The fix point THEORY

POSET (Partially ordered set, PO)

$(P, \sqsubseteq)$ $\qquad \sqsubseteq \subseteq P \times P$

reflexive $\forall p \in P. \qquad p \sqsubseteq p$

antisymmetry $\forall p, q \in P. \qquad p \sqsubseteq q \ \wedge \ q \sqsubseteq p \ \Rightarrow \ p = q$

transitive $\forall p, q, r \in P. \ p \sqsubseteq q \ \wedge \ q \sqsubseteq r \ \Rightarrow \ p \sqsubseteq r$

$p \sqsubseteq q$ means that p is less than (or equal to) q

$p \sqsubset q$ means $p \sqsubseteq q \ \wedge \ p \neq q$

Total Order

A PO $(P, \sqsubseteq)$ is total iff

$$\forall p, q \in P. \quad p \sqsubseteq q \ \vee \ q \sqsubseteq p$$

A PO where every two elements are comparable

Examples

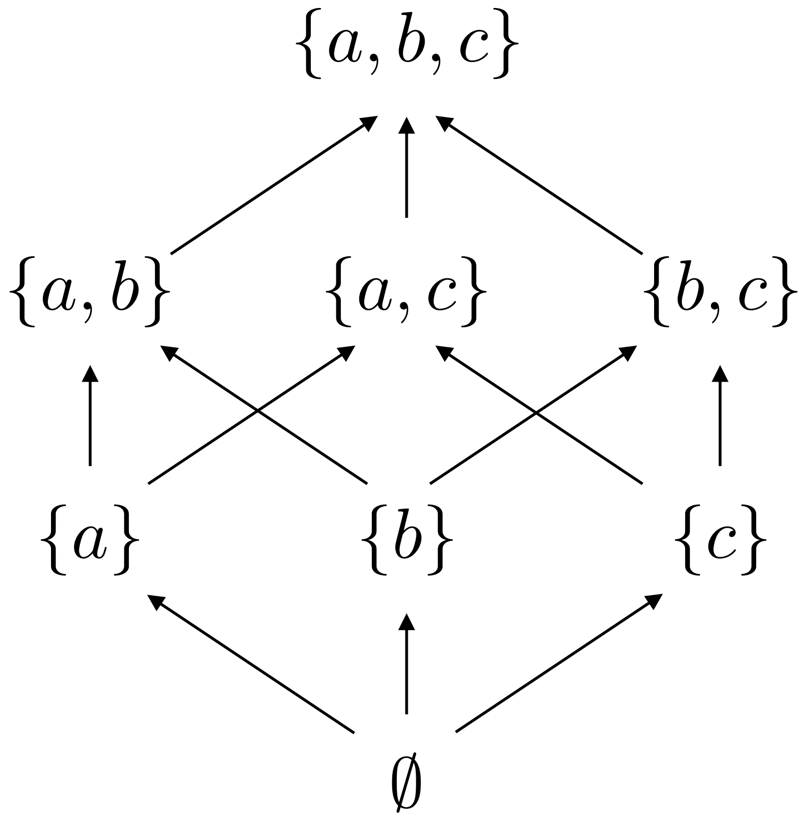
	PO?	Total?
$(\mathbb{N}, \leq)$	Yes	Yes

Hasse diagram notation
(omit: reflexive arcs,
transitive arcs)

Examples

$(\wp(S), \subseteq)$

	PO?	Total?
$(\wp(S), \subseteq)$	Yes	|S|<2



Examples

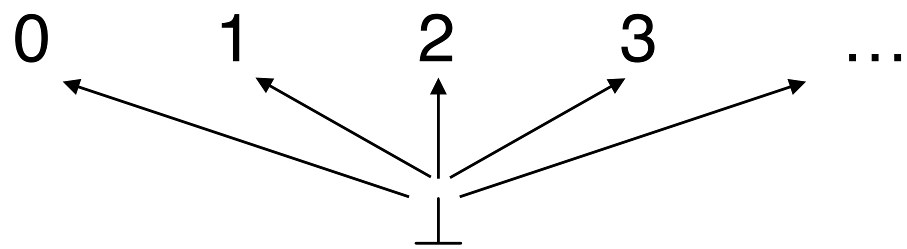
	PO?	Total?
$(\mathbb{N}, =)$	Yes	No

0 1 2 3 …

Examples

$$(\mathbb{N} \cup \{\bot\}, \{(\bot, n) \mid n \in \mathbb{N}\} \ \ )$$

PO?	Total?
Yes	No



PO with bottom

A PO $(P, \sqsubseteq)$ that has a least element e, i.e,

$$\forall p \in P. e \sqsubseteq p$$

e is often indicates as $\perp$

Examples

PO with $\bot$?

$(\mathbb{N}, \leq)$

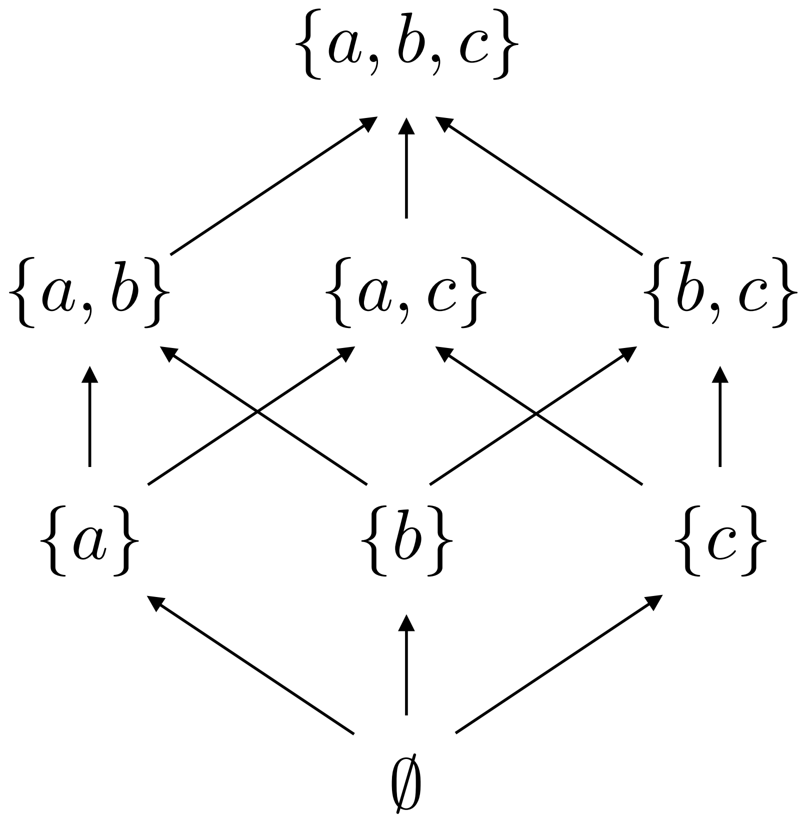
Yes

Examples

$(\wp(S), \subseteq)$

PO with $\bot$?

