Overview

Introduction
Modelling parallel systems
Linear Time Properties

Regular Properties
regular safety properties
ω-regular properties
model checking with Büchi automata

Linear Temporal Logic
Computation-Tree Logic
Equivalences and Abstraction
idea: define regular LT properties to be those languages of infinite words over the alphabet $2^{AP}$ that have a representation by a finite automata
Regular LT properties

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• regular safety properties:
  NFA-representation for the bad prefixes
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- regular safety properties:
  
  NFA-representation for the bad prefixes

- representation other regular LT properties by
  
  $\omega$-automata, i.e., acceptors for infinite words
Regular LT properties

*idea:* define regular LT properties to be those languages of infinite words over the alphabet $2^{AP}$ that have a representation by a finite automata

- regular safety properties:
  - NFA-representation for the bad prefixes

- representation other regular LT properties by
  - $\omega$-automata, i.e., acceptors for infinite words
  - $\omega$-regular expressions
Regular expressions

*remind:* syntax and semantics of regular expressions over some alphabet $\Sigma = \{A, B, \ldots\}$
Regular expressions over $\Sigma$

$$\alpha ::= \emptyset | \epsilon | A | \alpha_1 + \alpha_2 | \alpha_1 \alpha_2 | \alpha^*$$
Regular expressions over $\Sigma$

\[ \alpha ::= \emptyset \mid \epsilon \mid A \mid \alpha_1 + \alpha_2 \mid \alpha_1 \cdot \alpha_2 \mid \alpha^* \]

where $A \in \Sigma$
Regular expressions over $\Sigma$

$$\alpha ::= \emptyset \mid \epsilon \mid A \mid \alpha_1 + \alpha_2 \mid \alpha_1 . \alpha_2 \mid \alpha^*$$

where $A \in \Sigma$

semantics: $\alpha \mapsto \mathcal{L}(\alpha) \subseteq \Sigma^*$ language of finite words
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<table>
<thead>
<tr>
<th>$\mathcal{L}(\emptyset)$</th>
<th>$\mathcal{L}(\epsilon)$</th>
<th>$\mathcal{L}(A)$</th>
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<tbody>
<tr>
<td>$\emptyset$</td>
<td>${\epsilon}$</td>
<td>${A}$</td>
</tr>
</tbody>
</table>

$\mathcal{L}(\alpha_1 + \alpha_2) = \mathcal{L}(\alpha_1) \cup \mathcal{L}(\alpha_2)$ union

$\mathcal{L}(\alpha_1 \cdot \alpha_2) = \mathcal{L}(\alpha_1)\mathcal{L}(\alpha_2)$ concatenation

$\mathcal{L}(\alpha^*) = \mathcal{L}(\alpha)^*$ Kleene closure
ω-regular expressions

regular expressions:

\[ \alpha ::= \emptyset \mid \epsilon \mid A \mid \alpha_1 + \alpha_2 \mid \alpha_1 \cdot \alpha_2 \mid \alpha^* \]

ω-regular expressions:

regular expressions + ω-operator \( \alpha^\omega \)
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Kleene star: “finite repetition”
ω-operator: “infinite repetition”
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Kleene star: “finite repetition”
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for \( L \subseteq \Sigma^* \):

\[ L^\omega \overset{\text{def}}{=} \{ w_1 w_2 w_3 \ldots : w_i \in L \text{ for all } i \geq 1 \} \]
\( \omega \)-regular expressions

Regular expressions:

\[ \alpha ::= \emptyset \mid \epsilon \mid A \mid \alpha_1 + \alpha_2 \mid \alpha_1 \cdot \alpha_2 \mid \alpha^* \]

\( \omega \)-regular expressions:

regular expressions + \( \omega \)-operator \( \alpha^\omega \)

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for \( L \subseteq \Sigma^* \):

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Note: \( L^\omega \subseteq \Sigma^\omega \) if \( \epsilon \notin L \)
Syntax and semantics of $\omega$-regular expressions
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Syntax of $\omega$-regular expressions over alphabet $\Sigma$:

$$\gamma = \alpha_1 \cdot \beta_1^\omega + \ldots + \alpha_n \cdot \beta_n^\omega$$

where

$\alpha_i, \beta_i$ are regular expressions over $\Sigma$ s.t. $\epsilon \notin \mathcal{L}(\beta_i)$
syntax of $\omega$-regular expressions over alphabet $\Sigma$:

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where $\alpha_i, \beta_i$ are regular expressions over $\Sigma$ s.t. $\varepsilon \not\in \mathcal{L}(\beta_i)$

semantics: the language generated by $\gamma$ is:

$$\mathcal{L}_\omega(\gamma) \overset{\text{def}}{=} \bigcup_{1 \leq i \leq n} \mathcal{L}(\alpha_i) \mathcal{L}(\beta_i)^\omega$$
Syntax and semantics of $\omega$-regular expressions

**Syntax:** The syntax of $\omega$-regular expressions over alphabet $\Sigma$ is defined as:

$$\gamma = \alpha_1\beta_1^\omega + ... + \alpha_n\beta_n^\omega$$

where $\alpha_i, \beta_i$ are regular expressions over $\Sigma$ such that $\epsilon \notin L(\beta_i)$.

**Semantics:** The language generated by $\gamma$ is:

$$L_\omega(\gamma) \overset{\text{def}}{=} \bigcup_{1 \leq i \leq n} L(\alpha_i)L(\beta_i)^\omega \subseteq \Sigma^\omega$$
Syntax and semantics of $\omega$-regular expressions

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- Language of $(A^* \cdot B)^\omega$
Syntax and semantics of $\omega$-regular expressions

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- language of $(A^* . B)^\omega = \{ \text{set of all infinite words over } \Sigma = \{A, B\} \} \text{ containing infinitely many } B \text{’s}$
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- language of $(A^* . B)^\omega$ = set of all infinite words over $\Sigma = \{A, B\}$ containing infinitely many $B$’s
- language of $(A^* . B)^\omega + (B^* . A)^\omega$
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- **Language of** $(A^* . B)^\omega$ = set of all infinite words over $\Sigma = \{A, B\}$ containing infinitely many $B$’s
- **Language of** $(A^* . B)^\omega + (B^* . A)^\omega$ = set of all infinite words over $\Sigma$ with infinitely many $A$’s or $B$’s
Syntax and semantics of $\omega$-regular expressions

**Syntax** of $\omega$-regular expressions over alphabet $\Sigma$:

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**Semantics**: the language generated by $\gamma$ is:

$$L_\omega(\gamma) \overset{\text{def}}{=} \bigcup_{1 \leq i \leq n} L(\alpha_i) L(\beta_i)^\omega \subseteq \Sigma^\omega$$

- language of $(A^* . B)^\omega = \text{set of all infinite words over } \Sigma = \{A, B\} \text{ containing infinitely many } B \text{'s}$
- language of $(A^* . B)^\omega + (B^* . A)^\omega = \text{set of all infinite words over } \Sigma \text{ with infinitely many } A \text{'s or } B \text{'s} = \Sigma^\omega$
\( \omega \)-regular languages

**Syntax** of \( \omega \)-regular expressions over alphabet \( \Sigma \):

\[
\gamma = \alpha_1 \cdot \beta_1^\omega + \ldots + \alpha_n \cdot \beta_n^\omega \quad \text{where}
\]

\( \alpha_i, \beta_i \) are regular expressions over \( \Sigma \) s.t. \( \varepsilon \notin L(\beta_i) \)

**Semantics**: the language generated by \( \gamma \) is:

\[
L_\omega(\gamma) \overset{\text{def}}{=} \bigcup_{1 \leq i \leq n} L(\alpha_i)L(\beta_i)^\omega \subseteq \Sigma^\omega
\]

A language \( L \subseteq \Sigma^\omega \) is called \( \omega \)-regular iff there exists an \( \omega \)-regular expression \( \gamma \) s.t. \( L = L_\omega(\gamma) \)
Provide an $\omega$-regular expression for ...

alphabet $\Sigma = \{ A, B \}$

- set of all infinite words over $\Sigma$ containing only finitely many $A$’s
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$$(A + B)^* \cdot B^\omega$$
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- set of all infinite words where each $A$ is followed immediately by letter $B$
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\[(A + B)^* \cdot B^\omega\]

- set of all infinite words where each $A$ is followed immediately by letter $B$

\[(B^* \cdot A \cdot B)^* \cdot B^\omega + (B^* \cdot A \cdot B)^\omega\]
Provide an $\omega$-regular expression for ...

