Inexact Oracles in NonDifferentiable Optimization: Deflected Conditional Subgradient Methods and Generalized Bundle Methods

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48^{th} Workshop on Nonsmooth Analysis Optimization and Applications
Dedicated to V.F. Demyanov
Erice, May 14, 2008

Outline

1. Introduction, Motivation

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2. Subgradient methods: introduction
3. Polyak-type stepsize: the abstract case

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4. Polyak-type stepsize: the implementable case
5. Deflection-restricted rules

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6. Bundle methods

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Lagrangian Relaxation

- Difficult structured problem

$$z(P) = \sup_u \{ c(u) : h(u) \leq 0 , u \in U \} \quad (1)$$

with complicating constraints $h(u) \leq 0$ over easy set U

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- Assume Lagrangian relaxation of complicating constraints easy

$$f(x) = \sup_u \{ c(u) + xh(u) : u \in U \} \tag{2}$$

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- Equivalent to primal relaxation

$$\sup \{\ v\ :\ (u, v, 0) \in \mathcal{U}^{**}\ \} \tag{3}$$

where $\mathcal{U} = \{\ (u, v, r)\ :\ u \in U\ ,\ v \leq c(u)\ ,\ r \geq h(u)\ \}$

(a more palatable object if problem "affine enough")[1]

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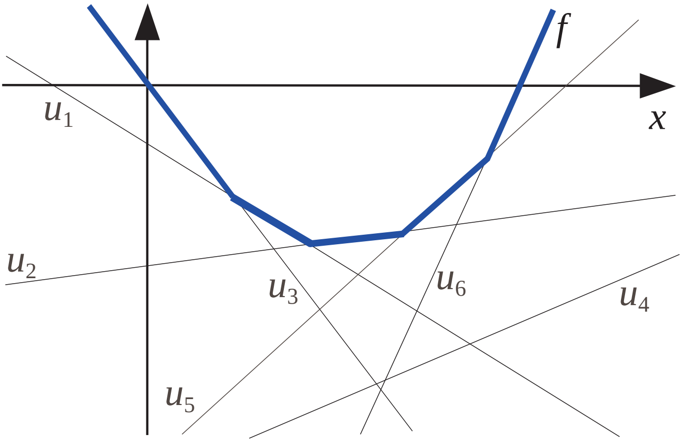
Lagrangian Relaxation (graphically)



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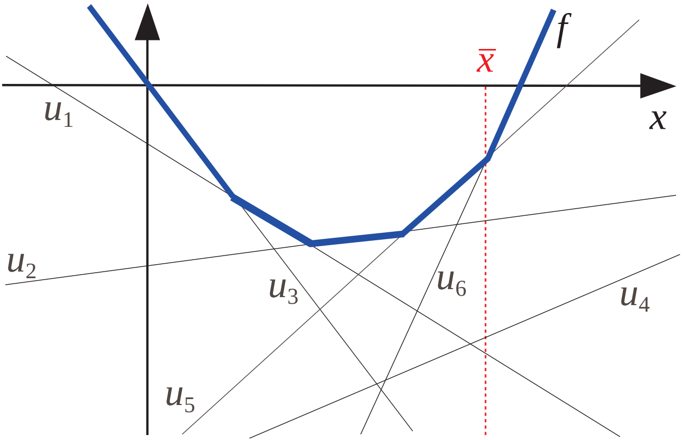

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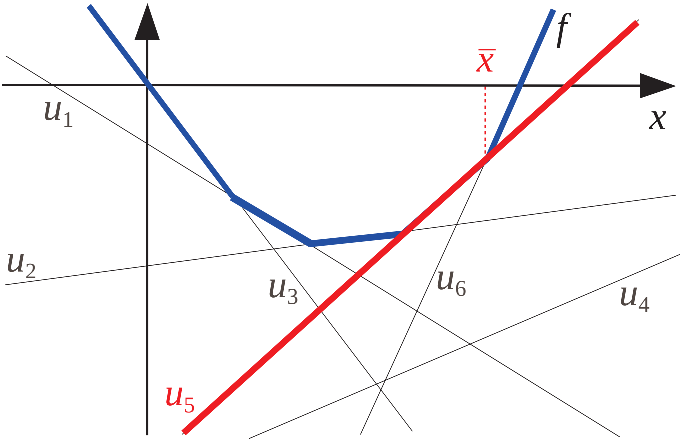

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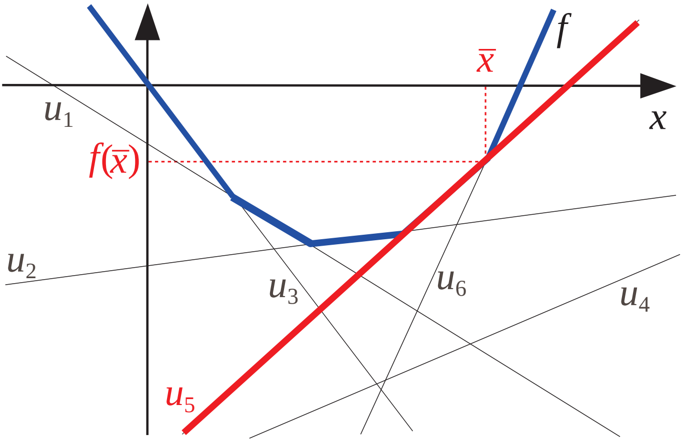

- Oracle to (efficiently) perform the maximization (structure inside)
- Solving exactly (2) provides both function value and subgradient

Lagrangian Relaxation: What For?

1. Primal "continuous" solutions useful to drive heuristics for (1)[2]

[2] F., Gentile, Lacalandra "Solving Unit Commitment Problems with General Ramp Contraints", IJEPES, 2008

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1. Primal "continuous" solutions useful to drive heuristics for (1)[2]

2. Mainly upper bounding: $z(\Pi) \geq z(P)$, "near" if (2) "not too easy"
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- Which may mean different things

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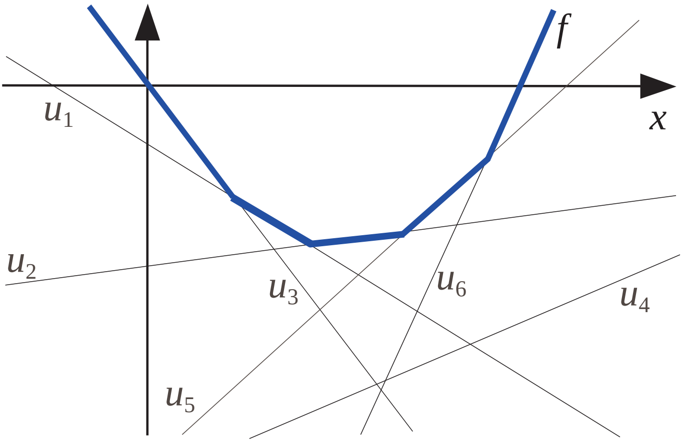