Yes



Examples

PO with $\perp$?

$(\mathbb{N}, =)$

No

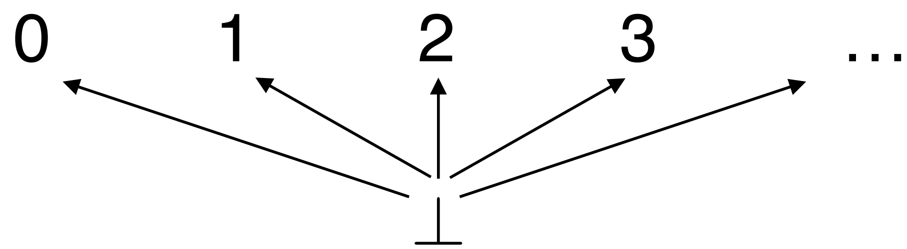
0 1 2 3 …

Examples

$$(\mathbb{N} \cup \{\bot\}, \{(\bot, n) \mid n \in \mathbb{N}\}\ \ )$$

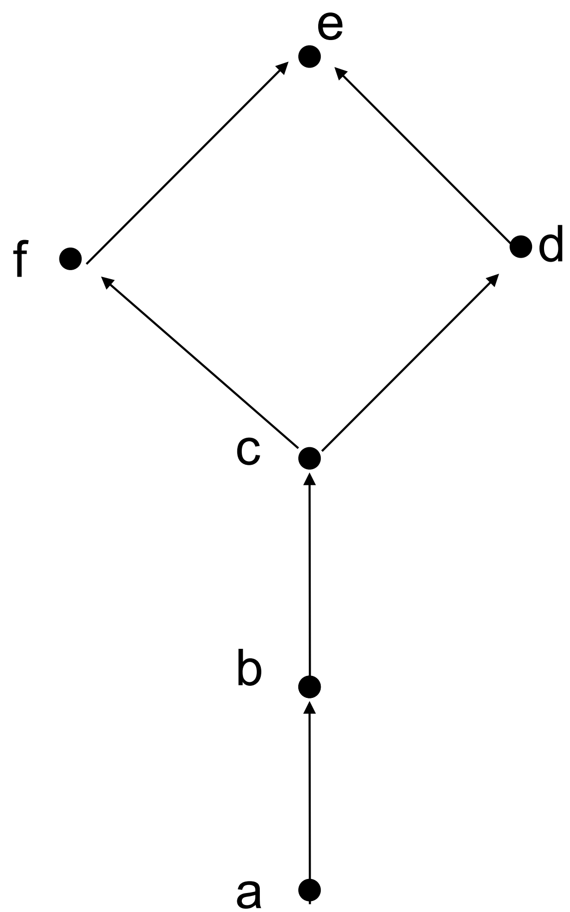
PO with $\bot$?

Yes



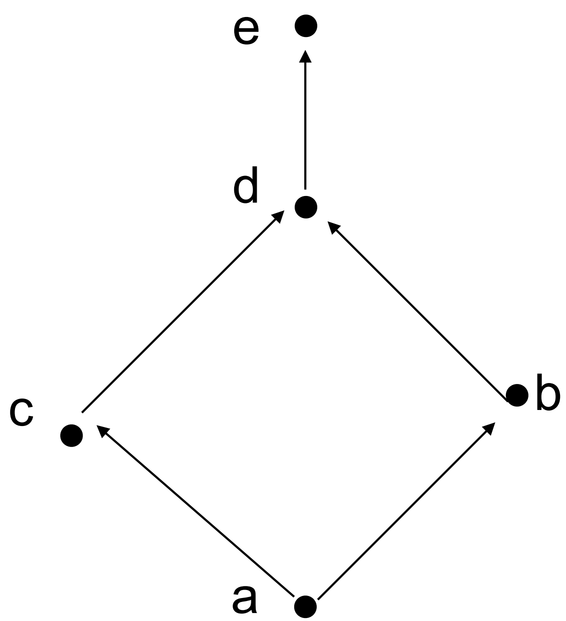
Lattice

A special structure arises when <u>every</u> pair of elements in a poset has a least upper bound (lub) and a a greatest lower (glb)



Lattice Definition

A lattice is a PO in which every pair of elements has both a lub and a glb



Examples

$(\mathbb{N}, \leq)$

Lattice ?

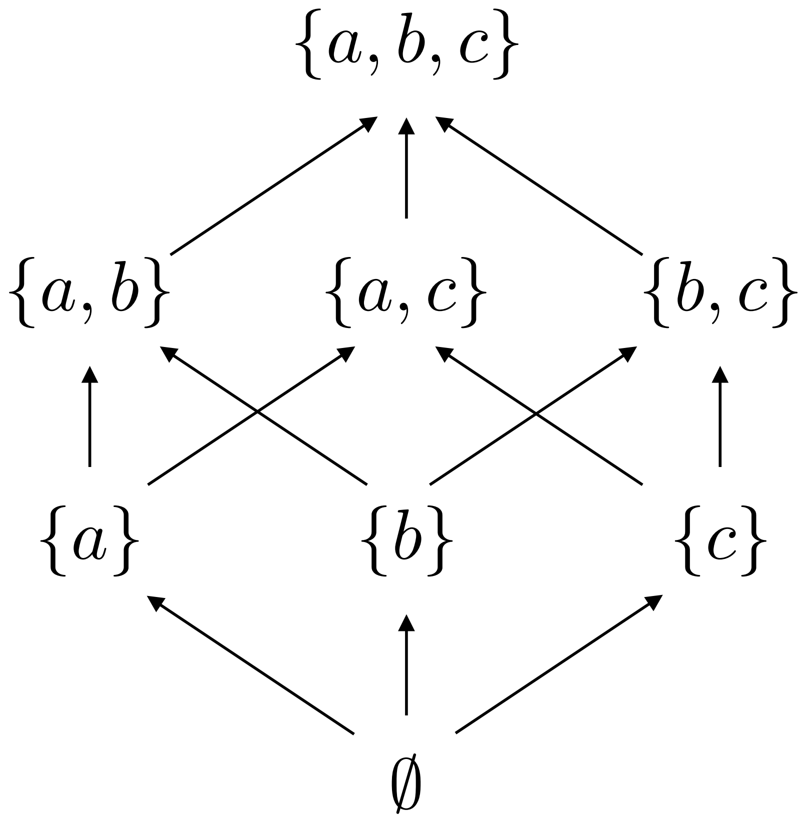
Yes

Examples

$(\wp(S), \subseteq)$

Lattice ?