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- set of all infinite words over $\Sigma$ containing only finitely many $A$’s
  $$(A + B)^*.B^\omega$$

- set of all infinite words where each $A$ is followed immediately by letter $B$
  $$(B*.A.B)^*.B^\omega + (B*.A.B)^\omega$$

- set of all infinite words where each $A$ is followed eventually by letter $B$
Provide an $\omega$-regular expression for ...

alphabet $\Sigma = \{A, B\}$

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  $$(B^*.A.B)^*.B^\omega + (B^*.A.B)^\omega$$

- set of all infinite words where each $A$ is followed eventually by letter $B$
  $$(B^*.A^+.B)^*.B^\omega + (B^*.A^+.B)^\omega$$

where $\alpha^+ \overset{\text{def}}{=} \alpha \cdot \alpha^*$. 
Provide an $\omega$-regular expression for ...

alphabet $\Sigma = \{A, B\}$

- set of all infinite words over $\Sigma$ containing only finitely many $A$’s
  
  $$(A + B)^* \cdot B^\omega$$

- set of all infinite words where each $A$ is followed immediately by letter $B$
  
  $$(B^* \cdot A \cdot B)^* \cdot B^\omega + (B^* \cdot A \cdot B)^\omega$$

- set of all infinite words where each $A$ is followed eventually by letter $B$
  
  $$(B^* \cdot A^+ \cdot B)^* \cdot B^\omega + (B^* \cdot A^+ \cdot B)^\omega \equiv (A^* \cdot B)^\omega$$

where $\alpha^+ \overset{\text{def}}{=} \alpha \cdot \alpha^*$. 
Let $E$ be an LT-property over $AP$, i.e., $E \subseteq (2^AP)^\omega$
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$E$ is called an $\omega$-regular property iff there exists an $\omega$-regular expression $\gamma$ over $2^{AP}$ s.t. $E = \mathcal{L}_\omega(\gamma)$
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Examples for $AP = \{a, b\}$

- invariant with invariant condition $a \lor \neg b$
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  $\omega$-

$(\emptyset + \{a\} + \{a, b\})^\omega$
ω-regular properties

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Each invariant is $\omega$-regular
ω-regular properties

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**Examples for $AP = \{a, b\}$**

- invariant with invariant condition $a \lor \neg b$

  $(\emptyset + \{a\} + \{a, b\})^\omega$

Each invariant is $\omega$-regular

Let $\Phi$ be an invariant condition and let

\[ \{A \subseteq AP : A \models \Phi\} = \{A_1, \ldots, A_k\} \]

Then: invariant “always $\Phi$” $\models (A_1 + \ldots + A_k)^\omega$
Let $E$ be an LT-property over $AP$, i.e., $E \subseteq (2^{AP})^\omega$

$E$ is called an $\omega$-regular property iff there exists an $\omega$-regular expression $\gamma$ over $2^{AP}$ s.t. $E = L_\omega(\gamma)$

Examples for $AP = \{a, b\}$

- invariant with invariant condition $a \lor \neg b$
  
  $(\emptyset + \{a\} + \{a, b\})^\omega$

  Indeed: each invariant is $\omega$-regular

- “infinitely often $a$”
\( \omega \)-regular properties

Let \( E \) be an LT-property over \( AP \), i.e., \( E \subseteq (2^{AP})^\omega \)

\( E \) is called an \( \omega \)-regular property iff there exists an \( \omega \)-regular expression \( \gamma \) over \( 2^{AP} \) s.t. \( E = \mathcal{L}_\omega(\gamma) \)

**Examples** for \( AP = \{ a, b \} \)

- invariant with invariant condition \( a \lor \neg b \)

\[
(\emptyset + \{a\} + \{a, b\})^\omega
\]

Indeed: each invariant is \( \omega \)-regular

- “infinitely often \( a \)”

\[
((\emptyset + \{b\})^*.(\{a\} + \{a, b\}))^\omega
\]
Let $E$ be an LT-property over $AP$, i.e., $E \subseteq 2^{AP}$.

$E$ is called an $\omega$-regular property iff there exists an $\omega$-regular expression $\gamma$ over $2^{AP}$ s.t. $E = \mathcal{L}_\omega(\gamma)$.

Examples for $AP = \{a, b\}$:

- “always $a$” (or any other invariant)
- “infinitely often $a$”
- “eventually $a$”
Let $E$ be an LT-property over $AP$, i.e., $E \subseteq 2^AP$. $E$ is called an $\omega$-regular property iff there exists an $\omega$-regular expression $\gamma$ over $2^AP$ s.t. $E = \mathcal{L}_\omega(\gamma)$.

Examples for $AP = \{a, b\}$:

- “always $a$” (or any other invariant)
- “infinitely often $a$”
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$$(2^AP)^*.(\{a\} + \{a, b\}).(2^AP)$$

where $2^AP \equiv \emptyset + \{a\} + \{b\} + \{a, b\}$
Let $E$ be an LT-property over $AP$, i.e., $E \subseteq 2^{AP}$. $E$ is called an $\omega$-regular property iff there exists an $\omega$-regular expression $\gamma$ over $2^{AP}$ s.t. $E = \mathcal{L}_\omega(\gamma)$.

Examples for $AP = \{a, b\}$:

- “always $a$” (or any other invariant)
- “infinitely often $a$”
- “eventually $a$”
- $(2^{AP})^*.((\{a\} + \{a, b\}).(2^{AP})^\omega$
- “from some moment on $a$”
Let \( E \) be an LT-property over \( AP \), i.e., \( E \subseteq 2^AP \). \( E \) is called an \( \omega \)-regular property iff there exists an \( \omega \)-regular expression \( \gamma \) over \( 2^AP \) s.t. \( E = L_\omega(\gamma) \).

**Examples for \( AP = \{a, b\} \):**

- “always \( a \)” (or any other invariant)
- “infinitely often \( a \)”
- “eventually \( a \)”
  
  \[ (2^AP)^*.(\{a\} + \{a, b\}).(2^AP)^\omega \]

- “from some moment on \( a \)”
  
  \[ (2^AP)^*.(\{a\} + \{a, b\})^\omega \]
symbolic notation for $\omega$-regular properties

... using formulas instead of sums ....
Symbolic notation

Examples for $AP = \{a, b\}$

- invariant with invariant condition $a \lor \neg b$

  $$(\emptyset + \{a\} + \{a, b\})^\omega$$
Examples for $AP = \{a, b\}$

- invariant with invariant condition $a \lor \neg b$

$$(a \lor \neg b)\omega \equiv (\emptyset + \{a\} + \{a, b\})\omega$$
Symbolic notation

Examples for $\mathbf{AP} = \{a, b\}$

- invariant with invariant condition $a \lor \neg b$

  $$(a \lor \neg b)\omega \equiv (\emptyset + \{a\} + \{a, b\})\omega$$

- “infinitely often $a$”

  $$((\neg a)^*.a)\omega \equiv ((\emptyset + \{b\})^*. (\{a\} + \{a, b\}))\omega$$
Examples for $AP = \{a, b\}$

- invariant with invariant condition $a \lor \neg b$
  
  $$(a \lor \neg b)^\omega \equiv (\emptyset + \{a\} + \{a, b\})^\omega$$

- “infinitely often $a$”
  
  $$((\neg a)^* \cdot a)^\omega \equiv (((\emptyset + \{b\})^* \cdot (\{a\} + \{a, b\}))^\omega$$

- “from some moment on $a$”:
Symbolic notation

Examples for $AP = \{a, b\}$

- invariant with invariant condition $a \lor \neg b$
  \[(a \lor \neg b)^\omega \cong (\emptyset + \{a\} + \{a, b\})^\omega\]

- “infinitely often $a$”
  \[((\neg a)^*.a)^\omega \cong (((\emptyset + \{b\})^*(\{a\} + \{a, b\}))^\omega\]

- “from some moment on $a$”:
  \[true^*.a^\omega\]
Examples for $AP = \{a, b\}$

- invariant with invariant condition $a \lor \neg b$

  $$(a \lor \neg b)^\omega \equiv (\emptyset + \{a\} + \{a, b\})^\omega$$

- “infinitely often $a$”

  $$(\neg a^* \cdot a)^\omega \equiv ((\emptyset + \{b\})^* (\{a\} + \{a, b\}))^\omega$$

- “from some moment on $a$”:

  $$true^* \cdot a^\omega$$

- “whenever $a$ then $b$ will hold somewhen later”
Symbolic notation

Examples for $AP = \{a, b\}$

- invariant with invariant condition $a \lor \neg b$
  
  $$(a \lor \neg b)\omega \models (\emptyset + \{a\} + \{a, b\})\omega$$

- “infinitely often $a$”

  $$((\neg a)^* . a)\omega \models ((\emptyset + \{b\})^*. (\{a\} + \{a, b\}))\omega$$

- “from some moment on $a$”:

  $$true^* . a\omega$$

- “whenever $a$ then $b$ will hold somewhen later”

  $$((\neg a)^* . a. true^* . b)^* . (\neg a)\omega + ((\neg a)^* . a. true^* . b)\omega$$
Nondeterministic Büchi automata (NBA)
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syntax as for NFA

↑
nondeterministic finite automata
Nondeterministic Büchi automata (NBA)

syntax as for NFA

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nondeterministic finite automata

semantics: language of infinite words
Nondeterministic Büchi automata (NBA)

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$
Nondeterministic Büchi automata (NBA)

\[ \mathcal{A} = (Q, \Sigma, \delta, Q_0, F) \]

- \( Q \) finite set of states
Nondeterministic Büchi automata (NBA)

NBA $A = (Q, \Sigma, \delta, Q_0, F)$

- $Q$ finite set of states
- $\Sigma$ alphabet
Nondeterministic Büchi automata (NBA)

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

- $Q$ finite set of states
- $\Sigma$ alphabet
- $\delta : Q \times \Sigma \rightarrow 2^Q$ transition relation
Nondeterministic Büchi automata (NBA)

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

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- $\delta : Q \times \Sigma \rightarrow 2^Q$ transition relation
- $Q_0 \subseteq Q$ set of initial states
Nondeterministic Büchi automata (NBA)