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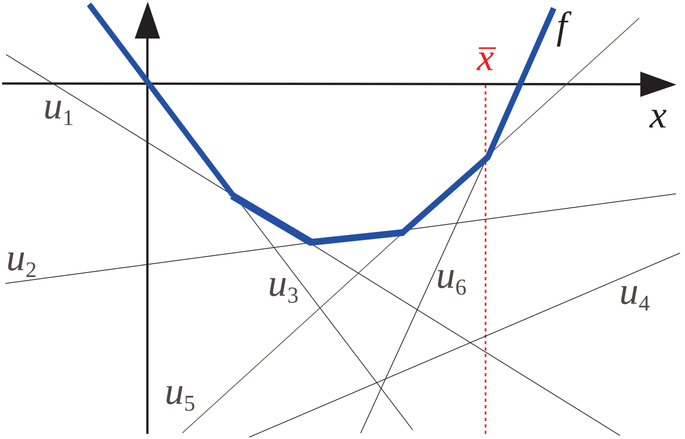

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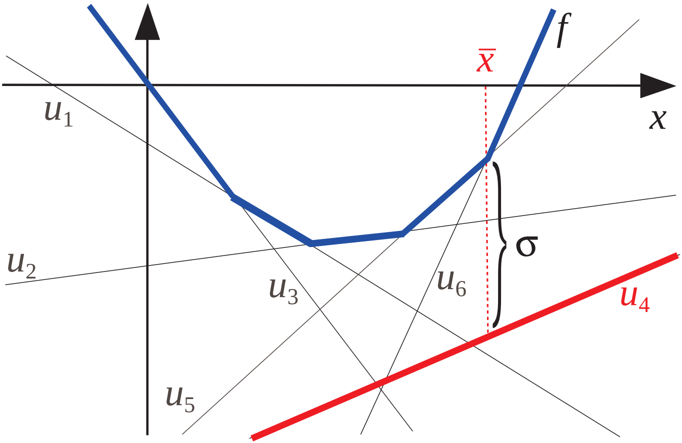

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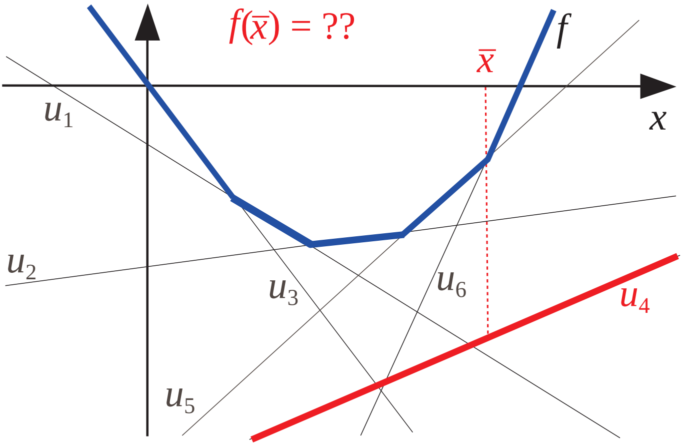

- Approximate solution ⇒ σ-subgradient, $\sigma \geq 0$
- Heuristics ⇒ no measure of σ ⇒ useless for bounding purposes

Approximate Lagrangian Relaxation II

- Heuristics have no (or too weak in practice) performance guarantee
- Different approach: an exact algorithm for solving (2)

[4] Beltran, Tadonki, Vial "Solving the p-Median Problem with a Semi-Lagrangian Relaxation", COAP, 2006

Approximate Lagrangian Relaxation II

- Heuristics have no (or too weak in practice) performance guarantee
- Different approach: an exact algorithm for solving (2)
- Three main components:
  - a heuristic producing $\bar{u} \in U \Rightarrow c(\bar{u}) + xh(\bar{u}) \leq f(x)$
  - an upper bound $\bar{f}(x) \geq f(x)$ (further relaxation)
  - enumeration to squeeze the two together (branching)
- Iterative process where $c(\bar{u}) + xh(\bar{u}) \rightarrow f(x) \leftarrow \bar{f}(x)$

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- The gap $\sigma = \bar{f}(x) - c(\bar{u}) - xh(\bar{u}) \geq 0$ may decrease rather slowly

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- The gap $\sigma = \bar{f}(x) - c(\bar{u}) - xh(\bar{u}) \geq 0$ may decrease rather slowly
- For bounding purposes, $\bar{f}(x)$ "is" $f(x)$

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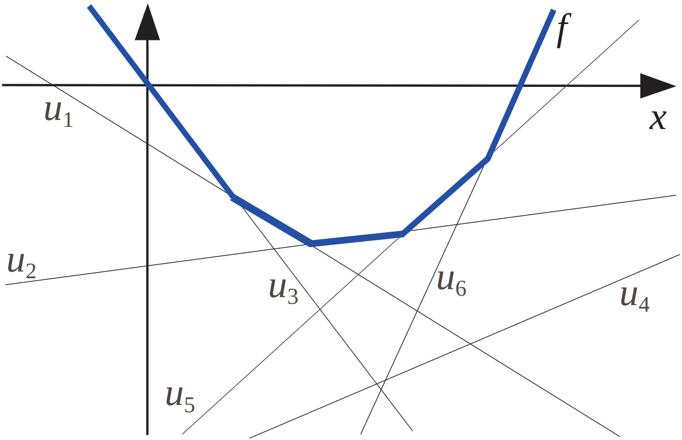

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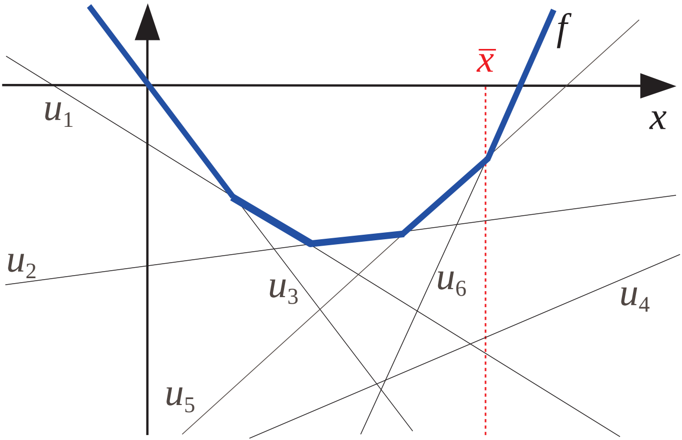