Yes



Examples

$(\mathbb{N}, =)$

Lattice ?

No

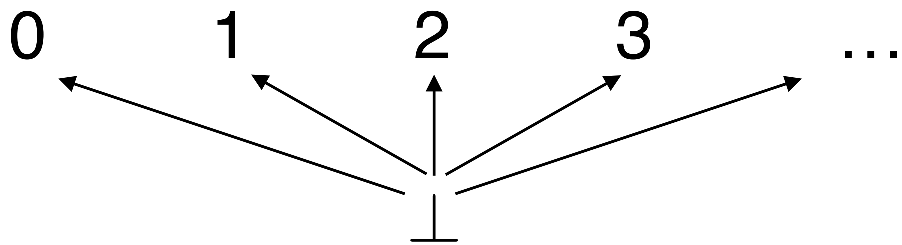
0 1 2 3 …

Examples

$$(\mathbb{N} \cup \{\bot\}, \{(\bot, n) \mid n \in \mathbb{N}\} \ \ )$$

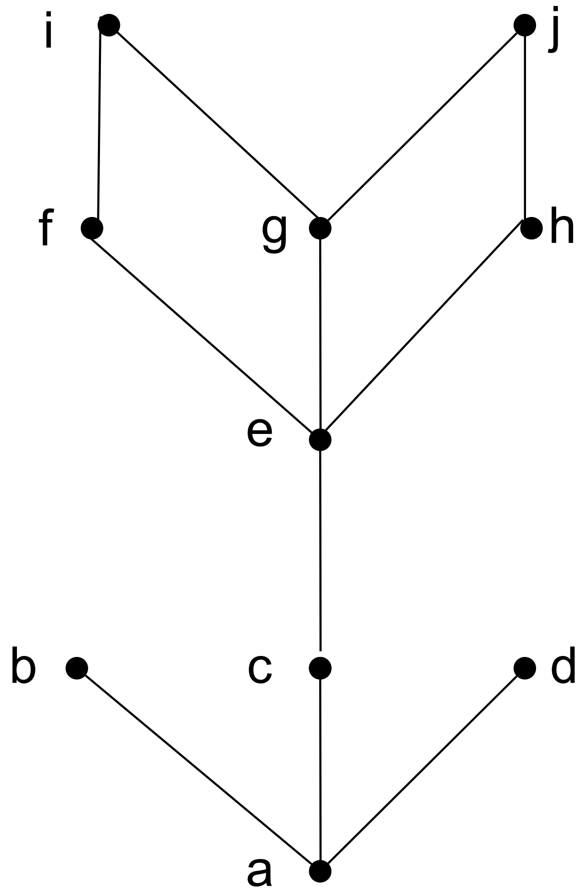
Lattice ?

No



Example

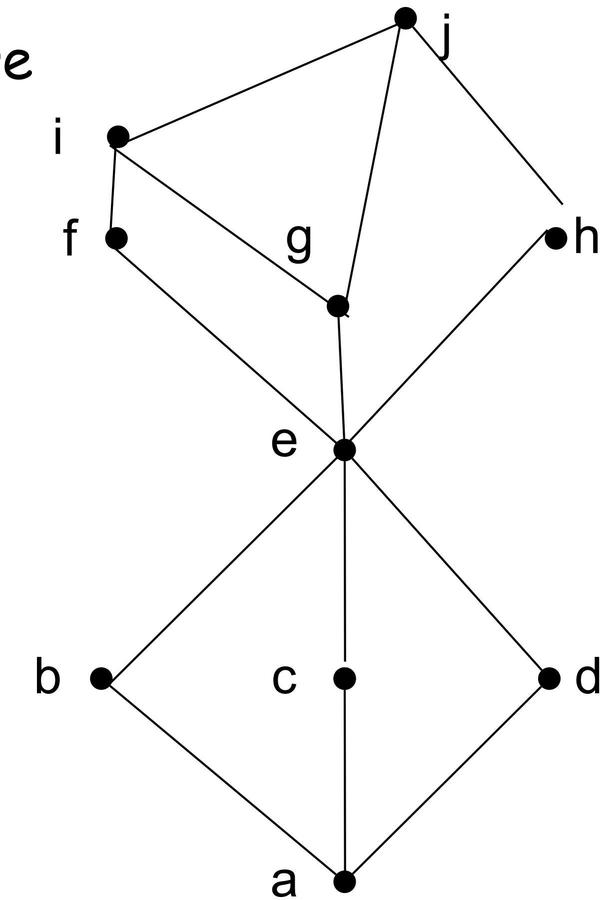
- Is this a lattice?



- **No, because the pair {b,c} does not have a least upper bound**

Example

- What if we modified it as shown here



  - Yes, because for any pair, there is a lub & a glb

Ascending chains

- A sequence $(l_n)_{n \in N}$ of elements in a partial order L is an ascending chain if

$$n \leq m \Rightarrow l_n \leq l_m$$

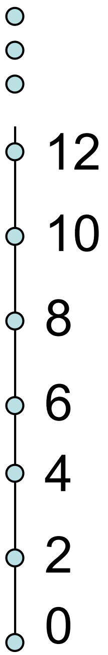
- A sequence $(l_n)_{n \in N}$ converges if and only if

$$\exists\, n_0 \in N : \forall\, n \in N : n_0 \leq n \Rightarrow l_{n_0} = l_n$$

- A partial order $(L,\leq)$ satisfes the ascending chain condition (ACC) iff each ascending chain converges.

