquad D$

$$f(\bot) = f(0) = 0$$
$$f(1) = 1$$

$\bot \sqsubseteq 1$ $\qquad f(\bot) = 0 \not\sqsubseteq 1 = f(1)$

Composition

TH. Any composition of monotone function is monotone

$(D, \sqsubseteq_D)$ PO

$(E, \sqsubseteq_E)$ PO

$(F, \sqsubseteq_F)$ PO

$f : D \to E$ monotone

$g : E \to F$ monotone

$\Rightarrow$ $h = g \circ f : D \to F$ monotone

Continuous functions

Continuous function

$(D, \sqsubseteq_D)$ CPO $(E, \sqsubseteq_E)$ CPO $f : D \to E$ monotone

f is continuous if $\forall \{d_i\}_{i \in \mathbb{N}}.\ f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = \bigsqcup_{i \in \mathbb{N}} f(d_i)$

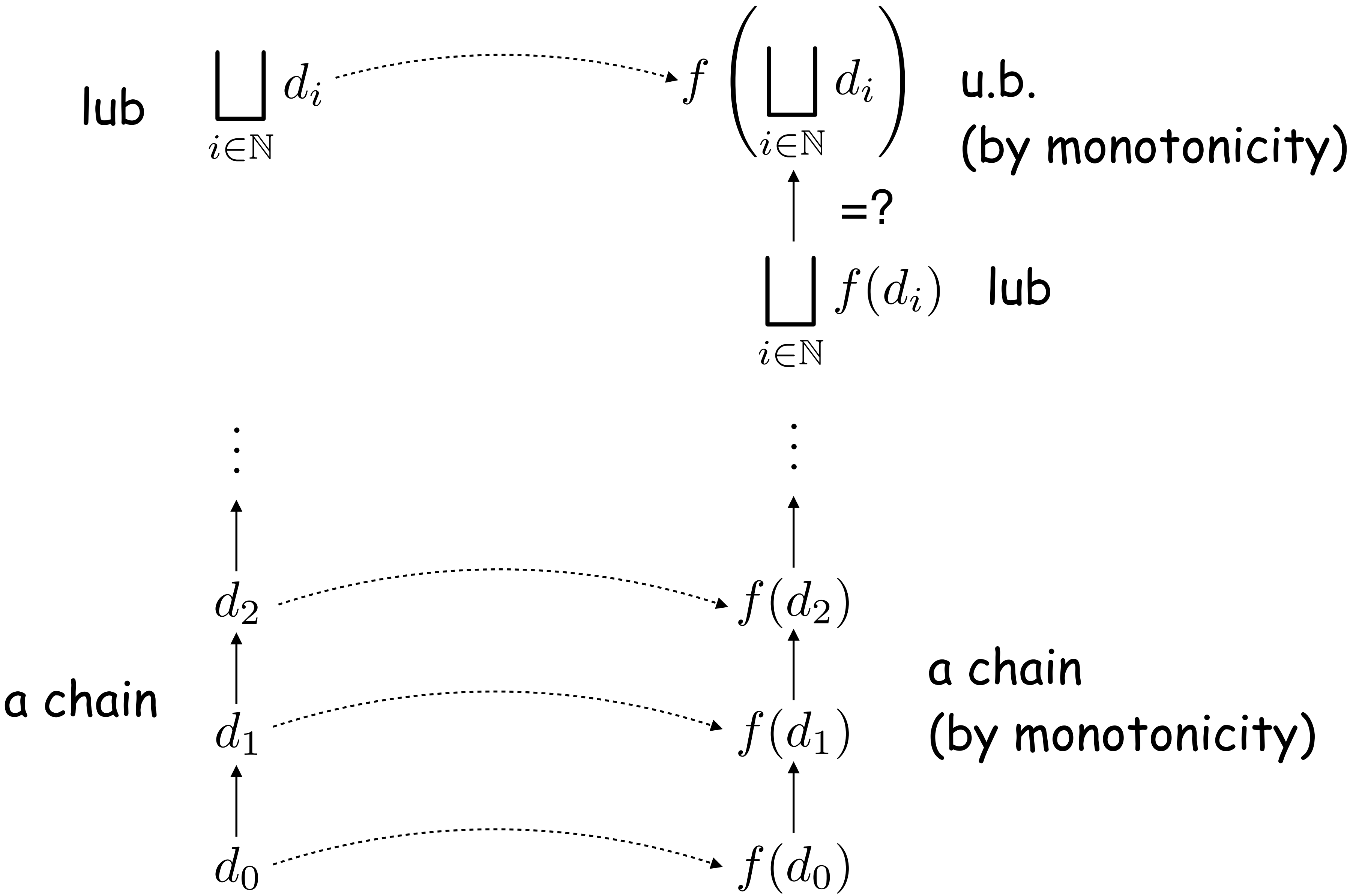
chain

limit in D

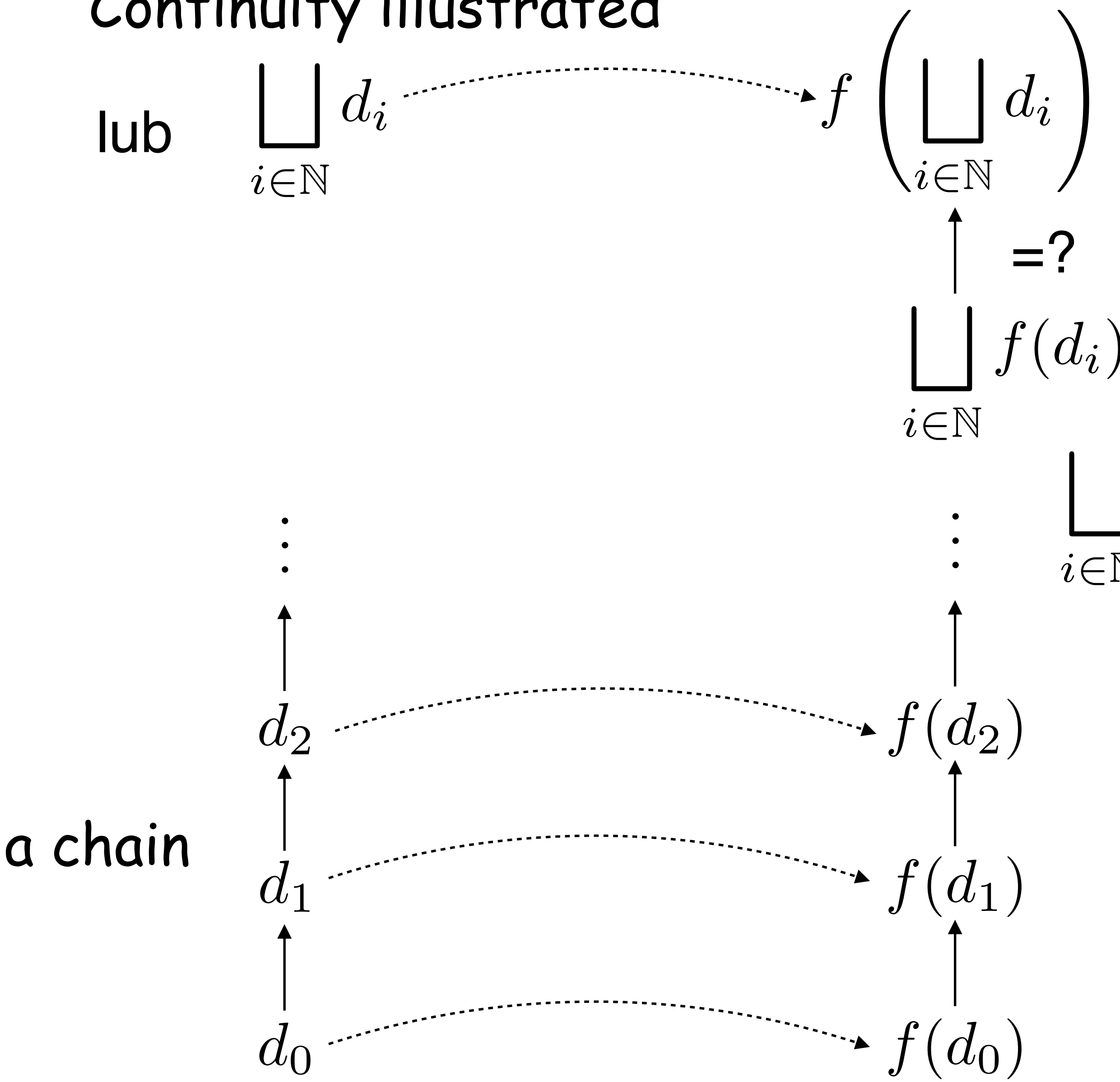
limit in E

Continuous = limit preserving

Continuity Illustrated



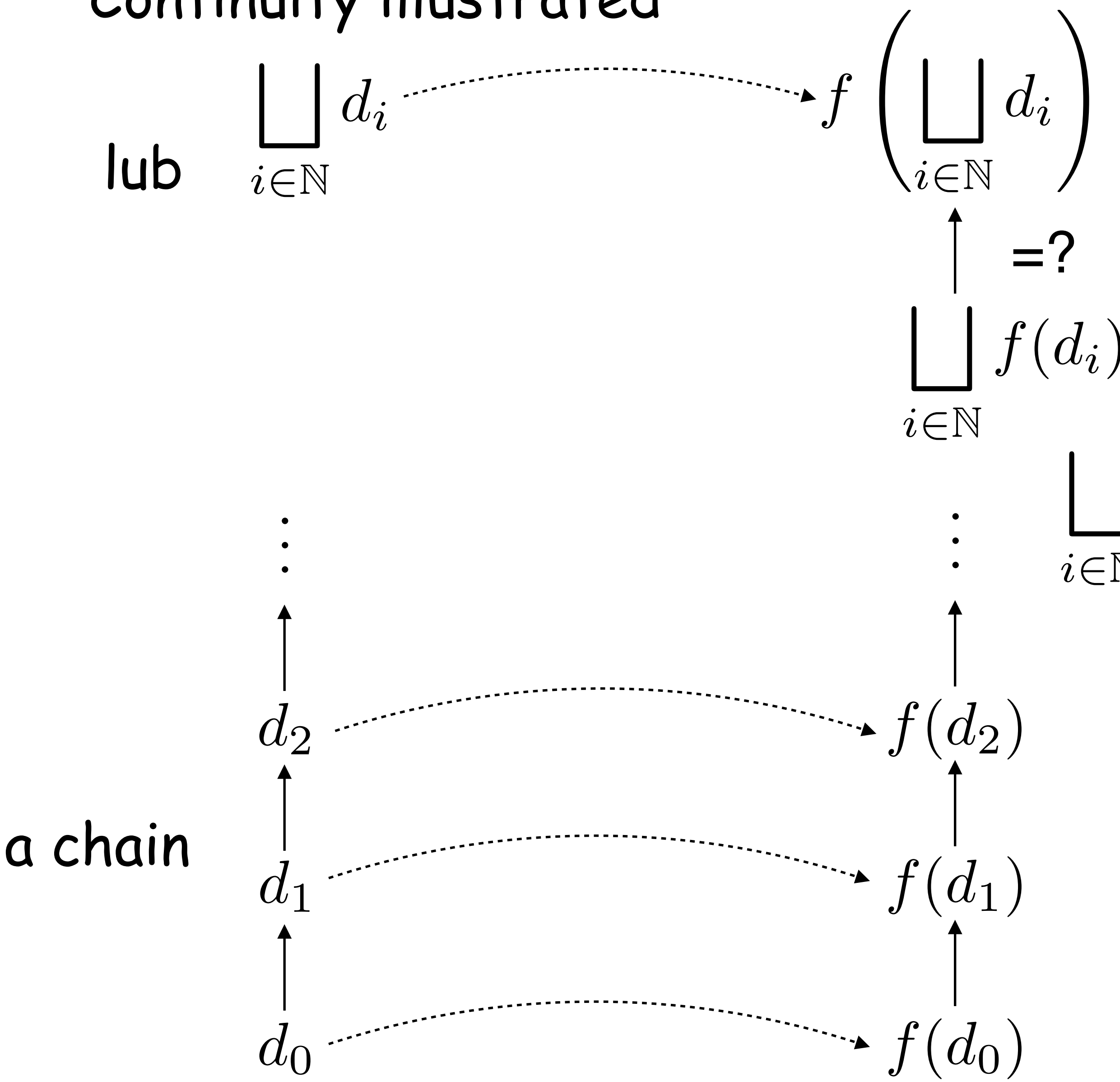
Continuity illustrated



$$\bigsqcup_{i\in\mathbb{N}} f(d_i) \sqsubseteq_E f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)$$

follows from monotonicity (and CPO)

Continuity illustrated



$$\bigsqcup_{i\in\mathbb{N}} f(d_i) \overset{?}{\sqsupseteq}_E f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)$$