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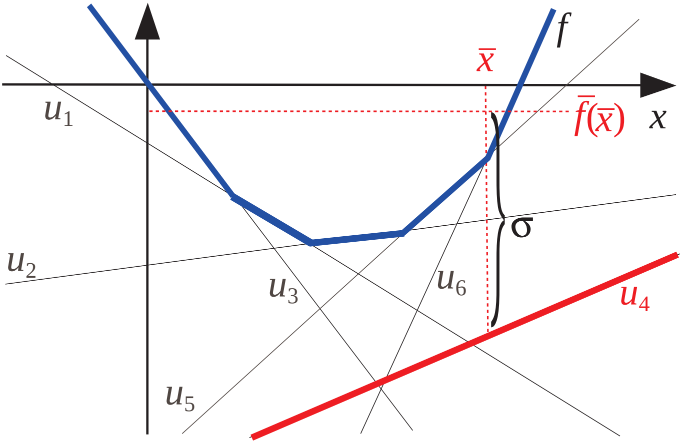

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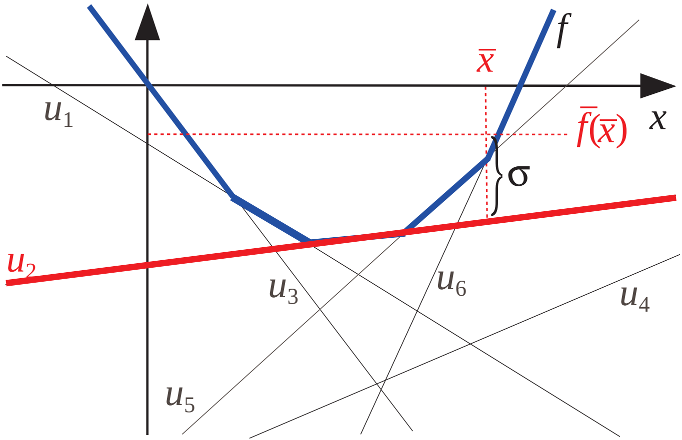

- The upper bound $\bar{f}(x)$ "is" the function value
- σ decreases if either $\bar{f}(x)$ decreases or $c(\bar{u}) + xh(\bar{u})$ increases

A Somewhat Different (but related) Case

- The decomposable case:

$$u = (u^1, \ldots, u^k) \in U^1 \times \ldots \times U^k$$
$$c(u) = c^1(u^1) + \ldots + c^k(u^k)$$
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- Computing $f(x)$ decomposes into k independent subproblems

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- Then, of course, each subproblem can be solved approximately

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The Issue

- Minimizing f using a approximated subgradient (= oracle) possible [6]

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- Of course, it depends on what approach is used

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Subgradient Methods

(with Giacomo d'Antonio)

(approximate) Subgradient Methods

- General problem:

$$\inf_x \{ f(x) \; : \; x \in X \}$$

$f : \mathbb{R}^n \to \mathbb{R}$ convex = approximated oracle, $X \subseteq \mathbb{R}^n$ closed convex

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- Basic approximate subgradient method:

$$g_k \in \partial_{\sigma_k} f(x_k) \quad , \quad \widehat{x}_{k+1} = x_k - \nu_k g_k \quad , \quad x_{k+1} = P_X(\widehat{x}_{k+1})$$

P_X = orthogonal projection on X (assumed "cheap"), ν_k stepsize

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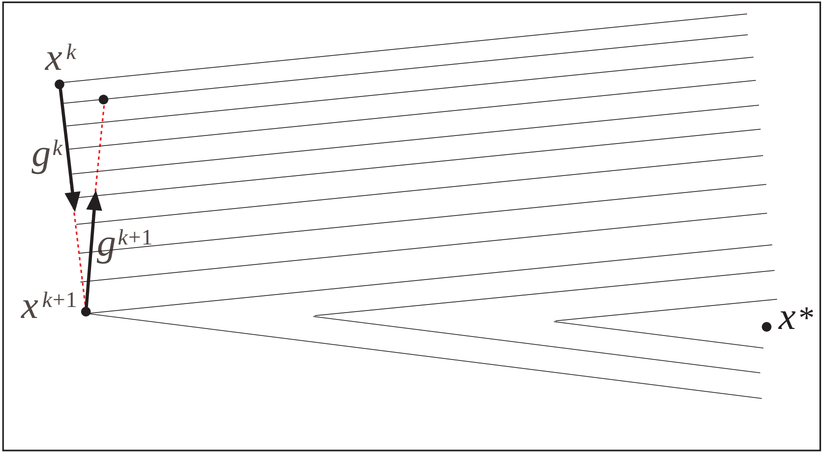
- Very simple, almost no overhead w.r.t. $f(x)$ computation
- Many variants (dilation methods, Bregman projections, ...)
- Typically rather slow, because:
  - a $(1 - \varepsilon)$th-order method, cannot be fast
  - zig-zagging I: in "deep and narrow valleys", successive subgradients almost orthogonal to each other
  - zig-zagging II: at ∂X, subgradients almost orthogonal to ∂X

Zig-Zagging I



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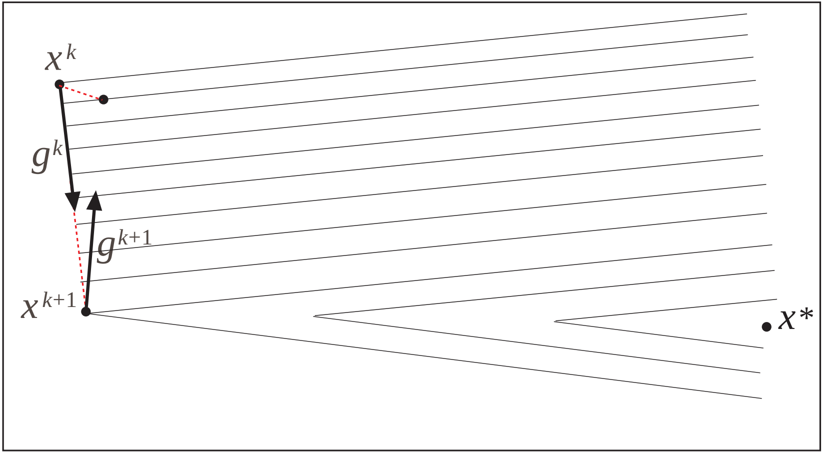

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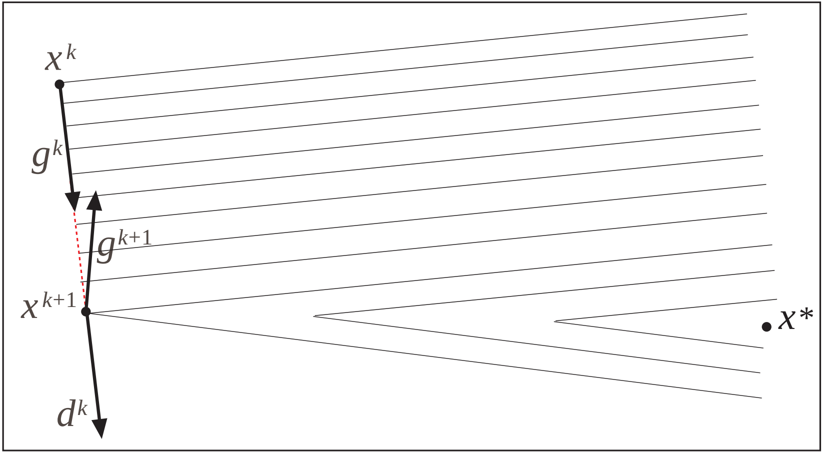

