On Some Network-Structured Mixed-Integer NonLinear Problems with Applications to IP Routing
(MINLP meets computer networks)

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Outline

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2. Delay Constrained Routing
3. SOCP models for SFSP-DCR
4. Combinatorial approaches
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7. Other Delay Formulae and Access Control
8. Computational results for WRP and FB
9. Extending the combinatorial approaches to WRP and FB
10. Conclusions
Motivation

- Trends in computer networks:
  - high-bandwidth applications
  - stringent Quality of Service (QoS) guarantees
- Packet-switching (IP, Ethernet) now dominant
- Issue: real-time guarantees (e.g. controlled end-to-end delay)
- Relevant in many WAN/LAN settings:
  - industrial control systems
  - remote sensing and surveillance systems
  - live Internet Protocol Television
  - IP Telephony
  - online gaming/MMORPGs
- Critical in embedded systems (automotive, avionics, ...)
- Packets, not circuits → requires traffic engineering
Delay Constrained Routing Problem

- IP Network \( \equiv \text{directed graph } G = (N, A) \quad (n = |N|, \; m = |A|) \), MTU \( L \)
- Set of flows \( K \): origin/destination \((s^k, d^k)\), deadline \( \delta^k \)
- Leaky-bucket traffic shaper \( \equiv \) Arrival curve \( A \equiv \) parameters \( \sigma^k \) (burst) and \( \rho^k \) (rate) \((A(t) = \sigma + t\rho)\)
- \((i, j) \in A\): link delay \( l_{ij} \), link speed \( w_{ij} \), reservable capacity \( c_{ij}^k \) \((\leq w_{ij})\)
- \( i \in N \): node delay \( n_i \)
- Linear capacity reservation cost \( f_{ij} \) (often \( = 1 \equiv \text{Equal Cost (EC)} \))

Delay Constrained Routing Problem (DCR)

Compute paths and reserve resources on arcs at minimum cost such that the maximum delay of each flow is \( \leq \) deadline

- Single-Flow Single-Path (SFSP) DCR: \(|K| = 1\), unsplittable flow
Worst-case delay modeling

- **One new flow to enter** (drop superscripts, $r_{ij}^k = $ existing flows, fixed)

- **Worst-case delay** of the flow depends on several factors:
  1. the selected \( s-d \) path \( P \) in \( G \);
  2. the reserved rate (capacity) \( r_{ij} \in [0, c_{ij}] \) for each \( (i, j) \in P \);
  3. the details of the scheduling protocol (requires network calculus)

- Necessary assumption for finite delay: \( r_{ij} \geq \rho \) for each \( (i, j) \in P \)
  (\( \rho \equiv \text{rate} \equiv \text{“steady-state” flow demand in classical flow models} \))

- General formula (**already nonlinear!**):

\[
\sigma \min\{ r_{ij} : (i, j) \in P \} + \sum_{(i,j)\in P} (\theta_{ij} + l_{ij} + n_i) \tag{1}
\]

where \( \theta_{ij} \equiv \text{protocol-specific arc delay (also nonlinear!)} \)

- (1) convex and SOCP-representable if \( \theta_{ij} \) is
Worst-case delay modeling (cont.)

- Exact formula for $\theta_{ij}$ depends on the scheduling protocol:

\[
\theta_{ij} = \frac{L}{r_{ij}} + \frac{L}{w_{ij}} \quad \text{Strictly Rate-Proportional (2)}
\]

\[
\theta_{ij} = \frac{L}{r_{ij}} + |P(i,j)| \frac{L}{w_{ij}} \quad \text{Weakly Rate-Proportional (3)}
\]

\[
\theta_{ij} = \frac{L}{w_{ij}} \frac{w_{ij} - r_{ij}}{\min\{r_{ij}, r_{ij}^{\text{min}}\}} + \frac{L}{r_{ij}} + |P(i,j)| \frac{L}{w_{ij}} \quad \text{Frame-Based (4)}
\]