continuity

Lemma

$(D, \sqsubseteq_D)$ CPO
no infinite chains

$(E, \sqsubseteq_E)$ PO

$f : D \to E$
monotone

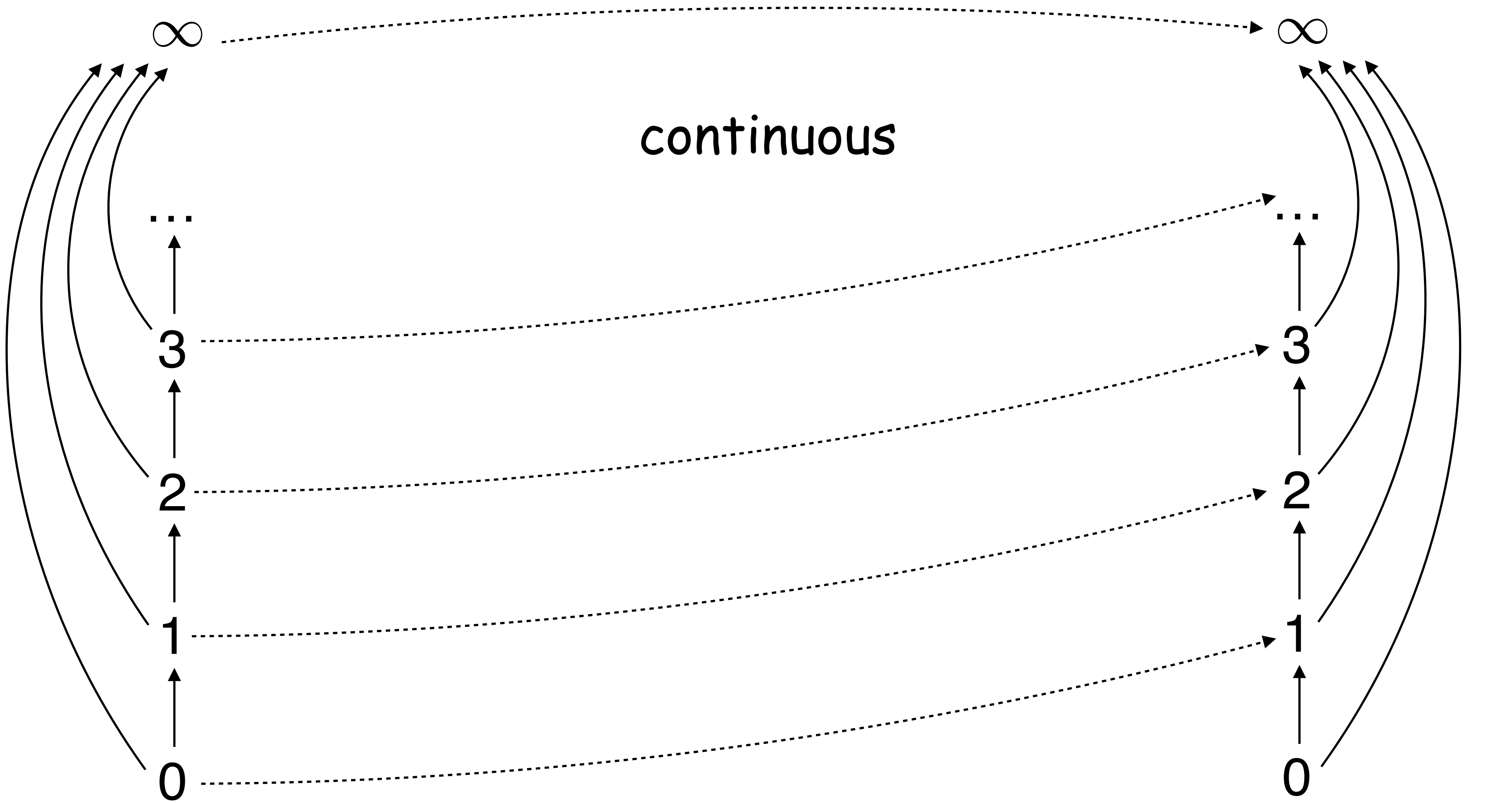
$\Rightarrow$

f
continuous

Example

$(\mathbb{N} \cup \{\infty\}, \leq)$ $\qquad$ $(\mathbb{N} \cup \{\infty\}, \leq)$