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

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- $Q_0 \subseteq Q$ set of initial states
- $F \subseteq Q$ set of final states, also called accept states
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- $\delta : Q \times \Sigma \rightarrow 2^Q$ transition relation
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run for a word $A_0A_1A_2 \ldots \in \Sigma^\omega$:

state sequence $\pi = q_0q_1q_2 \ldots$ where $q_0 \in Q_0$
and $q_{i+1} \in \delta(q_i, A_i)$ for $i \geq 0$
Nondeterministic Büchi automata (NBA)

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

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Run for a word $A_0 A_1 A_2 \ldots \in \Sigma^\omega$:

- state sequence $\pi = q_0 q_1 q_2 \ldots$ where $q_0 \in Q_0$
- and $q_{i+1} \in \delta(q_i, A_i)$ for $i \geq 0$

Run $\pi$ is accepting if $\exists i \in \mathbb{N}$. $q_i \in F$
Nondeterministic Büchi automata (NBA)

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

- $Q$ finite set of states
- $\Sigma$ alphabet
- $\delta : Q \times \Sigma \rightarrow 2^Q$ transition relation
- $Q_0 \subseteq Q$ set of initial states
- $F \subseteq Q$ set of final states, also called accept states

accepted language $\mathcal{L}_\omega(\mathcal{A}) \subseteq \Sigma^\omega$ is given by:

$$\mathcal{L}_\omega(\mathcal{A}) \overset{\text{def}}{=} \text{set of infinite words over } \Sigma \text{ that have an accepting run in } \mathcal{A}$$
Notations in pictures for NBA

Initial state

Nonfinal state

Final state

$q_0$ to $q_1$ transition with labels $B$ and $A$.
Notations in pictures for NBA

NBA with state space \( \{ q_0, q_1 \} \)

\( q_0 \) initial state
\( q_1 \) accept state

alphabet \( \Sigma = \{ A, B \} \)
Examples for NBA over $\Sigma = \{A, B\}$

accepted language: ?
Examples for NBA over $\Sigma = \{A, B\}$

accepted language:
set of all infinite words that contain infinitely many $A$'s
Examples for NBA over $\Sigma = \{A, B\}$

accepted language: set of all infinite words that contain infinitely many $A$'s

$$(B^*.A)^\omega$$
Examples for NBA over $\Sigma = \{A, B\}$

accepted language: set of all infinite words that contain infinitely many $A$'s

$(B^* . A)^\omega$
Examples for NBA over $\Sigma = \{A, B\}$

accepted language:
set of all infinite words that contain infinitely many $A$'s

$$(B^* . A)^\omega$$

accepted words:

$AAABBAAABBAAABB...$

$AAAAA AAAA AAAA AAAA ...$
Examples for NBA over $\Sigma = \{A, B\}$

accepted language: set of all infinite words that contain infinitely many $A$’s

$$(B^* . A)^\omega$$

accepted words:

$\{ AAA AAB AAB AAB \ldots, AAAAAA AAA AAA AAA \ldots \} \text{ accepted words}$$

accepted language:

“every $B$ is preceded by a positive even number of $A$’s”
Examples for NBA over $\Sigma = \{A, B\}$

accepted language:
set of all infinite words that contain infinitely many $A$’s

$$\left( B^* . A \right)^\omega$$

accepted language:
“every $B$ is preceded by a positive even number of $A$’s”

$$\left( \left( A . A \right)^+ . B \right)^\omega + \left( \left( A . A \right)^+ . B \right)^* . A^\omega$$
NBA for LT properties
NBA for LT properties

NBA \( \mathcal{A} = (Q, \Sigma, \delta, Q_0, F) \)

- \( Q \) finite set of states
- \( \Sigma \) alphabet
- \( \delta : Q \times \Sigma \rightarrow 2^Q \) transition relation
- \( Q_0 \subseteq Q \) set of initial states
- \( F \subseteq Q \) set of final states, also called accept states
NBA for LT properties

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

- $Q$ finite set of states
- $\Sigma$ alphabet
- $\delta : Q \times \Sigma \rightarrow 2^Q$ transition relation
- $Q_0 \subseteq Q$ set of initial states
- $F \subseteq Q$ set of final states, also called accept states

here: $\Sigma = 2^{AP}$
NBA for LT properties

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

- $Q$ finite set of states
- $\Sigma$ alphabet
- $\delta : Q \times \Sigma \to 2^Q$ transition relation
- $Q_0 \subseteq Q$ set of initial states
- $F \subseteq Q$ set of final states, also called accept states

accepted language $\mathcal{L}_\omega(\mathcal{A})$ is an LT-property:

$$\mathcal{L}_\omega(\mathcal{A}) = \text{set of infinite words over } 2^{AP} \text{ that have an accepting run in } \mathcal{A}$$
NBA for LT properties

\[ L_\omega(A) = \ ? \]

set of atomic propositions \( AP = \{a, b\} \)
NBA for LT properties

\[ L_\omega(A) \triangleq \text{true. } \neg a. \text{true}^\omega \]

set of atomic propositions \( AP = \{ a, b \} \)
NBA for LT properties

\[ L_\omega(A) \equiv \text{true. } \neg a. \text{ true}^\omega \]

set of atomic propositions \( AP = \{a, b\} \)
**NBA for LT properties**

\[ L_\omega(A) \equiv \text{true} \cdot \neg a \cdot \text{true}^\omega \]

Set of atomic propositions \( AP = \{a, b\} \)
NBA for LT properties
NBA for LT properties

\[ a \lor \neg b \rightarrow q_1 \]

\[ b \rightarrow q_1 \]

\[ a \rightarrow q_0 \]

\[ "always \ a" \equiv a^\omega \]
NBA for LT properties

\[ \text{“always } a \text{” } \equiv a^\omega \]
NBA for LT properties

```
q0 \rightarrow a \lor \neg b \rightarrow q1
```

“always \( a \)” \( \equiv a^{\omega} \)

```
q0 \rightarrow b \rightarrow q1
```

“infinitely often \( a \) and ...”
NBA for LT properties

“always $a$” $\equiv a^\omega$

“infinitely often $a$ and always $a \lor b$”

$\equiv ((a \lor b)^*.a)^\omega$
NBA for LT properties

“infinitely often $a$ and always $a \lor b$”

$$((a \lor b)^* \cdot a)^\omega$$

“infinitely often $a$”

$$((\neg a)^* \cdot a)^\omega$$
NBA for LT properties

"infinitely often \( a \) and always \( a \lor b \)"

\[ ((a \lor b)^* \cdot a)^\omega \]

"infinitely often \( a \)"

\[ ((\neg a)^* \cdot a)^\omega \]
From NBA to $\omega$-regular expressions
From NBA to $\omega$-regular expressions

For each NBA $A$ there is an $\omega$-regular expression $\gamma$ with $\mathcal{L}_\omega(A) = \mathcal{L}_\omega(\gamma)$
From NBA to $\omega$-regular expressions

For each NBA $A$ there is an $\omega$-regular expression $\gamma$ with $L_\omega(A) = L_\omega(\gamma)$

Proof. Let $A$ be an NBA $(Q, \Sigma, \delta, Q_0, F)$
From NBA to \( \omega \)-regular expressions

For each NBA \( \mathcal{A} \) there is an \( \omega \)-regular expression \( \gamma \) with \( \mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma) \)

**Proof.** Let \( \mathcal{A} \) be an NBA \( (Q, \Sigma, \delta, Q_0, F) \) and \( q, p \in Q \). Let \( \mathcal{A}_{q,p} \) be the NFA \( (Q, \Sigma, \delta, q, \{p\}) \).
From NBA to $\omega$-regular expressions

For each NBA $A$ there is an $\omega$-regular expression $\gamma$ with $\mathcal{L}_\omega(A) = \mathcal{L}_\omega(\gamma)$

Proof. Let $A$ be an NBA $(Q, \Sigma, \delta, Q_0, F)$ and $q, p \in Q$. Let $A_{q,p}$ be the NFA $(Q, \Sigma, \delta, q, \{p\})$. Then:

$$\mathcal{L}_\omega(A) = \bigcup_{q \in Q_0} \bigcup_{p \in F} \mathcal{L}(A_{q,p}) \cdot (\mathcal{L}(A_{p,p}) \setminus \{\varepsilon\})^\omega$$
From NBA to \(\omega\)-regular expressions

For each NBA \(A\) there is an \(\omega\)-regular expression \(\gamma\) with \(L_\omega(A) = L_\omega(\gamma)\)

**Proof.** Let \(A\) be an NBA \((Q, \Sigma, \delta, Q_0, F)\) and \(q, p \in Q\). Let \(A_{q,p}\) be the NFA \((Q, \Sigma, \delta, q, \{p\})\). Then:

\[
L_\omega(A) = \bigcup_{q \in Q_0} \bigcup_{p \in F} L(A_{q,p}) \left( L(A_{p,p}) \setminus \{\varepsilon\} \right)^\omega
\]

is \(\omega\)-regular as \(L(A_{q,p})\) and \(L(A_{p,p}) \setminus \{\varepsilon\}\) are regular.
Example: NBA $\rightsquigarrow \omega$-regular expression
Example: NBA $\rightsquigarrow \omega$-regular expression

$\mathcal{L}_\omega(A) = L_{12}(L'_{22})^\omega \cup L_{22}(L'_{22})^\omega$

$L_{12} = \mathcal{L}(A_{12})$
$L_{22} = \mathcal{L}(A_{22})$
$L'_{22} = L_{22} \setminus \{\varepsilon\}$
Example: NBA $\rightsquigarrow \omega$-regular expression