Example

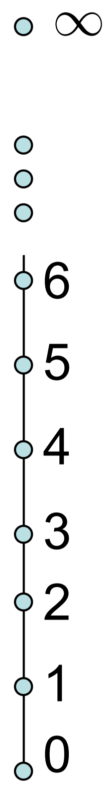
- The PO $(N, \sqsubseteq)$ does not satisfy the ascending chain condition,



Example

$$(N \cup \{\infty\}, \sqsubseteq)$$

satisfies the ACC condition



Complete Partial Order

A poset $(P, \sqsubseteq)$ is called a complete partial order (CPO) if and only if any of its chains has a lub

If $(P, \sqsubseteq)$ has a bottom element and any of its chains has a lub then $(P, \sqsubseteq)$ is called a complete partial order (CPO) with bottom

- $(N, \leq)$ has bottom 0 but is not complete:

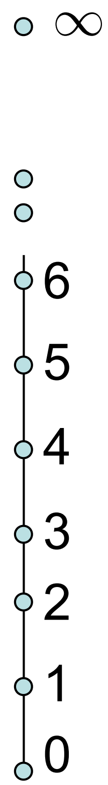
the chain $0 \leq 1 \leq 2 \leq \cdots \leq n \leq \cdots$ has no upper bound in N.

- $(N, \geq)$ is a CPO but has no bottom.

Example

$$(N \cup \{\infty\}, \sqsubseteq)$$

is CPO with bottom



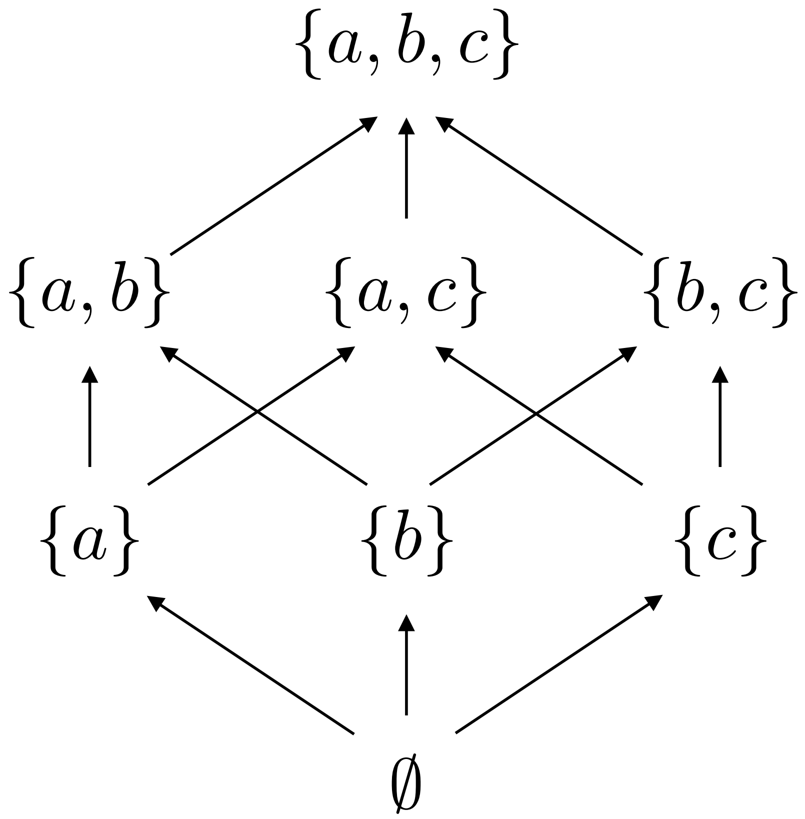
Some complete PO

If $(P, \sqsubseteq)$ has only finite chains it is complete

If $(P, \sqsubseteq)$ is finite then it is complete

CPO with bottom

$(\wp(S), \subseteq)$



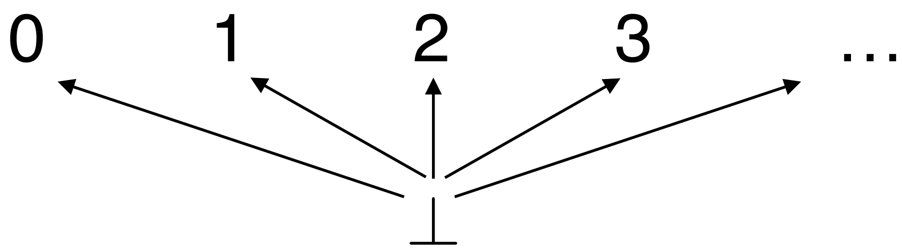
CPO without bottom

$(\mathbb{N}, =)$

0 1 2 3 …

CPO with bottom

$$(\mathbb{N} \cup \{\bot\}, \{(\bot, n) \mid n \in \mathbb{N}\} \ )$$



Equivalent Definitions of Complete Lattices

A lattice L is called a complete lattice if every subset S of L admits a lub in L.

↕

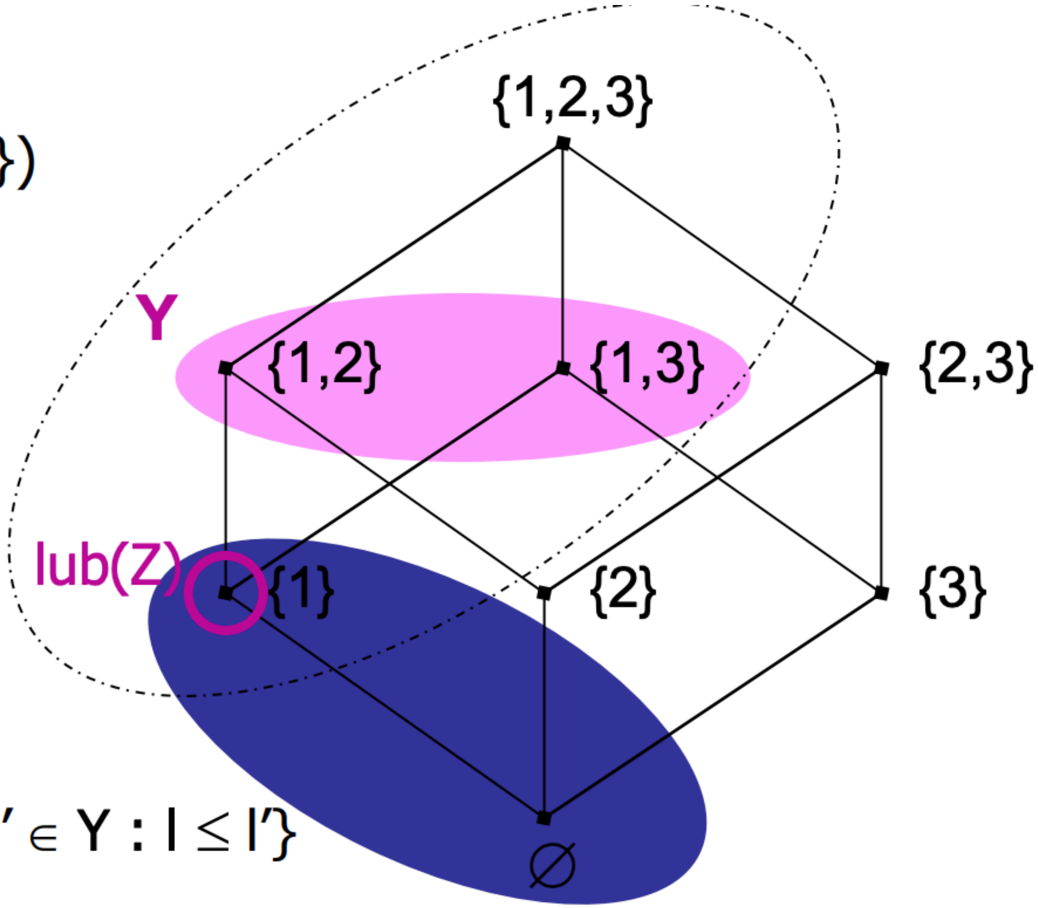
A lattice L is called a complete lattice if every subset S of L admits a glb in L.