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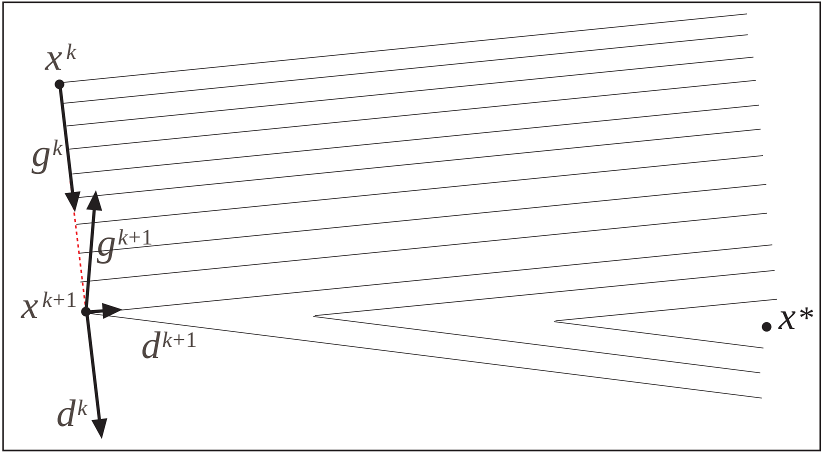

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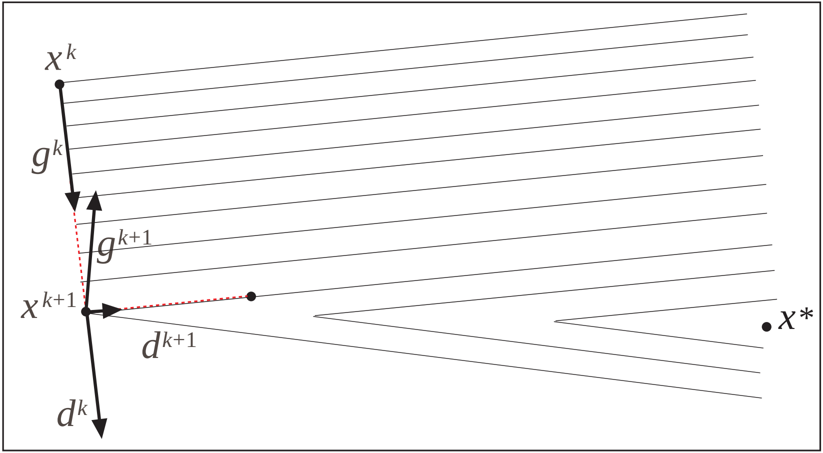

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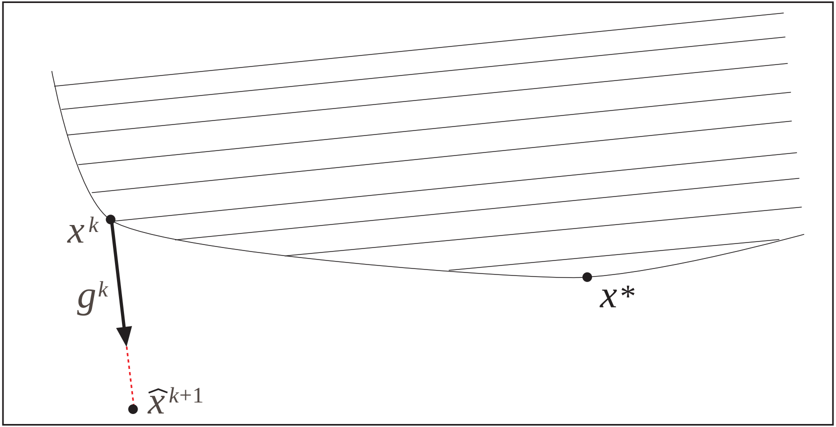
Zig-Zagging II



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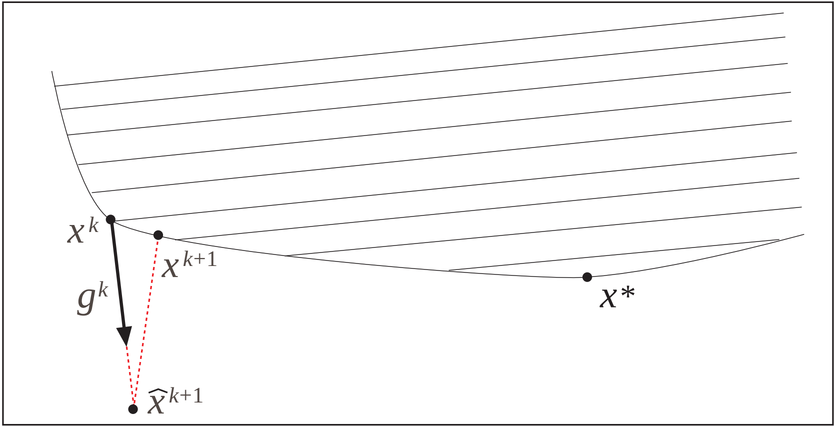

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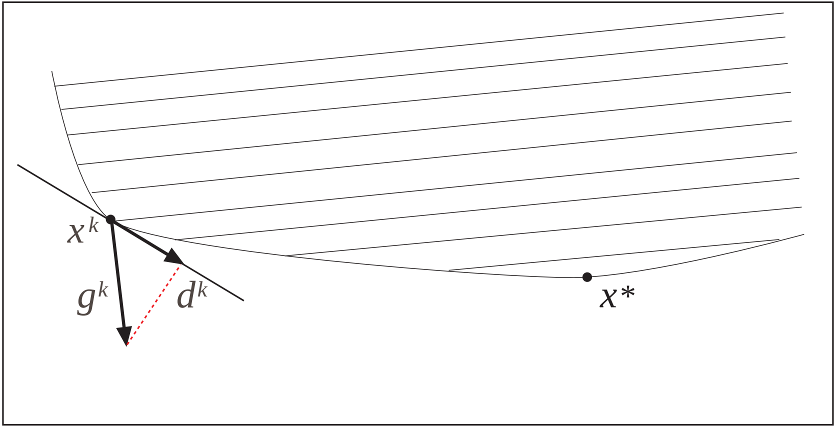