$\mathcal{L}_\omega(A) = L_{12}(L'_{22})^\omega \cup L_{22}(L'_{22})^\omega$

$L_{12} = \mathcal{L}(A_{12})$
$L_{22} = \mathcal{L}(A_{22})$
$L'_{22} = L_{22} \setminus \{\epsilon\}$
Example: NBA $\sim \omega$-regular expression

NBA $\mathcal{A}$

$L_\omega(\mathcal{A}) = L_{12}(L'_{22})^\omega \cup L_{22}(L'_{22})^\omega$

$L_{12} = \mathcal{L}(\mathcal{A}_{12})$

$L_{22} = \mathcal{L}(\mathcal{A}_{22})$

$L'_{22} = L_{22} \setminus \{\varepsilon\}$

$L_{12} \doteqdot A.(B.A + A.A.A)^*$

NFA $\mathcal{A}_{12}$
Example: NBA $\sim \omega$-regular expression

$\mathcal{L}_\omega(A) = L_{12}(L'_{22})^\omega \cup L_{22}(L'_{22})^\omega$

$L_{12} = \mathcal{L}(A_{12})$

$L_{22} = \mathcal{L}(A_{22})$

$L'_{22} = L_{22} \setminus \{\varepsilon\}$

$L_{12} \equiv A.(B.A + A.A.A)^*$

$L'_{22} \equiv (B.A + A.A.A)^+$
Example: NBA $\sim\omega$-regular expression

**NBA $\mathcal{A}$**

```
language of $\mathcal{A}$:
\[ A.(B.A + A.A.A)^\omega \]
\[ + (B.A + A.A.A)^\omega \]
```

**$L_{12} \equiv A.(B.A + A.A.A)^*$**

**$L'_{22} \equiv (B.A + A.A.A)^+$**

**NFA $\mathcal{A}_{12}$**
Example: NBA $\leadsto \omega$-regular expression

NBA $A$

language of $A$:

$$A.(B.A + A.A.A)\omega + (B.A + A.A.A)\omega \equiv (A + \varepsilon).(B.A + A.A.A)\omega$$

$L_{12} \equiv A.(B.A + A.A.A)^*$

$L'_{22} \equiv (B.A + A.A.A)^+$

NFA $A_{12}$
From $\omega$-regular expressions to NBA
From $\omega$-regular expressions to NBA

For each $\omega$-regular expression

$$\gamma = \alpha_1.\beta_1^\omega + \ldots + \alpha_n.\beta_n^\omega$$

there exists an NBA $A$ with $\mathcal{L}_\omega(A) = \mathcal{L}_\omega(\gamma)$. 
For each $\omega$-regular expression

$$\gamma = \alpha_1.\beta_1^\omega + \ldots + \alpha_n.\beta_n^\omega$$

there exists an NBA $A$ with $L_\omega(A) = L_\omega(\gamma)$.

**Proof.**
For each $\omega$-regular expression

$$\gamma = \alpha_1 \cdot \beta_1^\omega + \ldots + \alpha_n \cdot \beta_n^\omega$$

there exists an NBA $A$ with $L_\omega(A) = L_\omega(\gamma)$.

Proof. consider NFA $A_i$ for $\alpha_i$ and $B_i$ for $\beta_i$.
For each $\omega$-regular expression

$$\gamma = \alpha_1.\beta_1^\omega + \ldots + \alpha_n.\beta_n^\omega$$

there exists an NBA $A$ with $L_\omega(A) = L_\omega(\gamma)$.

Proof. consider NFA $A_i$ for $\alpha_i$ and $B_i$ for $\beta_i$

- construct NBA $B_i^\omega$ for $\beta_i^\omega$
For each $\omega$-regular expression

$$\gamma = \alpha_1 \cdot \beta^\omega_1 + \ldots + \alpha_n \cdot \beta^\omega_n$$

there exists an NBA $A$ with $\mathcal{L}_\omega(A) = \mathcal{L}_\omega(\gamma)$.

**Proof.** consider NFA $A_i$ for $\alpha_i$ and $B_i$ for $\beta_i$

- construct NBA $B^\omega_i$ for $\beta^\omega_i$
- construct NBA $C_i = A_i B^\omega_i$ for $\alpha_i \cdot \beta^\omega_i$
From $\omega$-regular expressions to NBA

For each $\omega$-regular expression

$$\gamma = \alpha_1.\beta_1^\omega + \ldots + \alpha_n.\beta_n^\omega$$

there exists an NBA $A$ with $L_\omega(A) = L_\omega(\gamma)$.

**Proof.** consider NFA $A_i$ for $\alpha_i$ and $B_i$ for $\beta_i$

- construct NBA $B_i^\omega$ for $\beta_i^\omega$
- construct NBA $C_i = A_iB_i^\omega$ for $\alpha_i.\beta_i^\omega$
- construct NBA for $\bigcup_{1 \leq i \leq n} L_\omega(C_i)$
For each \( \omega \)-regular expression \( \gamma = \alpha_1 \beta_1^\omega + \ldots + \alpha_n \beta_n^\omega \)
there exists an NBA \( A \) with \( \mathcal{L}_\omega(A) = \mathcal{L}_\omega(\gamma) \).

**Proof.** consider NFA \( A_i \) for \( \alpha_i \) and \( B_i \) for \( \beta_i \)

- construct NBA \( B_i^\omega \) for \( \beta_i^\omega \)
- construct NBA \( C_i = A_i B_i^\omega \) for \( \alpha_i \beta_i^\omega \)
- construct NBA for \( \bigcup_{1 \leq i \leq n} \mathcal{L}_\omega(C_i) \)
NBA are closed under union
NBA are closed under union
NBA are closed under union

NBA $A_1$

NBA $A_2$

NBA for $L_\omega(A_1) \cup L_\omega(A_2)$
From $\omega$-regular expressions to NBA

For each $\omega$-regular expression

$$\gamma = \alpha_1.\beta_1^\omega + \ldots + \alpha_n.\beta_n^\omega$$

there exists an NBA $A$ with $L_\omega(A) = L_\omega(\gamma)$.

Proof. consider NFA $A_i$ for $\alpha_i$ and $B_i$ for $\beta_i$

- construct NBA $B_i^\omega$ for $\beta_i^\omega$
- construct NBA $C_i = A_iB_i^\omega$ for $\alpha_i.\beta_i^\omega$
- construct NBA for $\bigcup_{1 \leq i \leq n} L_\omega(C_i)$
Concatenation of an NFA and an NBA
Concatenation of an NFA and an NBA

NFA $\mathcal{A}_1$

NBA $\mathcal{A}_2$
Concatenation of an NFA and an NBA

NFA $\mathcal{A}_1$

NBA $\mathcal{A}_2$

NBA for $\mathcal{L}(\mathcal{A}_1).\mathcal{L}_\omega(\mathcal{A}_2)$:
Concatenation of an NFA and an NBA

NFA \( A_1 \)

NBA \( A_2 \)

NBA for \( \mathcal{L}(A_1) \cdot \mathcal{L}_\omega(A_2) \):

accept states as in \( A_2 \)
Concatenation of an NFA and an NBA

NFA $A_1$

NBA $A_2$

NBA for $\mathcal{L}(A_1).\mathcal{L}_\omega(A_2)$:

accept states as in $A_2$
For each $\omega$-regular expression

$$\gamma = \alpha_1 \beta_1^\omega + \ldots + \alpha_n \beta_n^\omega$$

there exists an NBA $A$ with $L_\omega(A) = L_\omega(\gamma)$.

**Proof.** consider NFA $A_i$ for $\alpha_i$ and $B_i$ for $\beta_i$

- construct NBA $B_i^\omega$ for $\beta_i^\omega$
- construct NBA $C_i = A_i B_i^\omega$ for $\alpha_i \beta_i^\omega$
- construct NBA for $\bigcup_{1 \leq i \leq n} L_\omega(C_i)$
$\omega$-operator for NFA
ω-operator for NFA

NFA $\mathcal{A}$ for language $L \subseteq \Sigma^+$

$\mathcal{A}^\omega$ for language $L^\omega \subseteq \Sigma^\omega$
ω-operator for NFA

NFA $\mathcal{A}$ for language $L \subseteq \Sigma^+$

NBA $\mathcal{A}^\omega$ for language $L^\omega \subseteq \Sigma^\omega$
$\omega$-operator for NFA

NFA $\mathcal{A}$ for language $L \subseteq \Sigma^+$

$\mathcal{A}^\omega$ for language $L^\omega \subseteq \Sigma^\omega$

wrong!
$\omega$-operator for NFA

NFA $A$ for language $L \subseteq \Sigma^+$

$\rightsquigarrow$

NBA $A^\omega$ for language $L^\omega \subseteq \Sigma^\omega$

wrong!
\( \omega \)-operator for NFA

NFA \( \mathcal{A} \) for language \( L \subseteq \Sigma^+ \)

\[ \cdots \]

\[ q_0 \]

\[ \quad \quad \quad \quad \quad \quad \quad \quad A \quad B \]

\[ q_1 \quad q_2 \]

\[ \cdots \]

\[ q \]

NBA \( \mathcal{A}^\omega \) for language \( L^\omega \subseteq \Sigma^\omega \)

\[ \cdots \]

\[ q_0 \]

\[ \quad \quad \quad \quad \quad \quad \quad \quad A \quad B \]

\[ q_1 \quad q_2 \]

\[ \cdots \]

\[ q \]

wrong !