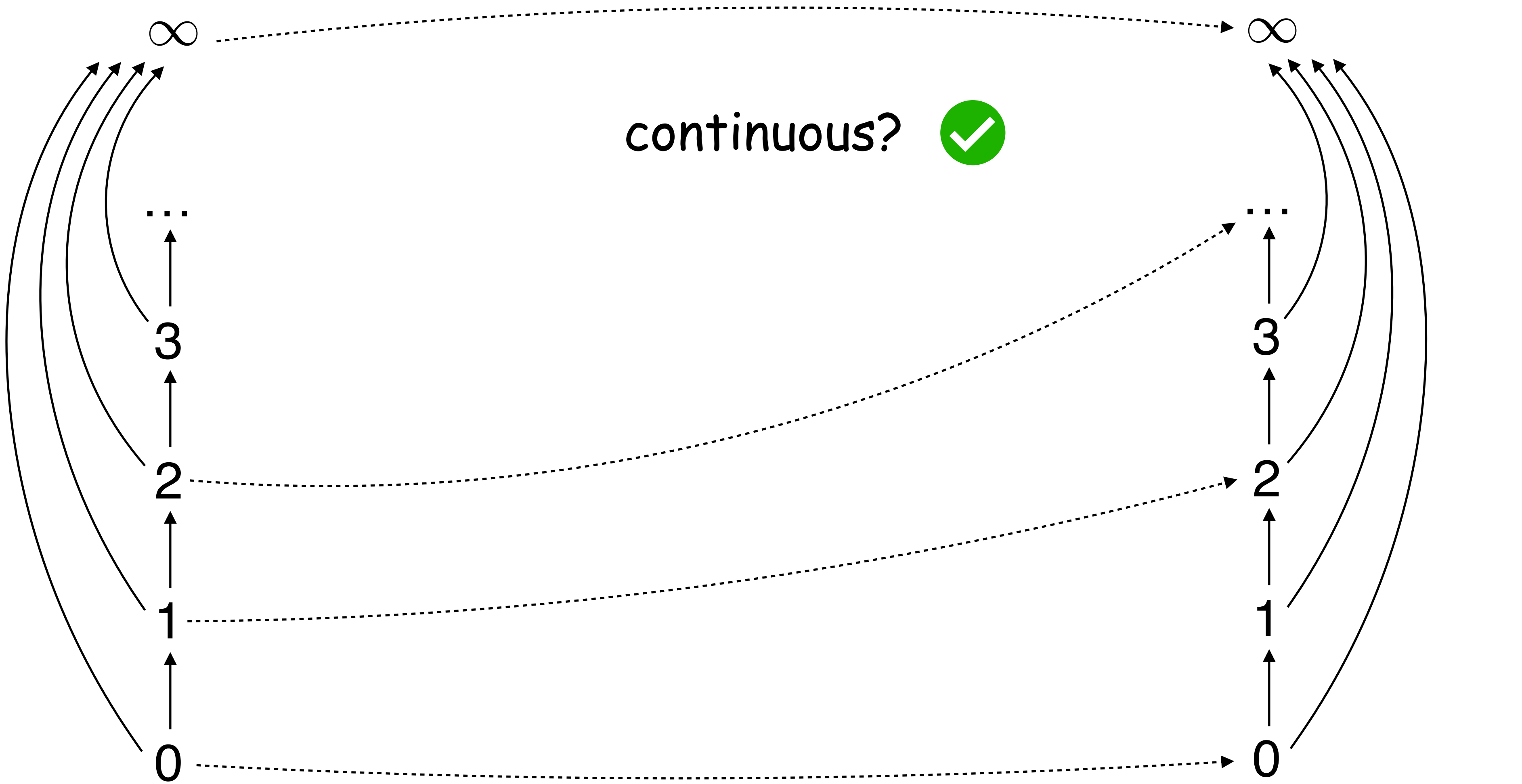
$$f(n) = n + 1$$
$$f(\infty) = \infty$$



Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$ $\qquad f(n) = 2 \cdot n$ $\qquad (\mathbb{N} \cup \{\infty\}, \leq)$