↕

A lattice L is called a complete lattice if every subset S of L admits a glb and a lub in L.

The idea

$glb(Y) = lub(\{l \in L \mid \forall l' \in Y : l \leq l'\})$



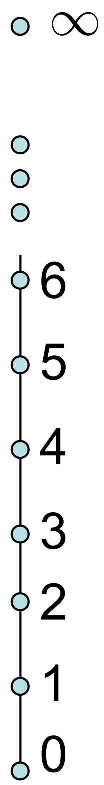
$Z = \{l \in L \mid \forall l' \in Y : l \leq l'\}$

Example

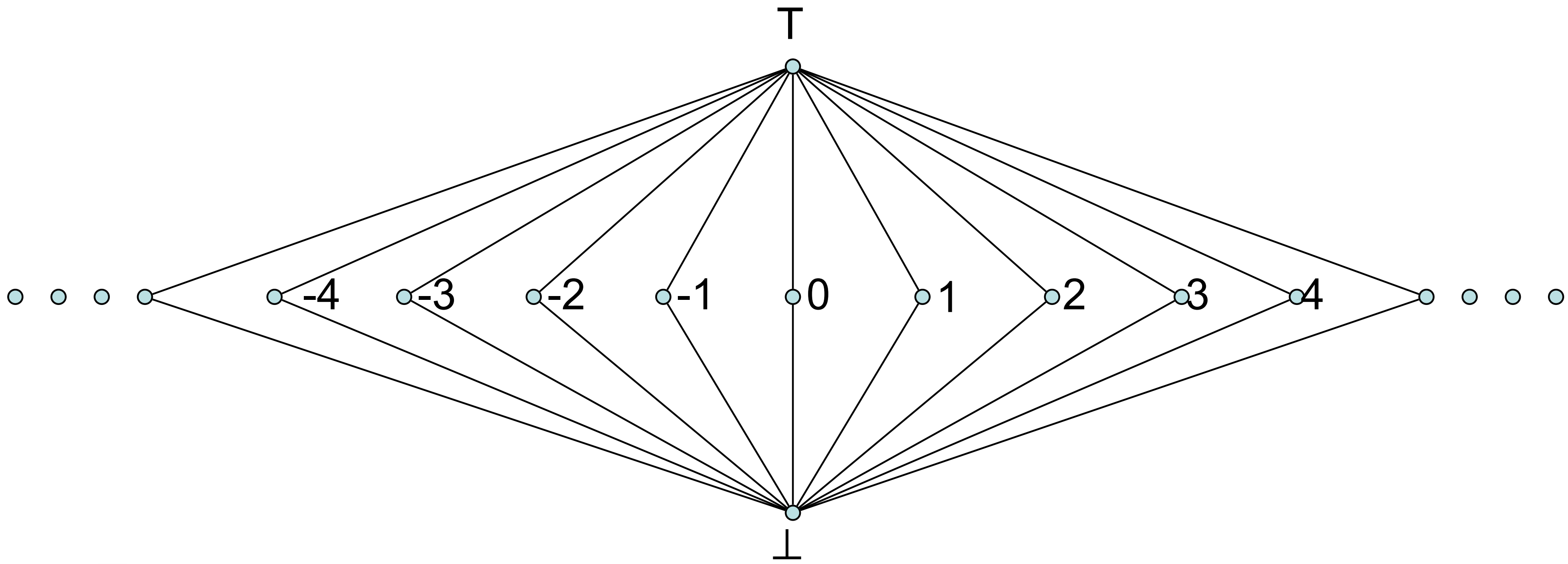
- L= N ∪ {}
- total order on N ∪ {∞}
- lub = max
- glb = min

$\leq$

This is a complete lattice



Example of complete lattice



- L= N∪ {T,⊥}
- ∀ n∈N : ⊥<n < T
- This is a complete lattice, with infinite elements

Lattices and ACC

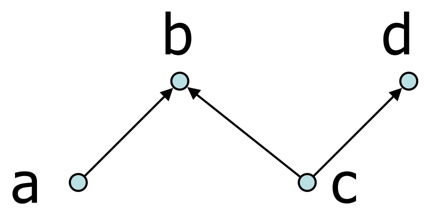
- If L is a lattice with a bottom element and ACC, then L is a complete lattice
- If L is a finite lattice then it satisfies the ACC and therefore it also is complete.

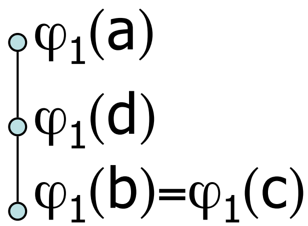
Monotone functions on partial orders

Let $(P,\leq_P)$ and $(Q,\leq_Q)$ be PO.