- $P(i,j)$ = set of paths passing through $(i,j)$ excluding new one
- $r_{ij}^{\text{min}} = \min\{r_{ij}^k : q^k \in P(i,j)\}$ ($\implies$ SRP \(\lesssim\) WRP \(\leq\) FB)

- (2) flow-independent, convex, SOCP-representable
- (3) \(\approx\) (2) but not flow-independent
- (4) (surprisingly) also convex but only for SFSP, less trivial
- (3) and (4) not flow-independent \(\implies\) have admission control issue
A SOCP model for SFSP-DCR [with SRP]

- **Path binary variables** $x_{ij}$, reserve continuous variables $r_{ij}$

$$\min \sum_{(i,j) \in A} f_{ij} r_{ij}$$

$$\sum_{(j,i) \in BS(i)} x_{ji} - \sum_{(i,j) \in FS(i)} x_{ij} = \begin{cases} -1 & \text{if } i = s \\ 1 & \text{if } i = d \\ 0 & \text{otherwise} \end{cases} \quad i \in N$$  

$$0 \leq r_{ij} \leq c_{ij} x_{ij} \quad (i,j) \in A$$  

$$\rho \leq r_{min} \leq r_{ij} + c_{max}(1 - x_{ij}) \quad (i,j) \in A$$  

$$t + \sum_{(i,j) \in A} \left( \theta_{ij} + \left( \frac{L}{w_{ij}} + l_{ij} + n_i \right) x_{ij} \right) \leq \delta$$

$$t r_{min} \geq \sigma \quad , \quad t \geq 0$$

$$x_{ij} \in \{0, 1\} \quad , \quad r_{ij} \in \mathbb{R} \quad (i,j) \in A$$

- **(10) rotated SOCP constraint** $\equiv t \geq \sigma / r_{min}$ (since $t \geq 0$)

- **Issue:** how to write “$x_{ij} = 1 \implies \theta_{ij} \geq L/r_{ij}, \ x_{ij} = 1 \implies \theta_{ij} = 0$”
Solution I: big-M

- Issue: can’t use $r_{ij} \theta_{ij} \geq L$ for that $\implies \theta_{ij} > 0$ always

- Solution: two extra sets of variables $s_{ij}$ and $r'_{ij}$

\[
0 \leq \theta_{ij} \leq Mx_{ij} \\
\theta_{ij} \geq s_{ij} - M(1 - x_{ij}) \\
s_{ij} r'_{ij} \geq L, \ s_{ij} \geq 0 \\
0 \leq r'_{ij} \leq r_{ij} + M(1 - x_{ij})
\]

- $\theta_{ij} \geq s_{ij}$ if $x_{ij} = 1$, while $\theta_{ij}$ and $s_{ij}$ are “free” if $x_{ij} = 0$

- $r'_{ij} \leq r_{ij}$ if $x_{ij} = 1$, while $r'_{ij}$ and $r_{ij}$ are “free” if $x_{ij} = 0$

- $s_{ij} \geq L / r'_{ij} \implies \theta_{ij} \geq s_{ij} \geq L / r'_{ij} \geq L / r_{ij}$ if $x_{ij} = 1$

- $M = \max(\sqrt{L}, L / \rho)$ suffices, still it’s big-M: can we do better?
Solution II: Perspective Reformulation

- **General Perspective Reformulation:** \( f : \mathbb{R}^q \rightarrow \mathbb{R} \) convex, two sets

  \[
P_0 = \{ 0 \} \quad , \quad P_1 = \{ v \in \mathbb{R}^q : l \leq v \leq u , f(v) \leq 0 \}
  \]

  the best possible convex approximation of their (nonconvex) union is

  \[
  \text{conv}(P_0 \cup P_1) = \left\{ v : \lambda l \leq v \leq \lambda u , \lambda f(v/\lambda) \leq 0 , \lambda \in [0, 1] \right\}
  \]