... correct, if \( \delta(q, x) = \emptyset \quad \forall q \in F \quad \forall x \in \Sigma \)
$\omega$-operator for NFA

NFA $A$ for language $L \subseteq \Sigma^+$

$\Rightarrow$

NFA $B$ for $L$ s.t. all final states are terminal
\( \omega \)-operator for NFA

NFA \( A \) for language \( L \subseteq \Sigma^+ \)

\[ \implies \]

NFA \( B \) for \( L \) s.t. all final states are terminal

... add a new final state \( p' \) ...
\*-operator for NFA

NFA \( A \) for language 
\( L \subseteq \Sigma^+ \) \( \iff \) NFA \( B \) for \( L \) s.t. all final states are terminal

\[-\downarrow\] NBA \( B^\omega \)

... add a new final state \( p' \) ...
ω-operator for NFA

NFA $A$ for language $L \subseteq \Sigma^+$

$\Rightarrow$

NFA $B$ for $L$ s.t. all final states are terminal

$\Downarrow$

NBA $B^\omega$

... add a new final state $p'$ ...
\( \omega \)-operator for NFA

NFA \( A \) for language \( L \subseteq \Sigma^+ \) \( \implies \) NFA \( B \) for \( L \) s.t. all final states are terminal

\[
\begin{align*}
q_0 & \quad B \\
q & \quad A \\
p & \quad \ldots \\
q & \quad \ldots \\
p & \quad \ldots
\end{align*}
\]

\[
\begin{align*}
q_0 & \quad B \\
q & \quad A \\
p & \quad \ldots \\
p' & \quad \ldots
\end{align*}
\]

... add a new final state \( p' \) ...

LTLMC3.2-31
ω-operator for NFA

NFA $A$ for language $L \subseteq \Sigma^+$

$\Rightarrow$

NFA $B$ for $L$ s.t. all final states are terminal

$\Downarrow$

NBA $B^\omega$

$L(A)^\omega = L_\omega(B^\omega)$
Example: $\omega$-operator for NFA

NFA $\mathcal{A}$ for $A.B^*$

$q_0$  

$\mathcal{A}$  

$A$  

$B$  

$p$
Example: $\omega$-operator for NFA

NFA $A$ for $A.B^*$

NFA $B$ for $A.B^*$
Example: $\omega$-operator for NFA

NFA $A$ for $A.B^*$

NFA $B$ for $A.B^*$

NBA $B^\omega$
Example: $\omega$-operator for NFA

NFA $A$ for $A.B^*$

$B \xrightarrow{B} p$  
$p \xrightarrow{A} q_0$  

NFA $B$ for $A.B^*$

$B \xrightarrow{B} p$  
$p \xrightarrow{A} A \xrightarrow{A} p'$

NBA $B^\omega$ for $(A.B^*)^\omega$

$q_0 \xrightarrow{A} A \xrightarrow{A} p'$  
$p \xrightarrow{B} p$  
$p \xrightarrow{B} A$  
$p' \xrightarrow{B} A$
Equivalence of $\omega$-regular expressions and NBA

(1) For each NBA $A$ there exists an $\omega$-regular expression $\gamma$ with $\mathcal{L}_\omega(A) = \mathcal{L}_\omega(\gamma)$

(2) For each $\omega$-regular expression $\gamma$ there exists an NBA $A$ with $\mathcal{L}_\omega(A) = \mathcal{L}_\omega(\gamma)$
Equivalence of $\omega$-regular expressions and NBA

(1) For each NBA $A$ there exists an $\omega$-regular expression $\gamma$ with $L_\omega(A) = L_\omega(\gamma)$

(2) For each $\omega$-regular expression $\gamma$ there exists an NBA $A$ with $L_\omega(A) = L_\omega(\gamma)$

Corollary:
If $E$ be an LT property then:

$$E \text{ is } \omega\text{-regular} \iff E = L_\omega(A) \text{ for some NBA } A$$
Equivalence of \( \omega \)-regular expressions and NBA

(1) For each NBA \( \mathcal{A} \) there exists an \( \omega \)-regular expression \( \gamma \) with \( L_\omega(\mathcal{A}) = L_\omega(\gamma) \)

(2) For each \( \omega \)-regular expression \( \gamma \) there exists an NBA \( \mathcal{A} \) with \( L_\omega(\mathcal{A}) = L_\omega(\gamma) \)

Corollary:

If \( E \) be an LT property, i.e., \( E \subseteq (2^{AP})^{\omega} \), then:

\( E \) is \( \omega \)-regular iff \( E = L_\omega(\mathcal{A}) \) for some NBA \( \mathcal{A} \) over the alphabet \( 2^{AP} \)
Closure properties of $\omega$-regular properties
Closure properties of $\omega$-regular properties

Remind: Kleene's theorem for regular languages:

The class of regular languages is closed under

- union, intersection, complementation
- concatenation and Kleene star
Closure properties of $\omega$-regular properties

**remind**: Kleene’s theorem for regular languages:

The class of regular languages is closed under:
- union, intersection, complementation
- concatenation and Kleene star

The class of $\omega$-regular languages is closed under union, intersection and complementation.
The class of \(\omega\)-regular languages is closed under union, intersection and complementation.

- **union:**

- **intersection:**

- **complementation:**
Closure properties of $\omega$-regular properties

The class of $\omega$-regular languages is closed under union, intersection and complementation.

- **union:**
  obvious from definition of $\omega$-regular expressions

- **intersection:**

- **complementation:**
The class of $\omega$-regular languages is closed under union, intersection and complementation.

- **union**: obvious from definition of $\omega$-regular expressions
- **intersection**: will be discussed later relies on a certain product construction for NBA
- **complementation**: 
Closure properties of $\omega$-regular properties

The class of $\omega$-regular languages is closed under union, intersection and complementation.

- **union:**
  obvious from definition of $\omega$-regular expressions

- **intersection:**
  will be discussed later
  relies on a certain product construction for NBA

- **complementation:**
  much more difficult than for NFA,
  via other types of $\omega$-automata
Nonemptiness for NBA
Nonemptiness for NBA

given: \( \text{NBA } A = (Q, \Sigma, \delta, Q_0, F) \)

question: does \( \mathcal{L}_\omega(A) \neq \emptyset \) hold?
Let $A = (Q, \Sigma, \delta, Q_0, F)$ be an NBA. Then:

\[ \mathcal{L}_\omega(A) \neq \emptyset \text{ iff } \exists q_0 \in Q_0 \ \exists p \in F \ \exists x \in \Sigma^* \ \exists y \in \Sigma^+. \]

\[ p \in \delta(q_0, x) \cap \delta(p, y) \]
Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ be an NBA. Then:

$$\mathcal{L}_\omega(\mathcal{A}) \neq \emptyset \iff \exists q_0 \in Q_0 \exists p \in F \exists x \in \Sigma^* \exists y \in \Sigma^+. \quad p \in \delta(q_0, x) \cap \delta(p, y)$$

there exists a reachable accept state $p \in F$ that belongs to a cycle.
Let $A = (Q, \Sigma, \delta, Q_0, F)$ be an NBA. Then:

\[ L_\omega(A) \neq \emptyset \quad \text{iff} \quad \exists q_0 \in Q_0 \ \exists p \in F \ \exists x \in \Sigma^* \ \exists y \in \Sigma^+ \ . p \in \delta(q_0, x) \cap \delta(p, y) \]

iff there exist finite words $x, y \in \Sigma^*$ s.t. $y \neq \varepsilon$ and $xy^\omega \in L_\omega(A)$
Nonemptiness for NBA

Let $A = (Q, \Sigma, \delta, Q_0, F)$ be an NBA. Then:

$\mathcal{L}_\omega(A) \neq \emptyset$ iff $\exists q_0 \in Q_0 \exists p \in F \exists x \in \Sigma^* \exists y \in \Sigma^+ . \ p \in \delta(q_0, x) \cap \delta(p, y)$

iff there exist finite words $x, y \in \Sigma^*$ s.t. $y \neq \varepsilon$ and $xy^\omega \in \mathcal{L}_\omega(A)$

“ultimatively periodic words”
Nonemptiness for NBA

Let $A = (Q, \Sigma, \delta, Q_0, F)$ be an NBA. Then:

$$\mathcal{L}_\omega(A) \neq \emptyset \iff \exists q_0 \in Q_0 \ \exists p \in F \ \exists x \in \Sigma^* \ \exists y \in \Sigma^+. \ p \in \delta(q_0, x) \cap \delta(p, y)$$

iff there exist finite words $x, y \in \Sigma^*$ s.t. $y \neq \varepsilon$ and $xy^\omega \in \mathcal{L}_\omega(A)$

The emptiness problem for NBA is solvable by means of graph algorithms in time $O(poly(A))$
Deterministic Büchi automata (DBA)
A DBA is an NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ such that

- $\mathcal{A}$ has a unique initial state,
- $|\delta(q, A)| \leq 1$ for all $q \in Q$ and $A \in \Sigma$
A DBA is an NBA \( \mathcal{A} = (Q, \Sigma, \delta, Q_0, F) \) such that

- \( \mathcal{A} \) has a unique initial state, i.e., \( Q_0 \) is a singleton
- \( |\delta(q, A)| \leq 1 \) for all \( q \in Q \) and \( A \in \Sigma \)
Deterministic Büchi automata (DBA)