$$f(\infty) = \infty$$

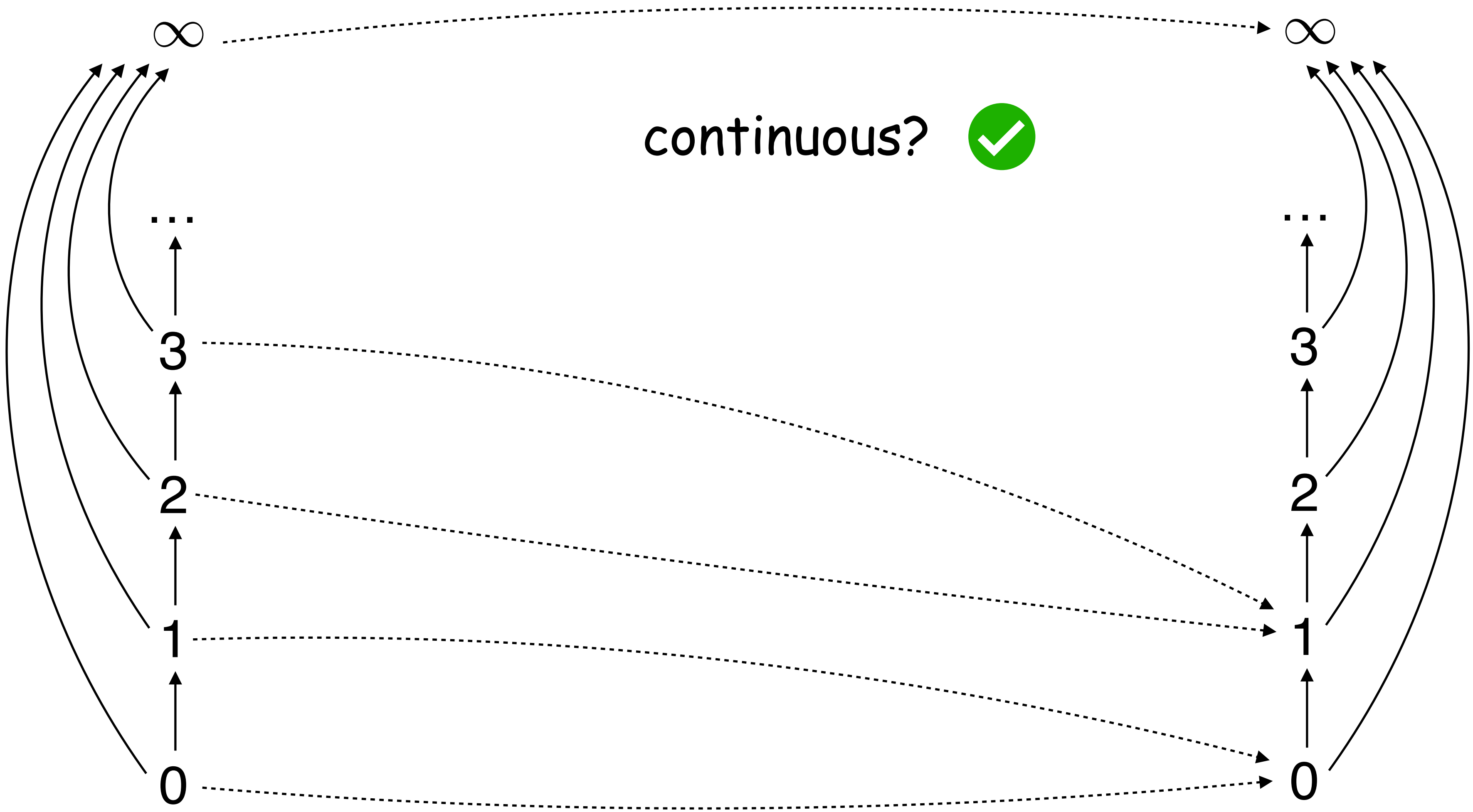


Example

$(\mathbb{N} \cup \{\infty\}, \leq)$

$f(n) = n/2$
$f(\infty) = \infty$

$(\mathbb{N} \cup \{\infty\}, \leq)$



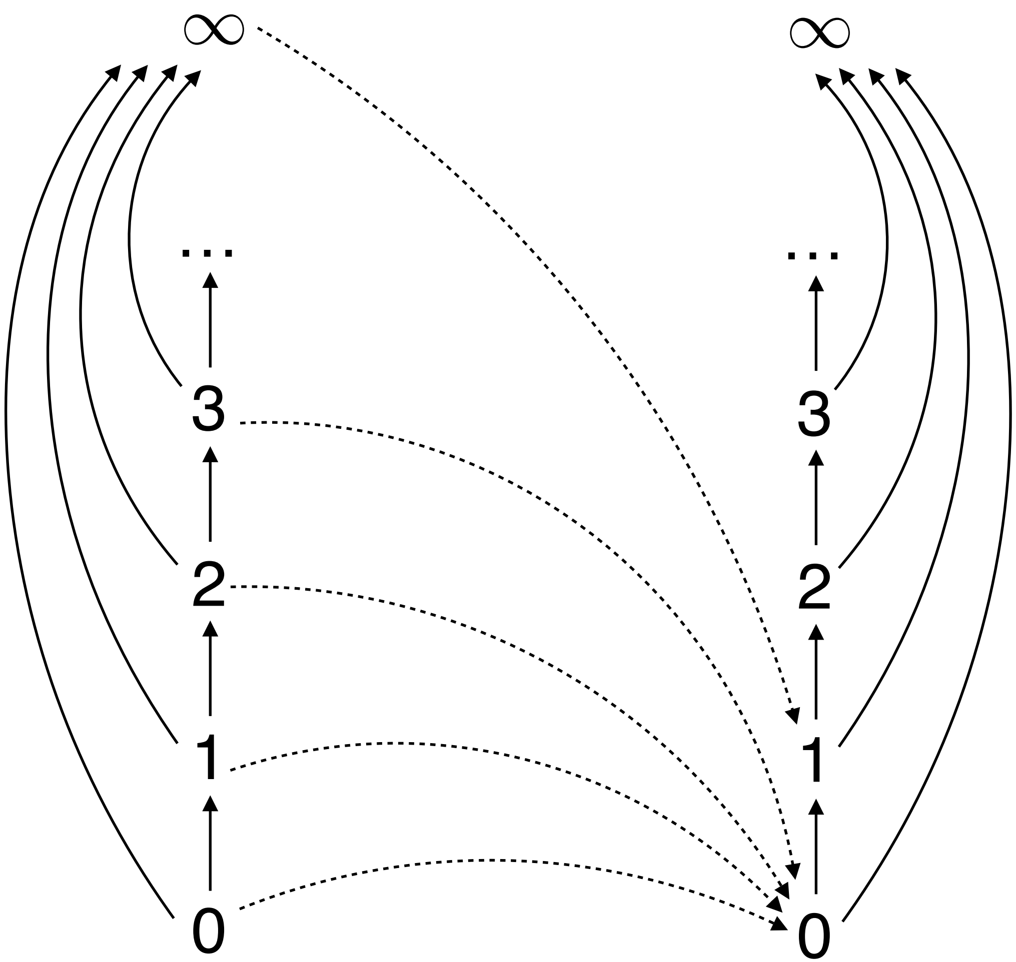
continuous? ✓

Example

$(\mathbb{N} \cup \{\infty\}, \leq)$

monotone function, not continuous

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{N} \\ 1 & \text{if } x = \infty \end{cases}$$



Composition

TH. Any composition of continuous function is continuous

$(D, \sqsubseteq_D)$ CPO

$(E, \sqsubseteq_E)$ CPO

$(F, \sqsubseteq_F)$ CPO

$f : D \to E$ continuous

$g : E \to F$ continuous

$\Rightarrow$ $h = g \circ f : D \to F$ continuous

Kleene's fixpoint theorem

Repeated application

$$f : D \to D$$

$$f^0(d) \triangleq d$$

$$f^{n+1}(d) \triangleq f(f^n(d))$$

$$f^n(d) = \overbrace{f(\cdots(f}^{n \text{ times}}(d))\cdots)$$

$$f^n : D \to D$$

Lemma

$(D, \sqsubseteq)$ $\mathrm{PO}_\perp$ $f : D \to D$ monotone $\Rightarrow$ $\{f^n(\perp)\}_{n \in \mathbb{N}}$ is a chain

Towards Kleene's theo.

when $(D, \sqsubseteq)$ is a $\mathrm{CPO}_{\perp}$

then $\{f^n(\perp)\}_{n\in\mathbb{N}}$ is a chain

it must have a limit

$\{f^n(d)\}_{n\in\mathbb{N}}$
not necessarily a chain!

Kleene's fix point theorem states that
if f is continuous, then the limit of the above chain
is the least fixpoint of f