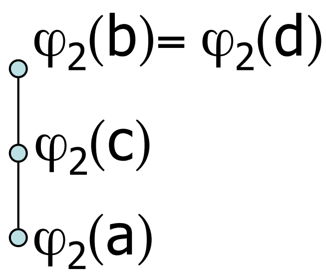
A function φ from P to Q is monotone iff

$$p1 \leq_P p2 \Rightarrow \phi(p1) \leq_Q \phi(p2)$$



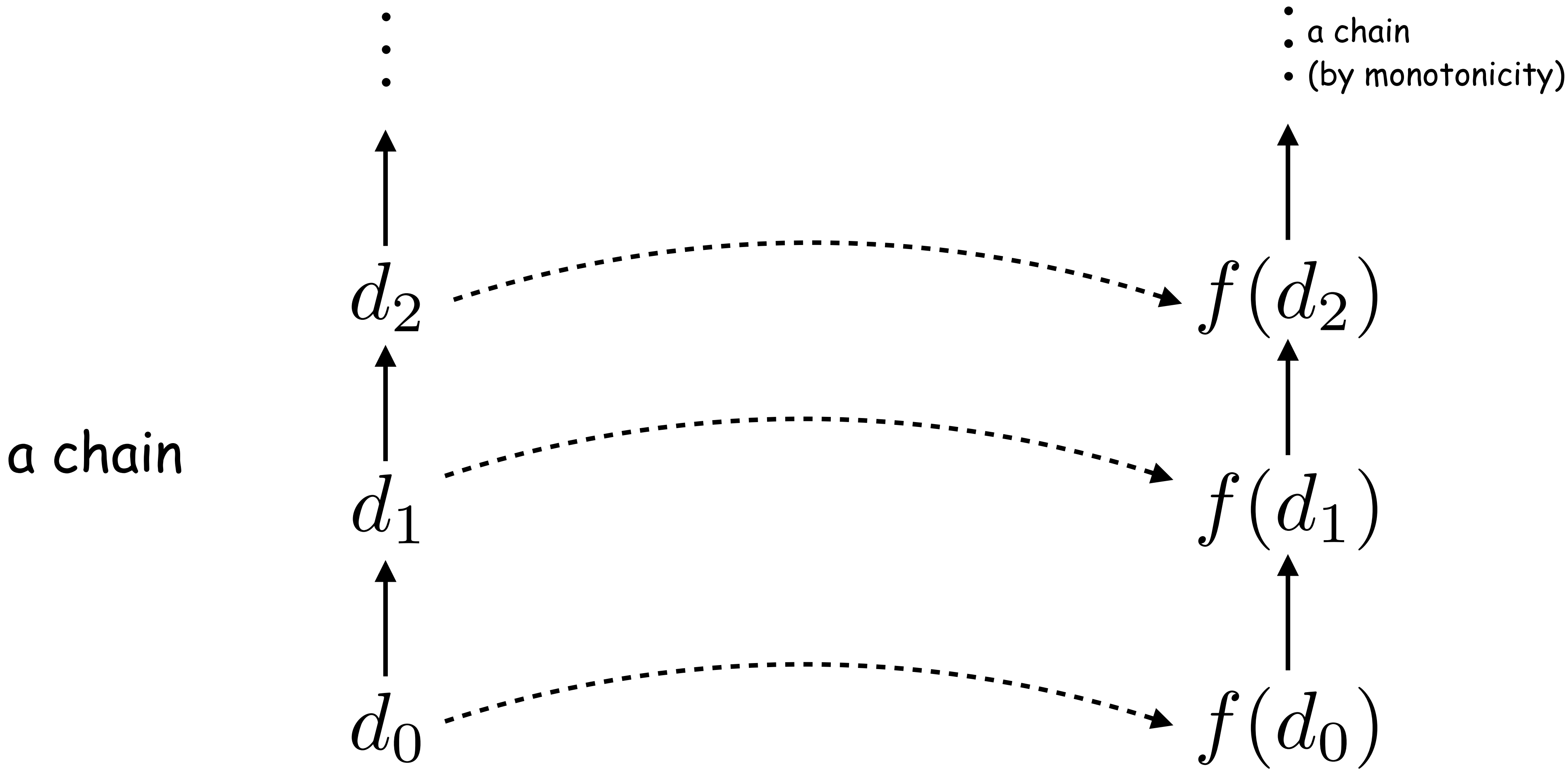


- φ_1 is not monotone



- φ_1 is monotone

Monotonicity

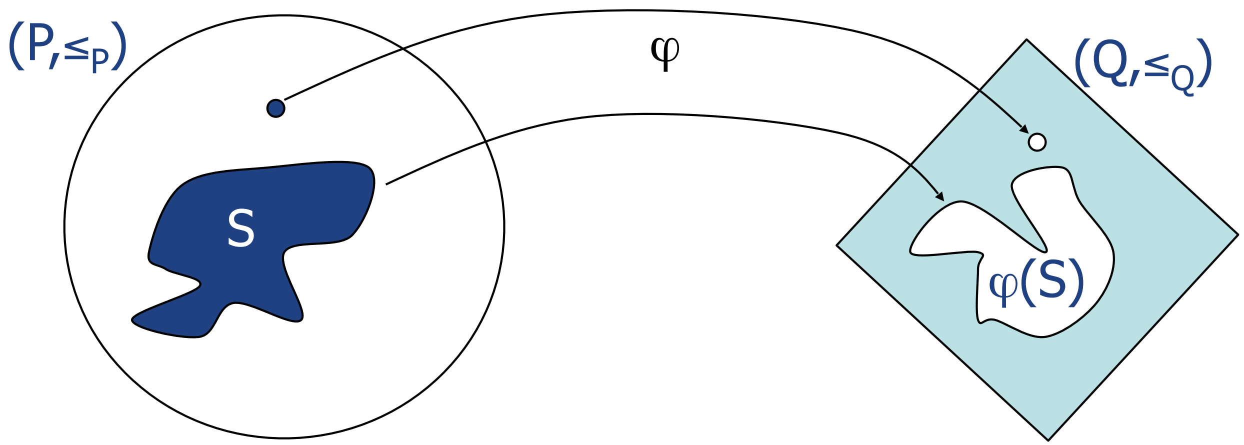


Continuity

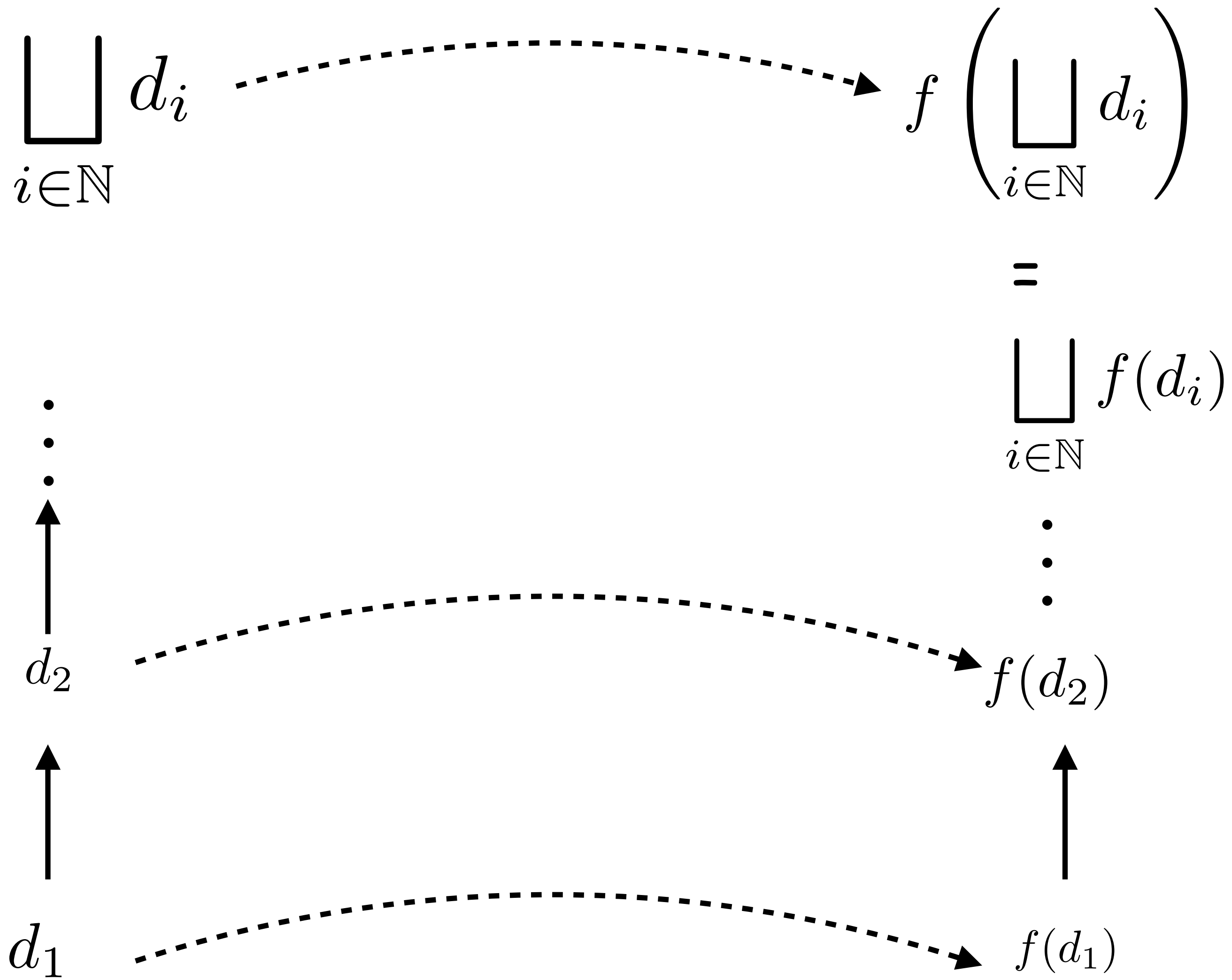
- Given two partial orders $(P, \leq_P)$ and $(Q, \leq_Q)$, a functoin φ from P to Q is continuous if for every chain S in P

$$\varphi(lub(S)) = lub\{ \varphi(x) \mid x \in S \}$$

  - if f is monotone on an *ACC* lattice then f is continuous



Continuous function



Fixpoints

- Consider a monotone function f: $(P,\leq_P) \rightarrow (P,\leq_P)$ on a partial order P
- An element x of P is a fixpoint of f if f(x)=x
- The set of fixpoints of f is a subset of P called Fix(f):