A DBA is an NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ such that

- $\mathcal{A}$ has a unique initial state, i.e., $Q_0$ is a singleton
- $|\delta(q, A)| \leq 1$ for all $q \in Q$ and $A \in \Sigma$

notation: $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ if $Q_0 = \{q_0\}$
A DBA is an NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ such that

- $\mathcal{A}$ has a unique initial state, i.e., $Q_0$ is a singleton
- $|\delta(q, A)| \leq 1$ for all $q \in Q$ and $A \in \Sigma$

**notation:** $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ if $Q_0 = \{q_0\}$

**alphabet** $\Sigma = \{A, B\}$
A Deterministic Büchi automata (DBA) $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ such that

- $\mathcal{A}$ has a unique initial state, i.e., $Q_0$ is a singleton
- $|\delta(q, A)| \leq 1$ for all $q \in Q$ and $A \in \Sigma$

**notation:** $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ if $Q_0 = \{q_0\}$

DBA for “infinitely often $B$”

alphabet $\Sigma = \{A, B\}$
Determinization by powerset construction

well-known:

the powerset construction for the determinization (and complementation) of finite automata (NFA)
Determinization by powerset construction

well-known:

the powerset construction for the determinization (and complementation) of finite automata (NFA)

question:

does the powerset construction also work for Büchi automata (NBA)?
Determinization by powerset construction

NBA for “eventually forever $a$”

$q_0$  $a$  $q_F$

true  $a$
Determinization by powerset construction

NBA for “eventually forever $a$”

$q_0 \xrightarrow{a} q_F \xrightarrow{\neg a} q_1$

$q_0$ true

$q_F$ $a$

$q_1$ true
Determinization by powerset construction

NBA for “eventually forever $a$”

```
q_0 \xrightarrow{a} q_F \xrightarrow{\neg a} q_1
```

true

powerset construction

```
q_0 \xrightarrow{a} q_0 \ 
eg a
q_0 \neg a
```

```
q_F \xrightarrow{a} q_0 \ 
eg a
q_F \neg a
```

```
q_1 \xrightarrow{a} q_0 \ 
eg a
q_1 \neg a
```

```
q_F \xrightarrow{a} q_0 \ 
eg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```

```
q_0 \neg a
q_1 \neg a
q_F \neg a
```
Determinization by powerset construction

**NBA** for “eventually forever $a$”

![NBA Diagram]

powerset construction

![Powerset Construction Diagram]

e.g., $\delta(q_0, a) = \{ q_0, q_F \}$ and $\delta(q_0, \neg a) = \{ q_0 \}$
Determinization by powerset construction

**NBA** for “eventually forever \( a \)"

\[
\begin{array}{c}
q_0 \\
\neg a & \rightarrow & a & \rightarrow & q_F & \neg a & \rightarrow & q_1
\end{array}
\]

\[
\begin{array}{c}
q_F \\
a & \rightarrow & a
\end{array}
\]

powerset construction

\[
\begin{array}{c}
q_0 \\
\neg a & \rightarrow & a & \rightarrow & q_0 \cup q_F & \neg a & \rightarrow & q_0 \cup q_1
\end{array}
\]

\[
\begin{array}{c}
q_1 \\
\neg a & \rightarrow & a & \rightarrow & a
\end{array}
\]

\[
\begin{array}{c}
q_F \\
a & \rightarrow & a
\end{array}
\]

e.g., \( \delta(q_0, a) = \{q_0, q_F\} \) and \( \delta(q_0, \neg a) = \{q_0\} \)
Determinization by powerset construction

**NBA** for “eventually forever \( a \)"

![Diagram showing NBA for “eventually forever a”]

powerset construction

**DBA** for “infinitely often \( a \)”

![Diagram showing DBA for “infinitely often a”]
Powerset construction ← fails for NBA

**NBA** for “eventually forever a”

```
\[ q_0 \xrightarrow{a} q_0, q_F \xrightarrow{a} q_1 \xrightarrow{a} \neg a \]
```

powerset construction

