Exercise 1

Let $\varphi$, $\psi$, $\pi$ be arbitrary LTL formulae. For each of the following pairs of LTL formulae, determine in which relation they are. More specifically, determine whether they are equivalent, one of them subsumes the other or they are incomparable. Prove your claims.

a) $\diamond \square \varphi$ and $\square \diamond \varphi$

b) $\diamond \square \varphi \land \diamond \square \psi$ and $\diamond (\square \varphi \land \square \psi)$

c) $\varphi \land \square (\varphi \rightarrow \diamond \varphi)$ and $\square \diamond \varphi$

d) $(\varphi \psi)U \pi$ and $\varphi U (\psi U \pi)$
Exercise 2

Consider the following transition system TS:

Determine whether TS $|= \varphi_i$ for each of the following properties. Justify your answers.

a) $\varphi_1 = \Box \Diamond a$

b) $\varphi_2 = \Diamond \Box a$

c) $\varphi_3 = a \rightarrow \Box \Box a$

d) $\varphi_4 = bR\varphi$ where $\varphi R\psi \overset{def}{=} \neg(\neg\varphi U \neg\psi)$

\[\text{Diagram of TS:}\]

- $S_0 \rightarrow S_1$: \{a, b\} \rightarrow \{a\}
- $S_1 \rightarrow S_2$: \{a\} \rightarrow \emptyset
- $S_2 \rightarrow S_3$: \emptyset \rightarrow \{a, b\}
- $S_3 \rightarrow S_4$: \{a, b\} \rightarrow \{b\}
- $S_4 \rightarrow S_5$: \{b\} \rightarrow \{a\}
- $S_5 \rightarrow S_0$: \{a\} \rightarrow \{a, b\}$
Exercise 3

Provide a sequence \((\varphi_n)\) of LTL formulae such that the LTL formulae \(\psi_n\) is in PNF (including weak-until), \(\varphi_n \equiv \psi_n\), and \(\psi_n\) is exponentially longer than \(\varphi_n\). Use the transformation rules from the lecture.
Exercise 4

Consider the transition system TS above with the set \( AP = \{ a, b, c \} \) of atomic propositions. Note that this is a single transition system with two initial states. Consider the LTL fairness assumption

\[
\text{fair} = (\Box \Diamond (a \land b) \rightarrow \Box \Diamond \neg c) \land (\Diamond \Box (a \land b) \rightarrow \Box \Diamond \neg b).
\]

a) Determine the fair paths in TS, i.e., the initial, infinite paths satisfying \( \text{fair} \).
b) For each of the following LTL formulae:

\[
\begin{align*}
\varphi_1 &= b U \neg b \\
\varphi_2 &= b W \neg b \\
\varphi_3 &= (\Box \Box b) U (\Box \neg b)
\end{align*}
\]

determine whether \( TS \models_{\text{fair}} \varphi_i \). In case \( TS \not\models_{\text{fair}} \varphi_i \), indicate a path \( \pi \in \text{Paths}(TS) \) for which \( \pi \not\models \varphi_i \).
c) Redo the previous task but ignore the fairness assumption.
Exercise 5

Let $\varphi = (a \land igcirc a) U (a \land \neg igcirc a)$ be an LTL-formula over $AP = \{a\}$.

1. Compute all elementary sets with respect to $\varphi$.
2. Construct the GNBA $G_\varphi$ according to the algorithm from the lecture such that $L_\omega(G_\varphi) = Words(\varphi)$.
3. Give an $\omega$-regular expression $E$ such that $L_\omega(G_\varphi) = L_\omega(E)$. 

Exercise 1 (LTL Operators): (2 points)

Let $\varphi_1$ and $\varphi_2$ be LTL formulae. Consider the following new operators:

a) "At next" $\varphi_1 A_{\varphi_1}$ for all $i \geq 0$ where $\varphi_1 A_{\varphi_1} = \varphi_1(i+1)$, for which there exists no $0 < j < i$ where $\varphi_1(j) \varphi_1(j+1)$...

b) "While" $\varphi_1 W_{\varphi_1}$ for all $i \geq 0$ where $\varphi_1(j) \varphi_1(j+1)$ for all $0 < j < i$,

A $A_{\varphi_1}(i+1)$...

c) "Before" $\varphi_1 B_{\varphi_1}$ for all $i \geq 0$ where $\varphi_1(i)$ $\varphi_1(i+1)$...

there exists some $0 < j < i$ where $\varphi_1(j) \varphi_1(j+1)$...

Show that these operators are LTL-definable by providing equivalent LTL formulae. You may use both the until and weak until operator.

Exercise 2 (LTL to Büchi): (3 points)

Let $\varphi = (a \land \bigcirc a) U (a \land \neg \bigcirc a)$ be an LTL–formula over $AP = \{a\}$.

1. Compute all elementary sets with respect to $\varphi$.
2. Construct the GNBA $G_\varphi$ according to the algorithm from the lecture such that $L_\omega(G_\varphi) = Words(\varphi)$.
3. Give an $\omega$-regular expression $E$ such that $L_\omega(G_\varphi) = L_\omega(E)$.

Exercise 3 (CTL Equivalences): (3 points)

Prove or disprove the following implications:

(a) Let $\varphi_1 = 8 a \land \bigcirc b$ and $\varphi_2 = 8 a \land \bigcirc b$.

(b) Now consider $\varphi_1 = 9 (a U 9 (b U c))$ and $\varphi_2 = 9 (9 (a U b) U c)$.

Exercise 4 (CTL Normal Forms): (2 points)

Transform the CTL-formula $\varphi = \neg 8 \varphi \land \bigcirc (8 \\ b) U (8 a)$ into an equivalent CTL-formula in

(a) existential normal form and

(b) positive normal form.

Exercise 5

Let $\varphi = (a \land igcirc a) U (a \land \neg igcirc a)$ be an LTL-formula over $AP = \{a\}$.

1. Compute all elementary sets with respect to $\varphi$.
2. Construct the GNBA $G_\varphi$ according to the algorithm from the lecture such that $L_\omega(G_\varphi) = Words(\varphi)$.
3. Give an $\omega$-regular expression $E$ such that $L_\omega(G_\varphi) = L_\omega(E)$.