- Application: after a little tedious algebra

  \[
  \rho x_{ij} \leq r_{ij} \leq c_{ij} x_{ij} \quad , \quad 0 \leq \theta_{ij} \leq (L/\rho) x_{ij} \quad , \quad \theta_{ij} r_{ij} \geq L x_{ij}^2
  \]

  (now \( \theta_{ij} \) can be 0 when \( x_{ij} = 0 \), \( x^2/r \) convex for \( r > 0 \))

- original variables + a(another rotated) SOCP constraint

- Looks much better: is it?
Instances

- **Real-world** IP network topologies (GARR, SNDlib, TopoZOO)

- Realistic random topologies (Waxman model)

- **Equal (reservation) Costs** $f_{ij} = 1$

- **FNSS** tool for **realistic traffic matrices** ($\mu(T) = 0.8$ Gbps and $\sigma^2(T) = 0.05$) and link-capacity assignment (1, 10, 40 Gbps)

- **DCR-generator** for the remaining network parameters ($L = 1500, n_i = l_{ij} = L/w_{ij}, \sigma = 3L$)

- Distributed at
  http://www.di.unipi.it/optimize/Data/MMCF.html#UMMCF

- Experiments with “unloaded networks”, but “loaded” case analogous
### SOCP models – Cplex

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Frangioni et al. (DI + DII, UniPI)
## SOCP models – Cplex (cont.)

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## SOCP models – GUROBI

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Combinatorial properties

- A MINLP with strong network structure, how to exploit it?
  \[ \sigma = L = 0 \implies r_{ij} = \rho x_{ij} \implies \text{Constrained Shortest Path (CSP)} \]
  (this gives more than one idea, and proves $\mathcal{NP}$-hardness)

- Checking feasibility is easy: delay is a decreasing function of $r_{ij}$
  \[ \implies \text{simply push the rates to the maximum} \]

- Modified arc costs $\overline{l}_{ij} = \frac{L}{c_{ij}} + \frac{L}{w_{ij}} + l_{ij} + n_i$

- For each $r \in C = \{ c_{ij} : (i, j) \in A \}$:
  - define reduced graph $G^r = (N, A^r)$ where $A^r = \{ (i, j) \in A : c_{ij} \geq r \}$
  - solve s-d shortest path $P$ on $G^r$ w.r.t. $\overline{l}$
  - if $\overline{l}(P) \leq \delta - \sigma / r$, then $P$ feasible
  - if no feasible $P$ found, then problem unfeasible
    (for fixed $P$, both LHS and RHS of (1) increase with $r$)

- Keep $f$-best solution found: (ERA-I heuristic)
Equal Rate Allocation

- Equal Rate Allocation: $r_{ij} = r \geq \rho$ for all $(i,j) \in P \implies r_{\text{min}} = r$

- It is easy to solve EC-ERA-SFSP-DCR for fixed $r$ ($f_{ij} = 1$)
  - run Bellman-Ford on $G^r$ with costs $l^r_{ij} = L/r + L/w_{ij} + l_{ij} + n_i$
  - each time $d$ extracted from $Q$, check if delay $\leq \delta - \sigma/r$

- Repeating the above for all $r \in C$ does not solve EC-ERA-SFSP-DCR
  counterexample: returned path $P$ with delay constraint not tight

- Obvious solution: for each $P$ reduce $r$ until constraint tight
  $\implies$ keep feasibility, improve objective function $\implies$ optimal

- Solves EC-ERA-SFSP-DCR, ERA-H heuristic for EC-SFSP-DCR

- Extensions:
  - Non-equal but integer $f_{ij}$: use dynamic programming instead of Bellman-Ford (pseudo-poly)
  - Non-equal continuous $f_{ij}$: classical approximation algorithm with cost rounding using the above pseudo-poly
### ERA-Based Heuristics: Experiments