```
\[ q_0 \xrightarrow{a} q_0, q_F, q_1 \rightarrow q_0, q_1, q_F, q_0, q_1, q_F \]
```

**DBA** for “infinitely often a”
Complementation of DBA
Complementation of DBA

well-known:

**DFA** can be complemented by complementation of the acceptance set

question:

does this also work for **DBA**?
Complementation of DBA

DBA for “infinitely often ¬a”
Complementation of DBA

DBA for "infinitely often \( \neg a \)"

complement automaton
Complementation of DBA

Complement automaton

DBA for “infinitely often \( \neg a \)”

DBA for “infinitely often \( a \)”
Complementation fails for DBA

DBA for “infinitely often \( \neg a \)”

complement automaton

DBA for “infinitely often \( a \)”
Complementation fails for DBA

There is no DBA for the LT-property “eventually forever $a$”

DBA for “infinitely often $\lnot a$”

DBA for “infinitely often $a$”
There is no DBA $A$ over the alphabet $\Sigma = \{A, B\}$ such that $L_\omega(A) = L_\omega((A + B)^* \cdot A^\omega)$.
There is no DBA $\mathcal{A}$ over the alphabet $\Sigma = \{A, B\}$ such that $L_\omega(A) = L_\omega((A + B)^* . A^\omega)$.

Hence: there is no DBA for the LT-property “eventually forever $a$”
There is no DBA $A$ over the alphabet $\Sigma = \{A, B\}$ such that $L_\omega(A) = L_\omega((A + B)^* . A^\omega)$.

**Hence:** there is no DBA for the LT-property “eventually forever $a$”

**Proof:** apply the above theorem for $A = \{a\}$, $B = \emptyset$
There is no DBA $A$ over the alphabet $\Sigma = \{A, B\}$ such that $L_\omega(A) = L_\omega((A + B)^* A^\omega)$.

Hence: there is no DBA for the LT-property “eventually forever $a$”

Proof: apply the above theorem for $A = \{a\}$, $B = \emptyset$

The class of DBA-recognizable languages is a proper subclass of the class of $\omega$-regular languages.
There is no DBA $\mathcal{A}$ over the alphabet $\Sigma = \{A, B\}$ such that $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega((A + B)^* \cdot A^{\omega})$.

Hence: there is no DBA for the LT-property “eventually forever $a$”

Proof: apply the above theorem for $A = \{a\}, B = \emptyset$

The class of DBA-recognizable languages is a proper subclass of the class of $\omega$-regular languages and is not closed under complementation.
There is no DBA $\mathcal{A}$ over the alphabet $\Sigma = \{A, B\}$ such that $L_\omega(A) = L_\omega((A + B)^*.A^\omega)$.

The class of DBA-recognizable languages is a proper subclass of the class of $\omega$-regular languages and is not closed under complementation.

$(A^*.B)^\omega$ “infinitely many $B$’s” DBA-recognizable

$(A + B)^*.A^\omega$ “only finitely many $B$’s” not DBA-recognizable
Generalized NBA (GNBA)
A generalized nondeterministic Büchi automaton is a tuple

\[ \mathcal{G} = (Q, \Sigma, \delta, Q_0, \mathcal{F}) \]

where \( Q, \Sigma, \delta, Q_0 \) are as in NBA, but \( \mathcal{F} \) is a set of accept sets, i.e., \( \mathcal{F} \subseteq 2^Q \).
A generalized nondeterministic Büchi automaton is a tuple

\[ G = (Q, \Sigma, \delta, Q_0, \mathcal{F}) \]

where \( Q, \Sigma, \delta, Q_0 \) are as in NBA, but \( \mathcal{F} \) is a set of accept sets, i.e., \( \mathcal{F} \subseteq 2^Q \).

A run \( q_0 q_1 q_2 \ldots \) for some infinite word \( \sigma \in \Sigma^\omega \) is called accepting if each accept set is visited infinitely often.
Generalized NBA (GNBA)

A generalized nondeterministic Büchi automaton is a tuple

\[ G = (Q, \Sigma, \delta, Q_0, \mathcal{F}) \]

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A run \( q_0 q_1 q_2 \ldots \) for some infinite word \( \sigma \in \Sigma^\omega \) is called accepting if each accept set is visited infinitely often, i.e.,

\[ \forall F \in \mathcal{F} \ \exists \ i \in \mathbb{N} \ \text{s.t.} \ q_i \in F \]
Accepted language of a GNBA

GNBA $\mathcal{G} = (Q, \Sigma, \delta, Q_0, \mathcal{F})$ as NBA, but $\mathcal{F} \subseteq 2^Q$

A run $q_0 q_1 q_2 \ldots$ for some infinite word $\sigma \in \Sigma^\omega$ is accepting if

$$\forall F \in \mathcal{F} \quad \exists \ i \in \mathbb{N} \ \text{s.t.} \ q_i \in F$$

accepted language:

$$\mathcal{L}_\omega(\mathcal{G}) \overset{\text{def}}{=} \{ \sigma \in \Sigma^\omega : \sigma \text{ has an accepting run in } \mathcal{G} \}$$
Example: GNBA for liveness property

GNBA $\mathcal{G}$ over $\Sigma = 2^{AP}$ where $AP = \{\text{crit}_1, \text{crit}_2\}$

$F = \{\{q_1\}, \{q_2\}\}$
Example: GNBA for liveness property

GNBA $\mathcal{G}$ over $\Sigma = 2^{AP}$ where $AP = \{\text{crit}_1, \text{crit}_2\}$

$F = \{\{q_1\}, \{q_2\}\}$

specifies the LT-property

“infinitely often $\text{crit}_1$ and infinitely often $\text{crit}_2$”
Example: GNBA for liveness property

GNBA $G$ over $\Sigma = 2^{AP}$ where $AP = \{\text{crit}_1, \text{crit}_2\}$

$F = \{\{q_1\}, \{q_2\}\}$

note: $q_0 \xrightarrow{A} q_1$ implies $A \models \text{crit}_1$

$q_0 \xrightarrow{A} q_2$ implies $A \models \text{crit}_2$
Example: GNBA for liveness property

GNBA $G$ over $\Sigma = 2^{AP}$ where $AP = \{\text{crit}_1, \text{crit}_2\}$

\[ \mathcal{F} = \{\{q_1\}, \{q_2\}\} \]

Note: $q_0 \xrightarrow{A} q_1$ implies $A \models \text{crit}_1$

$q_0 \xrightarrow{A} q_2$ implies $A \models \text{crit}_2$

Hence: if $A_0 A_1 A_2 \ldots \in L_\omega(G)$ then

\[ \exists i \geq 0. \text{crit}_1 \in A_i \land \exists i \geq 0. \text{crit}_2 \in A_i \]
Example: GNBA for liveness property

GNBA $\mathcal{G}$ over $\Sigma = 2^{AP}$ where $AP = \{\text{crit}_1, \text{crit}_2\}$

$\mathcal{G}$

$\exists i \geq 0. \text{crit}_1 \in A_i$ and $\exists i \geq 0. \text{crit}_2 \in A_i$ have an accepting run of the form:

$q_0 \cdots q_0 \ q_1 \ q_0 \cdots q_0 \ q_2 \ q_0 \cdots q_0 \ q_1 \ q_0 \cdots q_0 \ q_2 \cdots$
Examples: GNBA over $\Sigma = \{A, B\}$

$\mathcal{G}$

$F = \{\{q_1\}, \{q_2\}\}$
Examples: GNBA over $\Sigma = \{A, B\}$

$\mathcal{F} = \{\{q_1\}, \{q_2\}\}$

$L_\omega(G) = \ ?$
Examples: GNBA over $\Sigma = \{A, B\}$

$F = \{\{q_1\}, \{q_2\}\}$

$L_\omega(G) = \emptyset$
Examples: GNBA over $\Sigma = \{A, B\}$

GNBA $G$

$\mathcal{F} = \{\{q_1\}, \{q_2\}\}$

$L_\omega(G) = \emptyset$

GNBA $G'$ with $\mathcal{F}' = \{\{q_1, q_3\}, \{q_2, q_4\}\}$
Examples: GNBA over $\Sigma = \{A, B\}$

**GNBA $G$**

- $q_0$ to $q_1$: $A$
- $q_1$ to $q_2$: $B$
- $q_2$ to $q_1$: $A$

$\mathcal{F} = \{\{q_1\}, \{q_2\}\}$

$\mathcal{L}_\omega(G) = \emptyset$

**GNBA $G'$ with $\mathcal{F}' = \{\{q_1, q_3\}, \{q_2, q_4\}\}$**

- $q_0$ to $q_1$: $A$
- $q_1$ to $q_2$: $B$
- $q_2$ to $q_1$: $B$
- $q_2$ to $q_3$: $A$
- $q_3$ to $q_4$: $A$
- $q_4$ to $q_3$: $B$

accepted language: ?
Examples: GNBA over $\Sigma = \{A, B\}$

**GNBA $G$**

$$F = \{\{q_1\}, \{q_2\}\}$$

$$\mathcal{L}_\omega(G) = \emptyset$$

**GNBA $G'$ with $F' = \{\{q_1, q_3\}, \{q_2, q_4\}\}$$

accepted language: $A.B^\omega + A.B^+.A.(A.B)^\omega$
Empty acceptance condition
Empty acceptance condition

NBA $\mathcal{A}$ over $\Sigma = \{A, B\}$:

![Diagram of NBA $\mathcal{A}$]

acceptance set $F = \emptyset$

GNBA $\mathcal{G}$ over $\Sigma = \{A, B\}$:

![Diagram of GNBA $\mathcal{G}$]

set of acceptance sets $\mathcal{F} = \emptyset$
Empty acceptance condition

NBA $\mathcal{A}$ over $\Sigma = \{A, B\}$:

acceptance set $F = \emptyset$

$L_\omega(A) = \emptyset$

---

GNBA $\mathcal{G}$ over $\Sigma = \{A, B\}$:

set of acceptance sets $\mathcal{F} = \emptyset$
Empty acceptance condition

NBA $\mathcal{A}$ over $\Sigma = \{A, B\}$:

- Initial state: $q_0$
- Final state: $q_1$
- Acceptance set $F = \emptyset$
- $\mathcal{L}_\omega(\mathcal{A}) = \emptyset$

GNBA $\mathcal{G}$ over $\Sigma = \{A, B\}$:

- Initial state: $q_0$
- Final state: $q_1$
- Set of acceptance sets $\mathcal{F} = \emptyset$
- $\mathcal{L}_\omega(\mathcal{G}) = ?$
Empty acceptance condition

NBA $\mathcal{A}$ over $\Sigma = \{A, B\}$:

- Acceptance set $F = \emptyset$
- $L_\omega(\mathcal{A}) = \emptyset$

GNBA $\mathcal{G}$ over $\Sigma = \{A, B\}$:

- Set of acceptance sets $\mathcal{F} = \emptyset$
- $L_\omega(\mathcal{G}) = \{A^\omega\}$
Empty acceptance condition

NBA $\mathcal{A}$ over $\Sigma = \{A, B\}$:

- $q_0 \xrightarrow{A} q_1$
- $q_0 \xleftarrow{A} q_1$

acceptance set $F = \emptyset$

$\mathcal{L}_\omega(\mathcal{A}) = \emptyset$

---

GNBA $\mathcal{G}$ over $\Sigma = \{A, B\}$:

- $q_0 \xrightarrow{A} q_1$
- $q_0 \xleftarrow{A} q_1$

set of acceptance sets $\mathcal{F} = \emptyset$

$\mathcal{L}_\omega(\mathcal{G}) = \{A^\omega\}$

$\mathcal{L}_\omega(\mathcal{G}) = \{\text{set of all infinite words that have an infinite run}\}$
For every GNBA $G$ there exists a GNBA $G'$ such that

- $\mathcal{L}_\omega(G) = \mathcal{L}_\omega(G')$
- the set of acceptance sets of $G'$ is nonempty
Correct or wrong?

For every GNBA $G$ there exists a GNBA $G'$ such that:

- $L_\omega(G) = L_\omega(G')$
- the set of acceptance sets of $G'$ is nonempty

**Correct**

$$
\text{GNBA } G = (Q, \Sigma, \delta, Q_0, \emptyset) \\
\Downarrow \\
\text{GNBA } G' = (Q, \Sigma, \delta, Q_0, \{Q\})
$$
From GNBA to NBA
For each GNBA $G$ there exists an NBA $A$ with
\[ \mathcal{L}_\omega(G) = \mathcal{L}_\omega(A) \]
For each **GNBA** \( G \) there exists an **NBA** \( A \) with

\[
L_\omega(G) = L_\omega(A)
\]

**Proof.** Let \( G = (Q, \Sigma, \delta, Q_0, \mathcal{F}) \) with \( \mathcal{F} = \{F_1, \ldots, F_k\} \).
From GNBA to NBA

For each GNBA $G$ there exists an NBA $A$ with

\[ \mathcal{L}_\omega(G) = \mathcal{L}_\omega(A) \]

Proof. Let $G = (Q, \Sigma, \delta, Q_0, \mathcal{F})$ with $\mathcal{F} = \{F_1, \ldots, F_k\}$ and $k \geq 1$