$$Fix(f) = \{ l \in P \mid f(l)=l \}$$

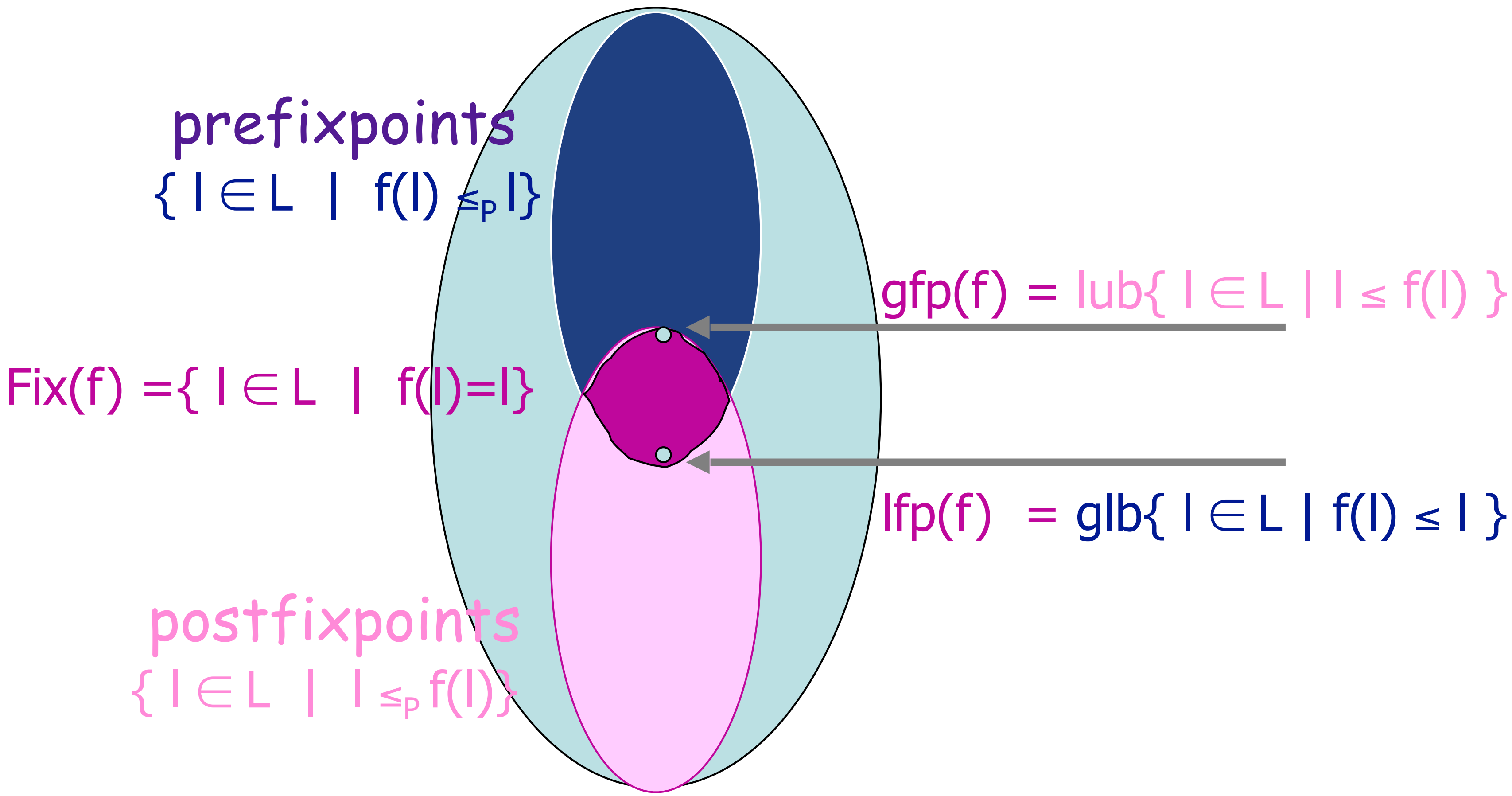
Fixpoint on Complete Lattices

- Consider a monotone function $f:L \rightarrow L$ on a complete lattice L.

- Tarski Theorem:
  Let L be a complete lattice. If $f:L \rightarrow L$ is monotone then

$$lfp(f) = glb\{\, l \in L \mid f(l) \leq l \,\}$$
$$gfp(f) = lub\{\, l \in L \mid l \leq f(l) \,\}$$

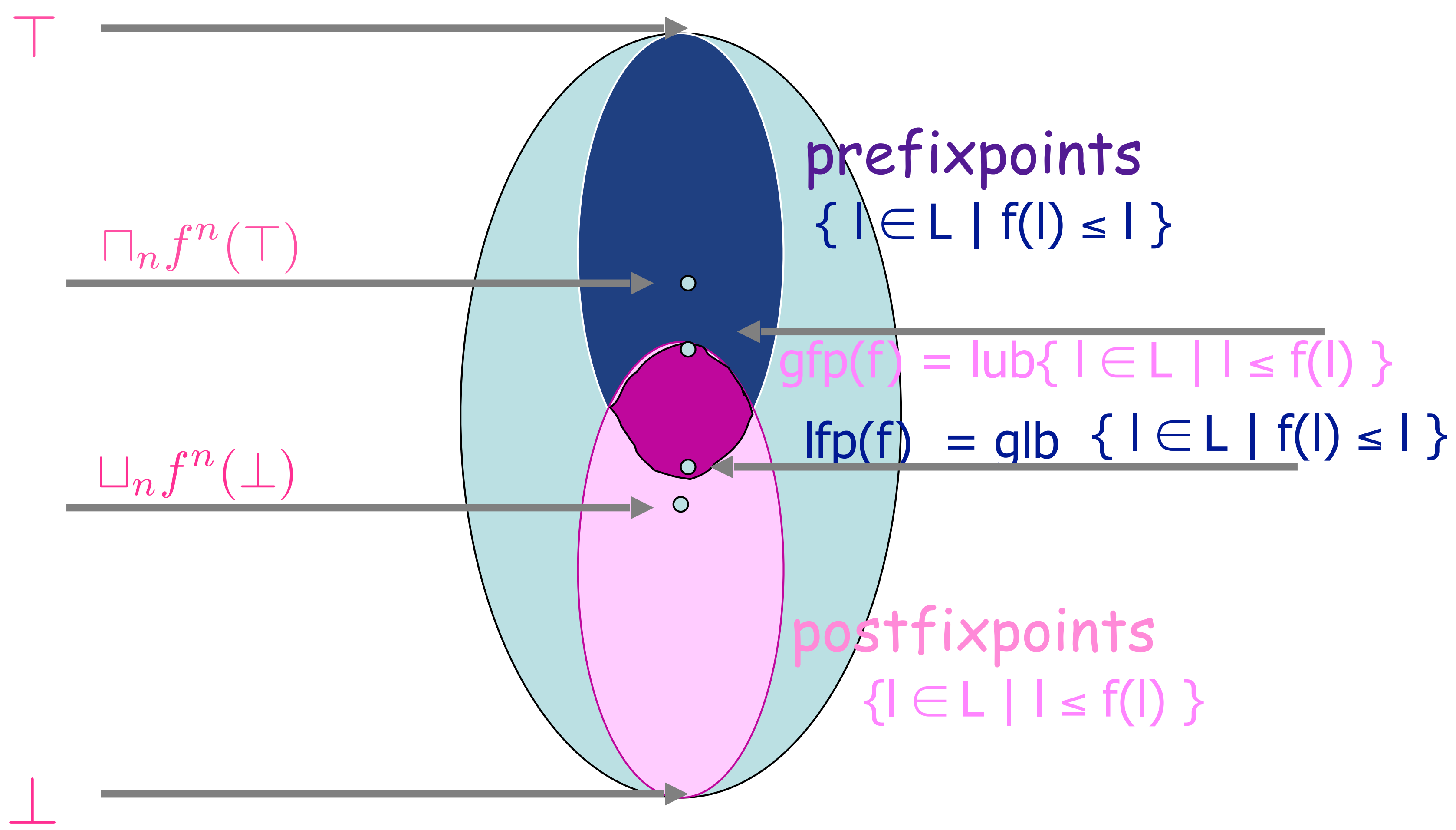
Fixpoints on Complete Lattices



Function monotone on complete lattice

- Let f be a monotone function: $(P, \leq_P) \rightarrow (P, \leq_P)$ on a complete lattice P.
  Let $\alpha = \bigsqcup_{n \geq 0} f^{n}(\perp)$
  - If $\alpha \in Fix(f)$ then $\alpha = lfp(f)$

Fixpoints on Complete Lattices



Kleene Theorem

Kleene Theorem

If f is continuous on a complete CPO with bottom then the least fixpoint of f exists and it is equal to α

recall that

- if f is monotone on an *ACC* lattice then f is continuous

Fixpoints on CPO with bottom when f is continuous