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- **inf** = fraction of feasible wrongly declared unfeasible
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3-Pronged Approach

- MI-SOCP approach **accurate but slow**
  ERA-* approaches **fast but inaccurate**

- Best of both worlds: **3-pronged approach**
  1. run **ERA-I**, if instance unfeasible terminate
  2. otherwise run **ERA-H**: if a solution found, report it and terminate
  3. if all else fails, then run **model P** and report its solution

- **So crude, does it really work?**
### 3-Pronged Approach: Experiments

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Frangioni et al. (DI + DII, UniPI)
### 3-Pronged Approach: Experiments (cont.)

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Frangioni et al. (DI + DII, UniPI)  SOCP for DCR  CWMINLP13 21 / 31
Does it really matter in practice?

- Simulating the network behavior, large number of path computations
- Exponential interarrival (avg = \( \lambda \)), exponential duration (avg = 1s)
- \( \sigma = 3 \) MTU and \( \delta \) random in \([d_{min}, d_{min} + \beta(d_{max} - d_{min})]\)
  \( d_{min} = \) minimum feasible deadline, \( d_{max} = \) delay constraint inactive
- Average of five independent replicas, and 95% confidence intervals
- Comparing all practical approaches known so far (2 new):
  1. ERA (equal rate allocation)
  2. SWPF-URA: shortest-widest-path + optimal (unequal) rate allocation
  3. WSPF-URA: widest-shortest-path + optimal (unequal) rate allocation
  4. SFSP-DCR: MI-SOCP model (perspective version)
  5. TPH: 3-pronged heuristic
- Same real-world topologies, realistic capacities
Simulation results: blocking probability

- ERA fails far too much (allocating the same rate a bad idea)
- both ERA and *-URA perform considerably worse than SFSP-DCR
- TPH performs quite close to the optimum

Similar on all topologies, \( \sigma \in \{1, 3, 10\} \) MTU, \( \beta \in \{0.2, 0.5, 1.0\} \)
Why does ERA fail so often?

- Hub-and-spoke-like network with well-connected core (40/100 Gb) but weaker links to the periphery (1 Gb)
- Path from a core node to a peripheral one has to cross a weak link
- ERA has to allocate the same rate to all links \(\Rightarrow\) no more than the weak link’s (residual) capacity \(\Rightarrow\) cannot meet the deadline
- The deadline can be met by reserving more capacity on core links
Simulation results: time

- SFSP-DCR slower but still affordable
- TPH much faster and almost as good
- “large” networks: $|N| = 70+$, $|A| = 230+$
- Path Computation Element makes this technically feasible
SOCP Model for WRP

\[ \theta_{ij} = \frac{L}{r_{ij}} + |P(i, j)| \frac{L}{w_{ij}} \approx (2) \implies \text{(basically) same model} \]

But requires access control: not to make existing flows unfeasible

Delay slack:

\[ \bar{\delta}^k = \delta^k - \frac{\sigma^k}{r_{\min}^k} - \sum_{(i,j) \in p^k} \left( \frac{L}{r_{ij}^k} + |P(i, j)| \frac{L}{w_{ij}} + l_{ij} + n_i \right) \]

Access control constraint, one for each \( k \in K \)

\[ \sum_{(i,j) \in p^k} \frac{L}{w_{ij}} x_{ij} \leq \bar{\delta}^k \]

Can be used to preprocess away some arcs

The coefficients are the same for each flow, can use path (+ RHS) dominance to detect redundant ones

Still, possibly many constraints (\(|K| \approx n^2\))
SOCP Model for FB

\[ \theta_{ij} = \frac{L}{r_{ij}} + (|P(i, j)| + \phi(r_{ij})) \frac{L}{w_{ij}} \approx \text{WRP} \]

where

\[ \phi(r) = \frac{(w_{ij} - r)}{\min\{r, \bar{r}_{ij}\}} \]