Pre-fixpoints

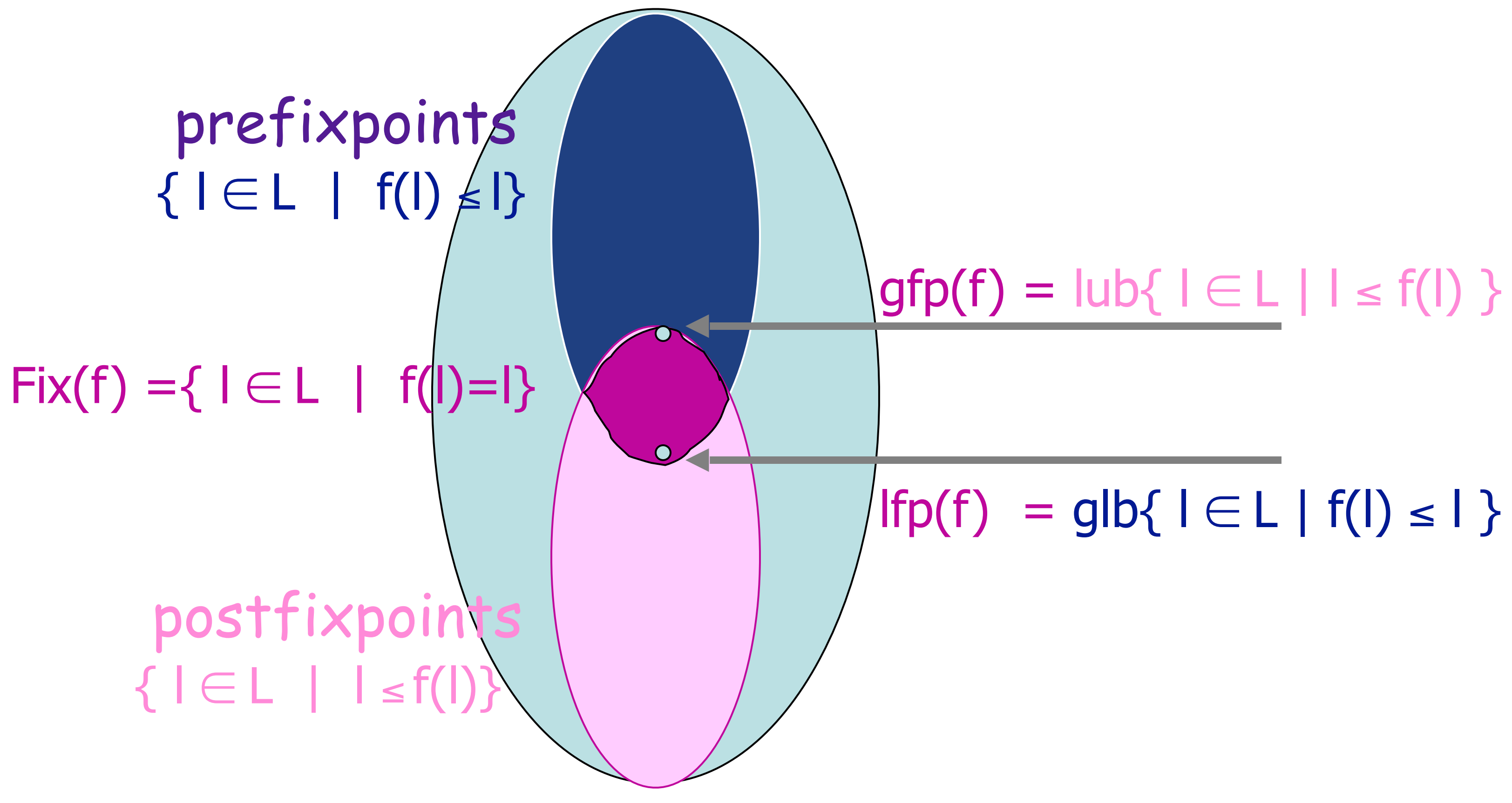
$(D, \sqsubseteq)$ PO $\qquad f : D \to D$ monotone