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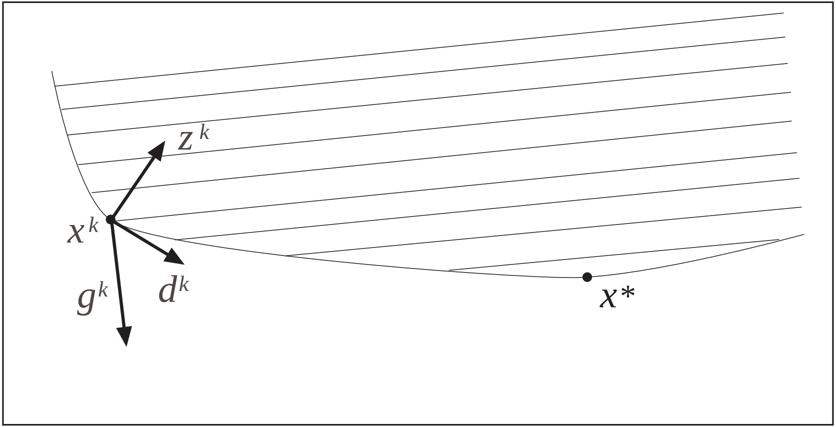

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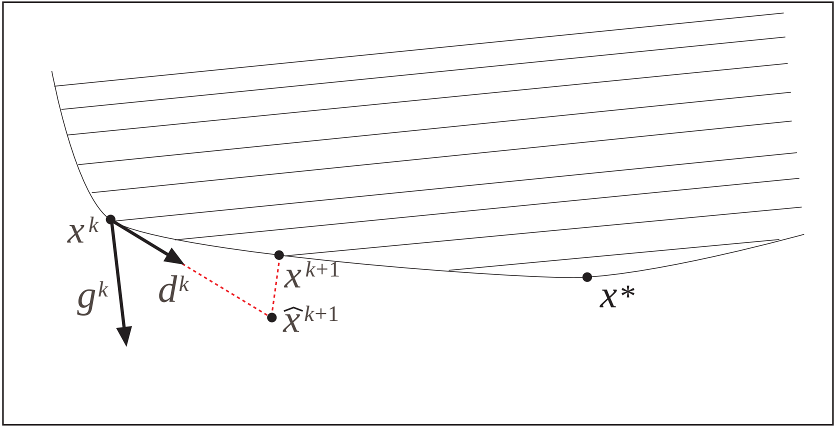

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- Conditional subgradient: $d_k = -P_{T_X(x_k)}(-g_k)^{11} \in \partial f_X(x^k)$

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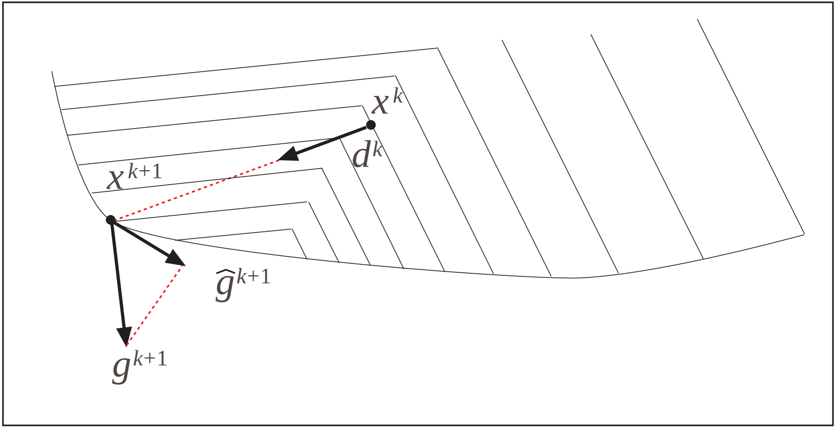

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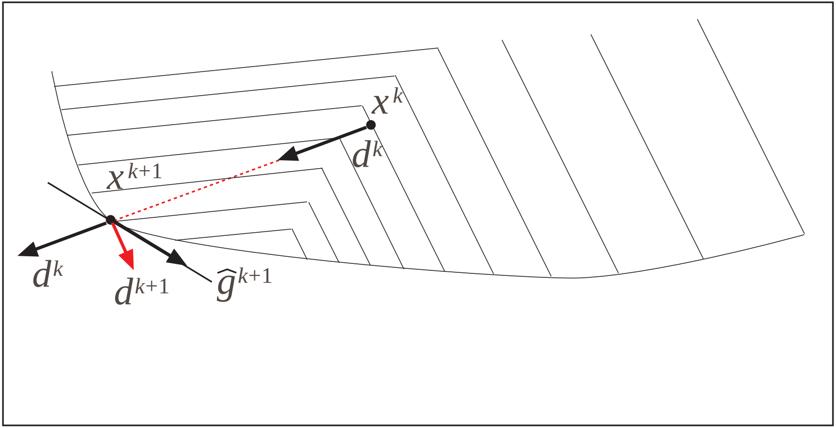

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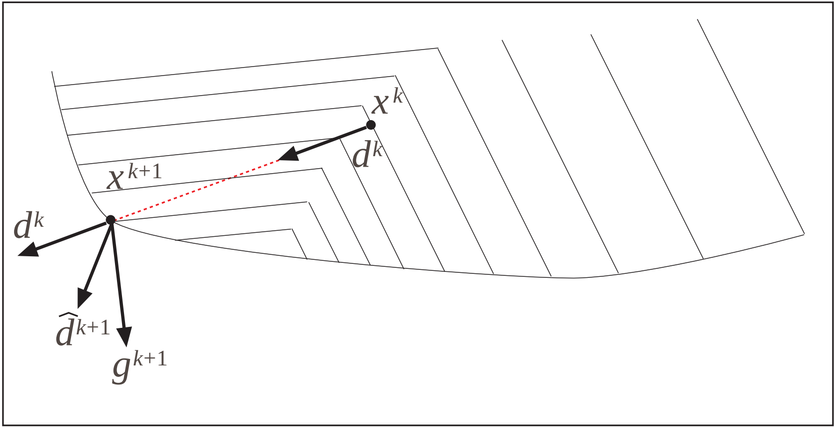

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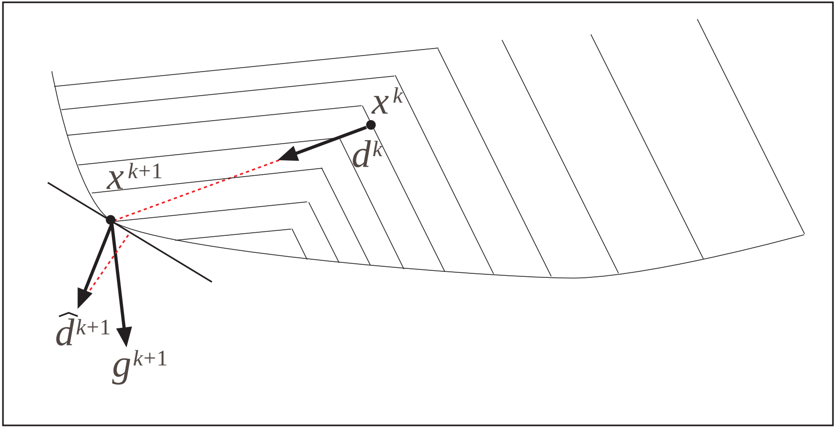