Since \( \bar{r}_{ij} \) is fixed, can be rewritten as

\[ \phi(r) = \begin{cases} 
\phi_1(r) = \frac{w_{ij}}{r} - 1 & \text{if } 0 < r \leq \bar{r}_{ij} \\
\phi_2(r) = \frac{(w_{ij} - r)}{\bar{r}_{ij}} & \text{if } \bar{r}_{ij} \leq r \leq c_{ij} \leq w_{ij} 
\end{cases} \]

Convex!: \( \phi'_1(\bar{r}_{ij}) \leq \phi'_2(\bar{r}_{ij}) \)

Can use the classical variable splitting reformulation

\[ \theta_{ij} = v_{ij} + v'_{ij} + \frac{L}{w_{ij}} \left[ |P(i, j)| x_{ij} - \frac{r_{ij}}{\bar{r}_{ij}} \right] \]

\[ r_{ij} = r'_{ij} + r''_{ij} \quad , \quad \rho x_{ij} \leq r'_{ij} \leq \bar{r}_{ij} x_{ij} \quad , \quad 0 \leq r''_{ij} \leq (c_{ij} - \bar{r}_{ij}) x_{ij} \]

\[ v_{ij} r_{ij} \geq L x_{ij}^2 \quad , \quad v_{ij} \geq 0 \quad , \quad v'_{ij} r'_{ij} \geq L x_{ij}^2 \quad , \quad v'_{ij} \geq 0 \]

two conic constraints to represent the same \( L/r_{ij} \): can we do better?
Actually we can: \( \phi_1(w_{ij}) = \phi_2(w_{ij}) \implies \phi(r) = \max\{ \phi_1(r), \phi_2(r) \}! \)

Can use the “cutting planes” representation of \( \phi \):

\[
\theta_{ij} = v_{ij} + v'_{ij} + \frac{L}{w_{ij}} (|P(i,j)| + 1) x_{ij}, \quad v_{ij}r_{ij} \geq Lx_{ij}^2, \quad v_{ij} \geq 0
\]

\[
v'_{ij} \geq v_{ij} - L/w_{ij}, \quad v'_{ij} \geq (L/\bar{r}_{ij}) x_{ij} - Lr_{ij} / (w_{ij}\bar{r}_{ij})
\]

only one conic constraint, less variables (is this better?)

Admission control constraint for FB:

\[
\sum_{(i,j) \in P^k} \frac{L}{w_{ij}} (x_{ij} + (w_{ij} - r^k_{ij}) z_{ij}) \leq \bar{\delta}^k
\]

\[
s_{ij} \leq r_{ij}, \quad s_{ij} \leq \bar{r}_{ij}, \quad s_{ij} z_{ij} \geq x_{ij}^2, \quad z_{ij} \geq 0
\]

+2\(|A|\) variables, +\(|A|\) conic constraints but shared among flows

Different coefficients (to share the \( z_{ij} \)), dominance more difficult

Arc-based preprocessing still possible (using \( r_{ij} = c_{ij} \))
Computational results for WRP and FB
Er . . . , not ready yet, sorry!
Extending the combinatorial approaches to WRP and FB
Er . . ., not ready yet, sorry!
Conclusions

• The world is indeed nonlinear, but surprisingly often nicely convex
• DCR: interesting generalization of classical “steady state” flows
• Relevant for applications, apparently good results
• MI-SOCP with substantial network structure = prototypical blend of nonlinear and combinatorial optimization
• MINLP techniques useful (Perspective Reformulation, SOCP, . . . )
• Combinatorial techniques useful (shortest paths, dynamic programming, approximation algorithms, . . . )
• Both are needed
• Still lots of work to do (WRP/FB, multi-flow, multi-path, network design, robust, . . . ), problems look pretty hard
• Lots of fun. Join in! :-)

Frangioni et al. (DI + DII, UniPI)