fixpoint $\qquad p \in D \qquad f(p) = p$

pre-fixpoint $p \in D \qquad f(p) \sqsubseteq p$

Clearly any fixpoint is also a pre-fixpoint

Tarsky's theorem



Kleene's theorem

$(D, \sqsubseteq)$ $\mathrm{CPO}_\perp$ $\quad f : D \to D$ continuous

let $\mathit{fix}(f) \triangleq \bigsqcup_{n \in \mathbb{N}} f^n(\perp)$

1. $\mathit{fix}(f)$ is a fix point of f

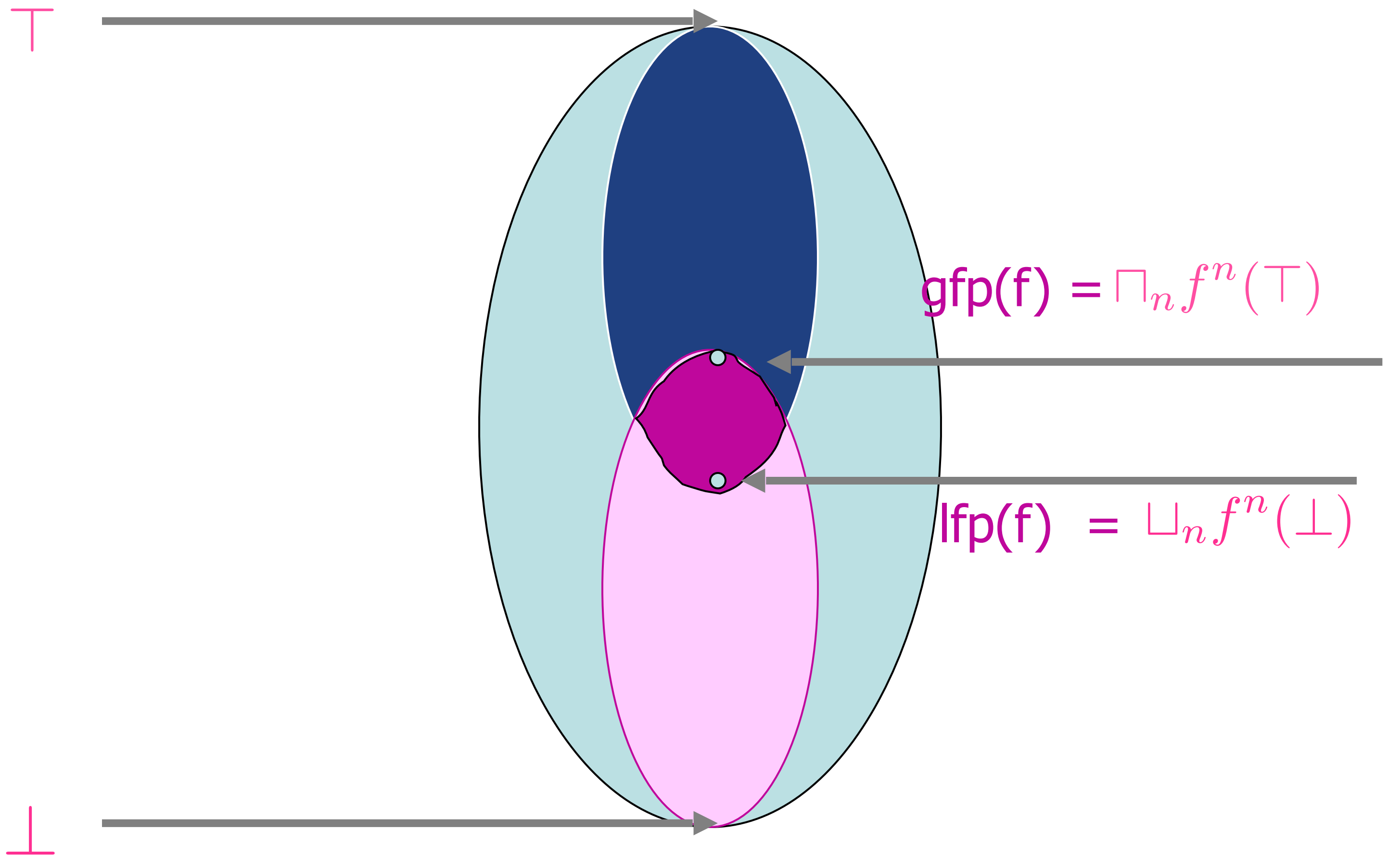
$$f(\mathit{fix}(f)) = \mathit{fix}(f)$$

2. $\mathit{fix}(f)$ is the least pre-fixpoint of f

$$\forall d \in D.\ f(d) \sqsubseteq d \Rightarrow \mathit{fix}(f) \sqsubseteq d$$

if d is a pre-fixpoint then $\mathit{fix}(f)$ is smaller than d

Kleene's theorem



Example

$n = 2 \cdot n$

$(\mathbb{N} \cup \{\infty\}, \leq)$ $\quad \bot = 0 \quad \mathrm{CPO}_\bot$

$f(n) = 2 \cdot n$
$f(\infty) = \infty$

monotone? ok
continuous? ok

$$f^0(0) = 0$$

$$f^1(0) = f(0) = 2 \cdot 0 = 0$$

fixpoint reached!

Example

$n = n + 1$ $(\mathbb{N} \cup \{\infty\}, \leq)$ $\bot = 0$ $\text{CPO}_\bot$

$$f(n) = n + 1$$
$$f(\infty) = \infty$$

monotone? ok

continuous? ok

$$f^0(0) = 0$$
$$f^1(0) = f(0) = 0 + 1 = 1$$
$$f^2(0) = f(f^1(0)) = f(1) = 1 + 1 = 2$$
$$f^3(0) = f(f^2(0)) = f(2) = 2 + 1 = 3$$

$$\bigsqcup_{n \in \mathbb{N}} f^n(0) = \bigsqcup_{n \in \mathbb{N}} n = \infty$$ fixpoint

Example

$X = X \cap \{1\}$ $(\wp(\mathbb{N}), \subseteq)$ $\bot = \emptyset$ $\mathrm{CPO}_\bot$

$f(X) = X \cap \{1\}$

monotone? ok

continuous? ok

$f^0(\emptyset) = \emptyset$

$f^1(\emptyset) = f(\emptyset) = \emptyset \cap \{1\} = \emptyset$

fixpoint reached!

Example

$X = \mathbb{N} \setminus X$ $\quad (\wp(\mathbb{N}), \subseteq)$ $\quad \bot = \emptyset$ $\quad \mathrm{CPO}_\bot$

$f(X) = \mathbb{N} \setminus X$

monotone? NO

the larger X the smaller $f(X)$

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \mathbb{N} \setminus \emptyset = \mathbb{N}$$

$$f^2(\emptyset) = f(f^1(\emptyset)) = f(\mathbb{N}) = \mathbb{N} \setminus \mathbb{N} = \emptyset$$

not a chain!

Example

$X = X \cup \{1\}$ $(\wp(\mathbb{N}), \subseteq)$ $\bot = \emptyset$ $\mathrm{CPO}_\bot$

$$f(X) = X \cup \{1\}$$

monotone? ok

continuous? ok

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \emptyset \cup \{1\} = \{1\}$$

$$f^2(\emptyset) = f(f^1(\emptyset)) = f(\{1\}) = \{1\} \cup \{1\} = \{1\}$$

fixpoint reached!