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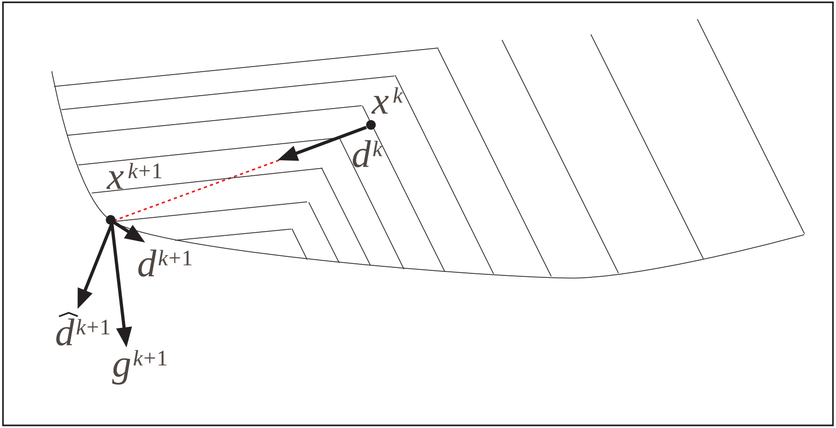

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$$\bar{d}_k(x - x_k) \leq \widehat{d}_k(x - x_k)$$

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- Crucial result (relying on $\alpha_k \in [0, 1]$): $\bar{d}_k \in \partial_{\varepsilon_k} f_X(x_k)$ with

$$\varepsilon_k = (1 - \alpha_k)\big(\ f_k - f_{k-1} - \bar{d}_{k-1}(x_k - x_{k-1}) + \varepsilon_{k-1}\ \big) + \alpha_k \sigma_k \quad (4)$$

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$\sigma^* = \limsup_{k \to \infty} \sigma_k < +\infty$ *(asymptotic maximum error of the oracle); no subgradient method can attain error* $< \sigma^*$ *(if* $f^* > -\infty$*)*

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Proof.

$\sigma_k \geq \sigma^*$ and $f(x_0) = f^* + \sigma^* \Rightarrow g_k$ can be 0 $\Rightarrow d_k = 0$: never moves! □

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$$\varepsilon_k \leq (1 - \alpha_k)(f_k - f^*) + \bar{\sigma}_k \qquad \text{where} \tag{5}$$

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- "Exact" convergence result at hand[7]: $\sigma_k \equiv 0 \Rightarrow$

$$\exists \xi \in [0, 1) \qquad \varepsilon_k \leq \xi(2 - \beta_k)(f_k - f^*)/2$$

$\Rightarrow \liminf_{k \to \infty} f_k = f^\infty \leq f^*$

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Without any assumption on deflection

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- However, knowing σ_k we can do better than that

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- Issue: $\sigma_k > f_k - f^* \Rightarrow \lambda_k < 0$. Solution:

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- Actually, a slightly stronger form is required:

$$\begin{array}{l} \lambda_k \geq 0 \ \Rightarrow \ (\alpha_k \ \geq) \ \beta_k \ \geq \ \beta^* > 0 \ , \\ \lambda_k < 0 \ \Rightarrow \ \alpha_k = 0 \ (\Rightarrow \beta_k = 0) \end{array}$$

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- Actually, a slightly stronger form is required:

$$\begin{aligned} \lambda_k \geq 0 \;&\Rightarrow\; (\alpha_k \;\geq)\; \beta_k \;\geq\; \beta^* > 0 \;, \\ \lambda_k < 0 \;&\Rightarrow\; \alpha_k = 0 \; (\Rightarrow \beta_k = 0) \end{aligned}$$

- (8) $\Rightarrow$ (5) + (7) with $\bar{\sigma}_k = \alpha_k \sigma_k$;
good deflecting "shaves away" a part of the error

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- Without any assumption on deflection: (8) $\Rightarrow$
  - $f^\infty \leq f^* + \sigma^*$
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- The general form: $\lambda_k = f_k - f^* - \gamma_k$

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- What if I do not know σ_k exactly?

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- Is (10) reasonable?

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- Small technical hurdle: all previous proofs require $f^* > -\infty$
- Solution: $f_{rec}^\infty = -\infty \Rightarrow f^* = -\infty$, otherwise feasible target $\bar{f} > -\infty$, $\bar{f} \ge f^*$, $\bar{f} \le f_{rec}^\infty$ ($\Rightarrow f_k - \bar{f} \ge 0$)

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- Abstract property:

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- Note: it may seem that "small ξ is good", but $\xi\sigma^* + \delta^* \geq \sigma^*$

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- Proof: again (almost) straightforward, $\gamma^* \geq \sigma^*$ ($\xi = 1$), minor quirks

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  - else, if $r_k > R$ (*target infeasibility condition*) then $\delta_k = \mu\delta_{k-1}$, $r_k = 0$;
  - otherwise, $f_{ref}^k = f_{ref}^{k-1}$, $\delta_k = \delta_{k-1}$, $r_k = r_{k-1} + \|\widehat{x}_{k+1} - x_k\|$
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- Proof: again (almost) straightforward, $\gamma^* \geq \sigma^*$ ($\xi = 1$), minor quirks
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[16] Lim, Sherali "Convergence . . . for Some Variable Target Value and Subgradient Deflection Methods", COAP, 2006

Vanishing Threshold

- Abstract property:

$$\text{either} \quad f_{ref}^{\infty} = f^* = -\infty \quad , \qquad \text{or} \quad \liminf_{k \to \infty} \delta_k = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \lambda_k / \|d_k\|^2 = \infty$$

- Implementation: $R > 0$ and $\mu \in [0, 1)$
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- Optimal error, extends known results[16] to projection and errors
- Weaker results than (8) ($f^{\infty} \to f_{ref}^{\infty}$, no convergence of $\{x_k\}$)

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Diminishing/Square Summable Stepsize

- Other main class of stepsize rules: diminishing/square summable

$$\sum_{k=1}^{\infty} \nu_k = \infty \quad , \quad \sum_{k=1}^{\infty} \nu_k^2 < \infty \tag{11}$$

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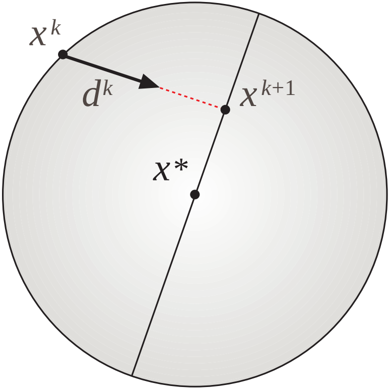
- Gives analogous to (5), (6)

$$\varepsilon_k \leq f_k - f^* + \bar{\sigma}_k \tag{12}$$

where $\bar{\sigma}_k = \alpha_k \sigma_k + (1 - \alpha_k)\bar{\sigma}_{k-1}$

Deflection Rule (geometrically)