For each **GNBA** $G$ there exists an **NBA** $A$ with

$$L_\omega(G) = L_\omega(A)$$

**Proof.** Let $G = (Q, \Sigma, \delta, Q_0, F)$ with $F = \{F_1, \ldots, F_k\}$ and $k \geq 1$

*note: if $k = 1$ then $G$ is an NBA*
For each **GNBA** $G$ there exists an **NBA** $A$ with

$$\mathcal{L}_\omega(G) = \mathcal{L}_\omega(A)$$

**Proof.** Let $G = (Q, \Sigma, \delta, Q_0, F)$ with $F = \{F_1, \ldots, F_k\}$ and $k \geq 2$

**note:** if $k = 1$ then $G$ is an NBA
For each GNBA $\mathcal{G}$ there exists an NBA $\mathcal{A}$ with

$$\mathcal{L}_\omega(\mathcal{G}) = \mathcal{L}_\omega(\mathcal{A})$$

Proof. Let $\mathcal{G} = (Q, \Sigma, \delta, Q_0, \mathcal{F})$ with $\mathcal{F} = \{F_1, \ldots, F_k\}$ and $k \geq 2$. NBA $\mathcal{A}$ results from $k$ copies of $\mathcal{G}$:
For each GNBA $\mathcal{G}$ there exists an NBA $\mathcal{A}$ with \[ \mathcal{L}_\omega(\mathcal{G}) = \mathcal{L}_\omega(\mathcal{A}) \]

**Proof.** Let $\mathcal{G} = (Q, \Sigma, \delta, Q_0, \mathcal{F})$ with $\mathcal{F} = \{F_1, \ldots, F_k\}$ and $k \geq 2$. NBA $\mathcal{A}$ results from $k$ copies of $\mathcal{G}$:
For each GNBA $G$ there exists an NBA $A$ with

$$\mathcal{L}_\omega(G) = \mathcal{L}_\omega(A)$$

**Proof.** Let $G = (Q, \Sigma, \delta, Q_0, \mathcal{F})$ with $\mathcal{F} = \{F_1, \ldots, F_k\}$ and $k \geq 2$. NBA $A$ results from $k$ copies of $G$:

Size of the NBA: $\text{size}(A) = \mathcal{O}(\text{size}(G) \cdot |\mathcal{F}|)$
Example: from GNBA to NBA

GNBA $\mathcal{G}$

alphabet $\Sigma = 2^{AP}$ where $AP = \{\text{crit}_1, \text{crit}_2\}$

infinitely often $\text{crit}_1$ and infinitely often $\text{crit}_2$
Example: from GNBA to NBA

GNBA $G$

$q_0 \rightarrow q_1$ with $q_0 \in q_1$
$q_0 \rightarrow q_2$ with $q_0 \in q_2$
$q_1 \rightarrow q_0$ with $q_1 \in q_0$
$q_2 \rightarrow q_0$ with $q_2 \in q_0$
$q_1 \rightarrow q_1$ with $q_2 \rightarrow q_2$
$q_2 \rightarrow q_2$

alphabet $\Sigma = 2^{AP}$ where $AP = \{\text{crit}_1, \text{crit}_2\}$

infinitely often $\text{crit}_1$ and infinitely often $\text{crit}_2$

NBA $A$

$q_0 \rightarrow q_1$ with $q_0 \in q_1$
$q_0 \rightarrow q_2$ with $q_0 \in q_2$
$q_1 \rightarrow q_0$ with $q_1 \in q_0$
$q_2 \rightarrow q_0$ with $q_2 \in q_0$
$q_1 \rightarrow q_1$ with $q_2 \rightarrow q_2$
$q_2 \rightarrow q_2$

$q_1 \rightarrow q_1$ with $q_2 \rightarrow q_2$
$q_2 \rightarrow q_2$
The class of \( \omega \)-regular languages is closed under union, intersection and complementation.
The class of $\omega$-regular languages is closed under union, intersection and complementation.

- **union**: obvious from definition of $\omega$-regular expressions
- **intersection**: via some product construction
- **complementation**: via other types of $\omega$-automata (not discussed here)
The class of $\omega$-regular languages is closed under union, intersection and complementation.

- **union:**
  obvious from definition of $\omega$-regular expressions

- **intersection:**
  using GNBA via some product construction

- **complementation:**
  via other types of $\omega$-automata (not discussed here)
Intersection for NBA

\[ A_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \]
\[ A_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \]

\[ \{ \text{two NBA} \} \]

**Goal:** Define an NBA \( A \) s.t. \( L_\omega(A) = L_\omega(A_1) \cap L_\omega(A_2) \)
Intersection for NBA

\[
\begin{align*}
\mathcal{A}_1 &= (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \\
\mathcal{A}_2 &= (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2)
\end{align*}
\]

two NBA

goal: define an NBA \( \mathcal{A} \) s.t. \( \mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2) \)

recall:

intersection for finite automata NBA \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) is realized by a product construction that

- runs \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) in parallel (synchronously)
- checks whether both end in a final state
Intersection for NBA

\[ A_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \]
\[ A_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \]

\( \{ \) \text{two NBA} \( \}

\text{goal: define an NBA } A \text{ s.t. } L_\omega(A) = L_\omega(A_1) \cap L_\omega(A_2)

\text{idea: define } A_1 \otimes A_2 \text{ as for NFA, i.e.,}

- \( A_1 \) and \( A_2 \) run in parallel (synchronously)
- and check whether both are accepting
Intersection for NBA

\[ A_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \]
\[ A_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \]

\{\text{two NBA}\}

**goal:** define an NBA \( A \) s.t. \( L_\omega(A) = L_\omega(A_1) \cap L_\omega(A_2) \)

**idea:** define \( A_1 \otimes A_2 \) as for NFA, i.e.,
- \( A_1 \) and \( A_2 \) run in parallel (synchronously)
- and check whether both are accepting

i.e., both \( F_1 \) and \( F_2 \) are visited infinitely often
Intersection for NBA

\[ \mathcal{A}_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \quad \mathcal{A}_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \]  
\{ \text{two NBA} \}

goal: define an NBA \( \mathcal{A} \) s.t. \( \mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2) \)

idea: define \( \mathcal{A}_1 \otimes \mathcal{A}_2 \) as for NFA, i.e.,

- \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) run in parallel (synchronously)
- and check whether both are accepting

\[ \Rightarrow \text{i.e., both } F_1 \text{ and } F_2 \text{ are visited infinitely often} \]

\( \Rightarrow \) product of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) yields a GNBA
Intersection for NBA

\[ \mathcal{A}_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \quad \{ \text{two NBA} \}
\]
\[ \mathcal{A}_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \]

goal: define an NBA \( \mathcal{A} \) s.t. \( \mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2) \)

GNBA \( \mathcal{G} = \mathcal{A}_1 \otimes \mathcal{A}_2 \)
Intersection for NBA

\[ \mathcal{A}_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \]
\[ \mathcal{A}_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \]

\{ \text{two NBA} \}

goal: define an NBA \( \mathcal{A} \) s.t. \( \mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2) \)

GNBA \( \mathcal{G} = \mathcal{A}_1 \otimes \mathcal{A}_2 \)

- state space \( Q = Q_1 \times Q_2 \)
Intersection for NBA

\[ \mathcal{A}_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \]
\[ \mathcal{A}_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \]

\{ two NBA \}

goal: define an NBA \( \mathcal{A} \) s.t. \( \mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2) \)

GNBA \( \mathcal{G} = \mathcal{A}_1 \otimes \mathcal{A}_2 \)

- state space \( Q = Q_1 \times Q_2 \)
- alphabet \( \Sigma \)
Intersection for NBA

\[ A_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \]
\[ A_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \]

\{ \text{two NBA} \}

goal: define an NBA \( \mathcal{A} \) s.t. \( L_\omega(\mathcal{A}) = L_\omega(\mathcal{A}_1) \cap L_\omega(\mathcal{A}_2) \)

GNBA \( \mathcal{G} = \mathcal{A}_1 \otimes \mathcal{A}_2 \)

- state space \( Q = Q_1 \times Q_2 \)
- alphabet \( \Sigma \)
- set of initial states: \( Q_0 = Q_{0,1} \times Q_{0,2} \)
Intersection for NBA

\[ A_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \]
\[ A_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \]
\[
\text{two NBA}
\]

goal: define an NBA \( A \) s.t. \( \mathcal{L}_w(A) = \mathcal{L}_w(A_1) \cap \mathcal{L}_w(A_2) \)

GNBA \( G = A_1 \otimes A_2 \)

- state space \( Q = Q_1 \times Q_2 \)
- alphabet \( \Sigma \)
- set of initial states: \( Q_0 = Q_{0,1} \times Q_{0,2} \)
- acceptance condition: \( F = \{ F_1 \times Q_2, Q_1 \times F_2 \} \)
Intersection for NBA

\[ \mathcal{A}_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \]
\[ \mathcal{A}_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \]

\{ two NBA \}

Goal: define an NBA \( \mathcal{A} \) s.t. \( \mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2) \)

**GNBA** \( \mathcal{G} = \mathcal{A}_1 \otimes \mathcal{A}_2 \)

- State space \( Q = Q_1 \times Q_2 \)
- Alphabet \( \Sigma \)
- Set of initial states: \( Q_0 = Q_{0,1} \times Q_{0,2} \)
- Acceptance condition: \( \mathcal{F} = \{ F_1 \times Q_2, Q_1 \times F_2 \} \)
- Transition relation:

\[ \delta(\langle q_1, q_2 \rangle, \mathcal{A}) = \{ \langle p_1, p_2 \rangle : p_1 \in \delta_1(q_1, \mathcal{A}), p_2 \in \delta_2(q_2, \mathcal{A}) \} \]
Intersection for NBA

\[ \mathcal{A}_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \]
\[ \mathcal{A}_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \]

\{ \text{two NBA} \}

goal: define an NBA \( \mathcal{A} \) s.t. \( L_\omega(\mathcal{A}) = L_\omega(\mathcal{A}_1) \cap L_\omega(\mathcal{A}_2) \)

GNBA \( \mathcal{G} = \mathcal{A}_1 \otimes \mathcal{A}_2 \) \( \xrightarrow{\sim} \) equivalent NBA \( \mathcal{A} \)

- state space \( Q = Q_1 \times Q_2 \)
- alphabet \( \Sigma \)
- set of initial states: \( Q_0 = Q_{0,1} \times Q_{0,2} \)
- acceptance condition: \( \mathcal{F} = \{F_1 \times Q_2, Q_1 \times F_2\} \)
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Summary: $\omega$-regular languages
The class of $\omega$-regular languages agrees with

- the class of languages given by $\omega$-regular expressions
- the class of NBA-recognizable languages
- the class of GNBA-recognizable languages
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The class of $\omega$-regular languages is closed under union, intersection and complementation.