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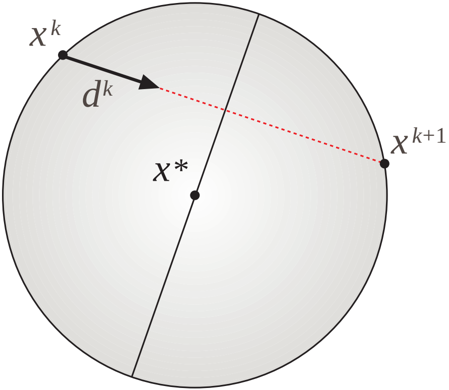

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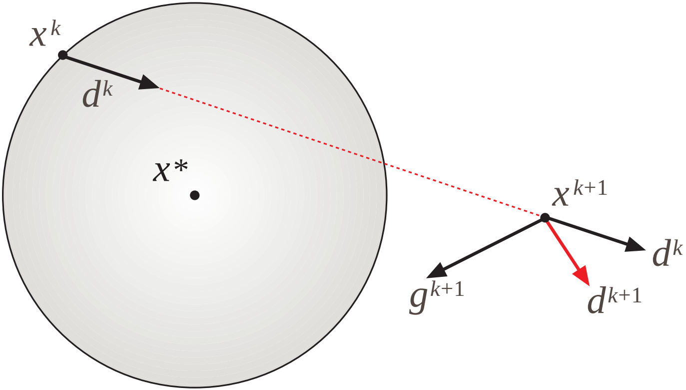

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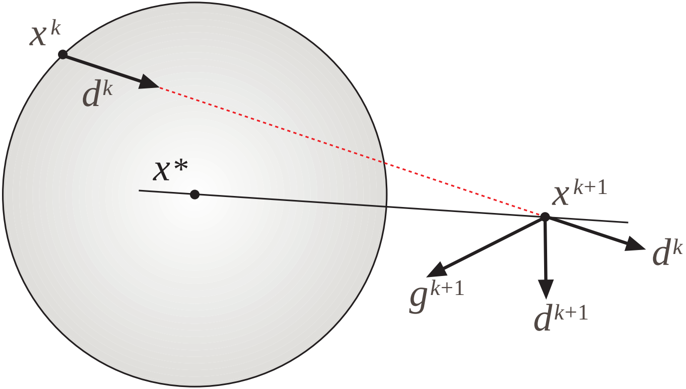

- Moving "towards x^*" is a short enough step and then any deflection
- . . . or any step and a proper deflection

Corrected Deflection Rule

- We learnt our lesson: corrected deflection rule

$$0 \leq \zeta_k = \frac{\nu_{k-1}\|d_{k-1}\|^2}{(\ f_k - f^* - \gamma_k\ ) + \nu_{k-1}\|d_{k-1}\|^2} \leq \alpha_k \leq 1$$

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- Avoid ζ_k is undefined ($\lambda_k = f_k - f^* - \gamma_k$):

$$\nu_{k-1}\|d_{k-1}\|^2 \leq \alpha_k(\lambda_k + \nu_{k-1}\|d_{k-1}\|^2) \tag{13}$$

- Avoid negative λ_k: makes (13) impossible

$$\begin{array}{l} \lambda_k \geq 0 \;\Rightarrow\; \alpha_k \geq \alpha^* > 0 \\ \lambda_k < 0 \;\Rightarrow\; \alpha_k = 0\ (\Rightarrow \nu_k = 0) \end{array} \tag{14}$$

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- Now ε_k is controlled: (12) holds with

$$\bar{\sigma}_k = \alpha_k(\sigma_k - \gamma_k) + (1 - \alpha_k)\bar{\sigma}_{k-1}$$

- Yields the crucial technical relationship, similar to (7)

$$\bar{d}_k(\bar{x} - x_k) \leq f(\bar{x}) - f^* + \bar{\sigma}_k$$

Convergence Results

- Relationships between σ^* and $\bar{\sigma}^*$:
  - in general, $\bar{\sigma}^* \leq \sigma^* + \bar{\gamma}$
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- As usual, f^* not available (and may be $-\infty$) $\Rightarrow$ same trick

Target Value Deflection

- Target value deflection rule

$$0 \leq \zeta_k = \frac{\nu_{k-1} \| d_{k-1} \|^2}{(f_k - f_{lev}^k) + \nu_{k-1} \| d_{k-1} \|^2} \leq \alpha_k \leq 1$$

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$$\delta_{k+1} \in \begin{cases} [\ \Delta_{r(k)+1}\ ,\ \infty\ ) & \text{if } f(x_{k+1}) \leq f_{lev}^k \\ \{\Delta_{k+1}\} & \text{if } f(x_{k+1}) > f_{lev}^k \end{cases}$$

where $r(k) = \#h \leq k$ s.t. $f_{h+1} \leq f_{lev}^h$ and

$$\Delta_k > 0 \qquad , \qquad \liminf_{k \to \infty} \Delta_k = 0 \qquad , \qquad \sum_{k=1}^{\infty} \Delta_k = \infty$$

Target Value Deflection (cont.d)

- Similar technical hurdles (reference value, ...)
- Convergence: either $f_{ref}^{\infty} = -\infty = f^*$, or $f_{ref}^{\infty} \leq f^* + \sigma^*$
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1. If σ^* is your error, then $f^* + \sigma^*$ is your target
2. Knowing σ_k, even approximately, is useful

Bundle Methods

(with Giovanni Giallombardo)

(exact) Bundle Methods: the Basic Ideas

- Any iterative algorithm produces a sequence $\{x_k\}$ of tentative points $\Rightarrow$ the f-values sequence $\{f_k\}$ and the bundle $\mathcal{B} = \{z_k \in \partial f(x_k)\}$

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$$\hat{f}^k_{\mathcal{B}}(x) = \sup_{\bar{z}} \{ \bar{z}x - f^*(\bar{z}) \ : \ \bar{z} \in \mathcal{B} \}$$

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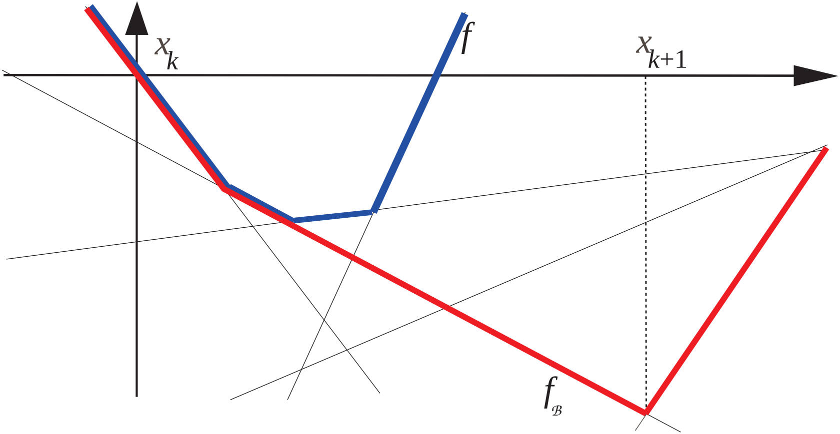
Instability and Stabilization



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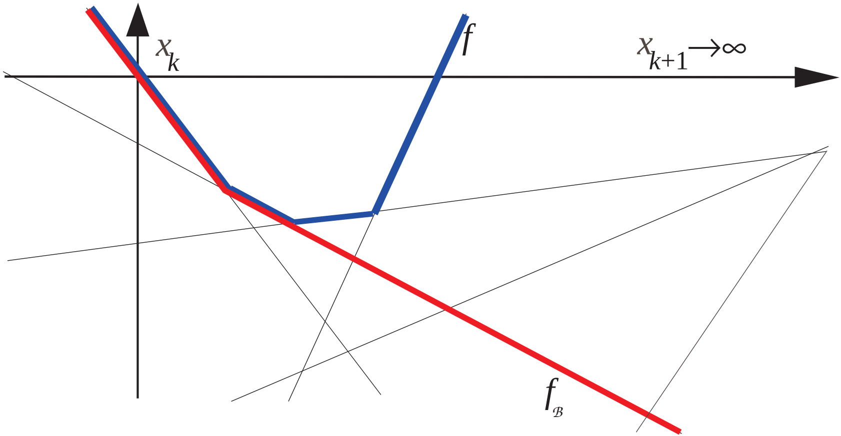

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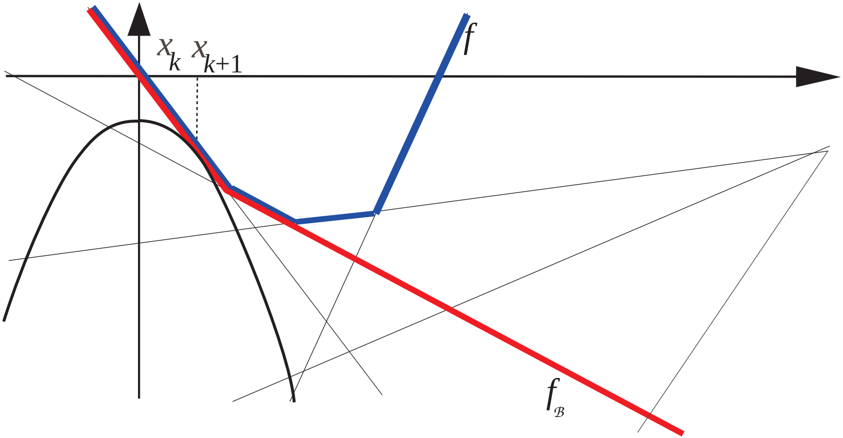

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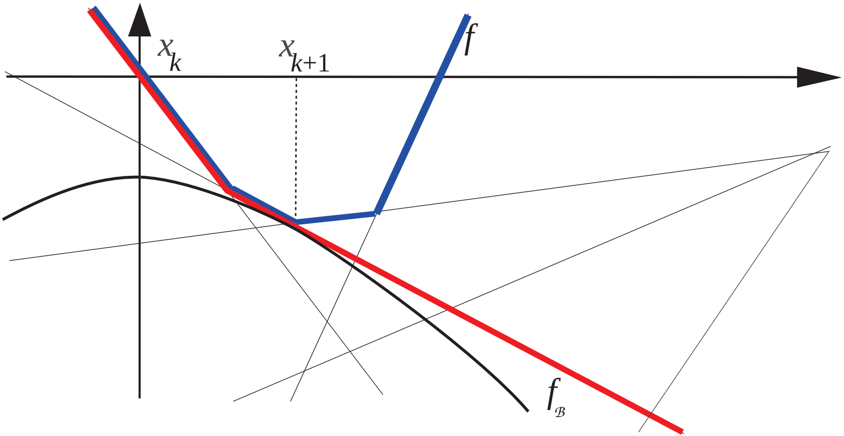

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Primal View of (Generalized) Bundle Methods

- Stabilization: stabilized primal problem ($X = \mathbb{R}^n$ for simplicity)

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- Stabilized primal master problem

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$$(\Pi_{\bar{x},t}) \qquad \phi_t(\bar{x}) = \inf_d \{ \; f(\bar{x} + d) + D_t(d) \; \} \tag{15}$$

  - current point $\bar{x}$
  - ϕ_t = (generalized) Moreau–Yosida regularization of f
  - D_t = stabilizing term ($\approx$ norm), t = proximity weight

- With proper D_t, good properties (e.g. smooth)

- But computing ϕ_t with an oracle for f is difficult $\Rightarrow$ approximation

- Stabilized primal master problem

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  - $x_{k+1} = \bar{x} + d^*$, compute f_{k+1}, $\mathcal{B} = \mathcal{B} \cup \{z_{k+1}\}$

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Dual View of (Generalized) Bundle Methods

- Dual of (Π)[18]: $\quad (\Delta) \qquad f^*(0) \;=\; \inf_z \left\{ \, f^*(z) \;:\; z = 0 \, \right\}$

[18] F. "Generalized Bundle Methods", SIOPT, 2002

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- Illustration: $f_{\mathcal{B}} = \hat{f}_{\mathcal{B}}$, $g(u) = Au - b$, $x \geq 0$

$$(\Delta_{\mathcal{B},\bar{x},t}) \equiv \sup_u \left\{ \begin{array}{l} c(u) + \bar{x}z - D^*_t(-z) \\ z = b + \omega - Au \; , \; \omega \geq 0 \; , \; u \in co\,\mathcal{B} \subseteq U \end{array} \right.$$

$\Rightarrow$ actually solving the weird convexification (3)

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The Decomposable Case

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[19] Bacaud, Lemaréchal, Renaud, Sagastizábal "Bundle methods in stochastic optimal power management: a disaggregated approach using preconditioners" COAP, 2001

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- No incremental version as yet

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Approximate Bundle Methods

- Proposal exist only using lower bound [8] [9] or for finite min-max[20]
- Unify and extend these.

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Definition

Incremental inexact oracle for f: inputs $\bar{x} \in \Re^n$, outputs:

- $\underline{f} \leq f(\bar{x})$, $z \in \Re^n$ s.t. $\underline{f} + z(x - \bar{x}) \leq f(x)$ $\forall x$ (lower linearization)
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Can be called repeatedly on the same $\bar{x}$.

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Conclusions

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- Lots of work still to be done
  - incremental subgradient
  - "dual" subgradient convergence[21]
  - incremental bundle
  - software development/refinement, numerical testing

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