# The Stabilized Structured Dantzig-Wolfe Method: a Bundle Method with a Different Model

Antonio Frangioni

Dipartimento di Informatica, Università di Pisa

joint work with

Bernard Gendron

CIRRELT, Université de Montréal

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# Outline

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where we know something about the set $U$, whereas
we know precious little of $U \cap \{ u \ : \ Au = b \}$

  - for sure we know some formulation, e.g., $U = \{ u \in \mathbb{Z}^n \ : \ Eu \leq d \}$
  - linearity used for simplicity, has some (but not paramount) impact, convexity is the issue
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- What does "we know something" means? Two possible things:
  - we know a good formulation for $U$
  - we know how to efficiently optimize upon $U$

# If we know a good formulation for $U$

- The best possible convex formulation

$$\tilde{E}u \leq \tilde{d} \quad \text{such that} \quad \left\{ u \in \mathbb{R}^n \ : \ \tilde{E}u \leq \tilde{d} \right\} = conv(U)$$

$\Rightarrow$ the best possible (convex = solvable) relaxation

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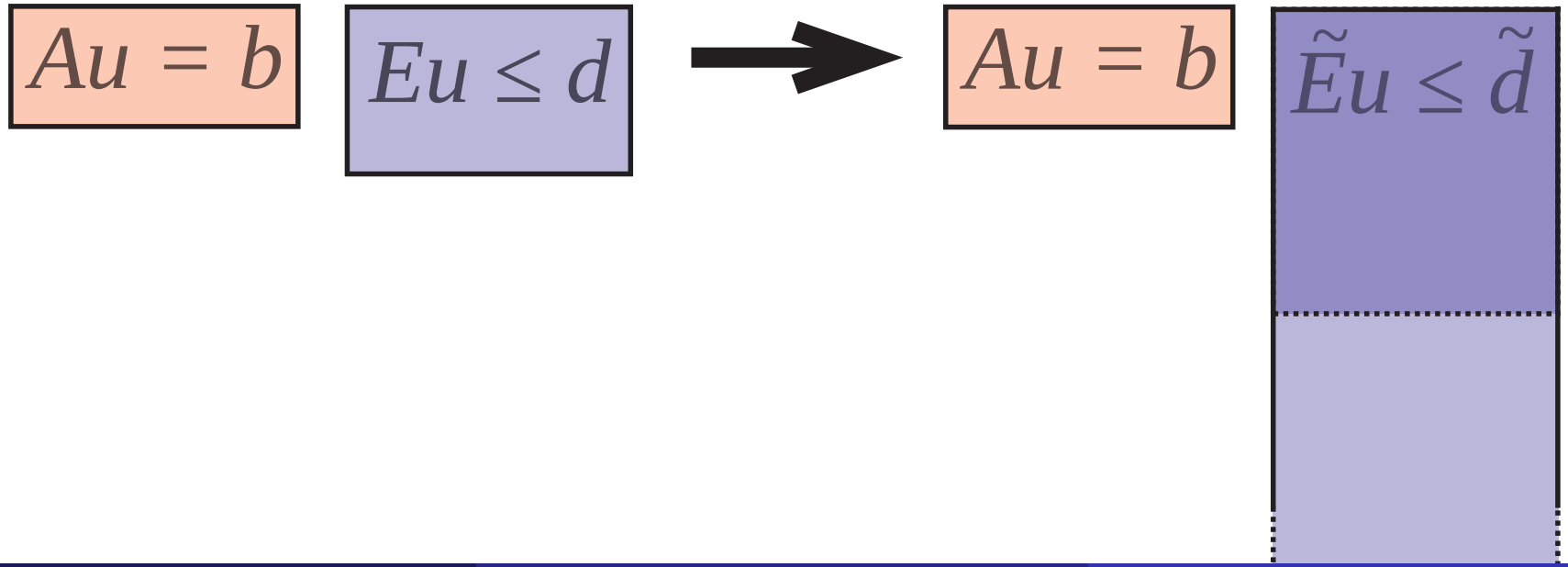
- Except, practically all good formulations are too large
  $\Rightarrow$ Row Generation (polyhedral methods)



# If we know how to efficiently optimize upon $U$

- Efficiently optimize upon $U \Rightarrow$ generate (extreme) points of $U$

  $\Rightarrow$ represent $conv(U)$ by points instead of by faces

$$conv(U) = \left\{ u = \sum_{\bar{u} \in U} \bar{u}\theta_{\bar{u}} \ : \ \sum_{\bar{u} \in U} \theta_{\bar{u}} = 1 \ , \ \theta_{\bar{u}} \geq 0 \quad \bar{u} \in U \right\}$$

  $\Rightarrow$ reformulate $(\bar{\Pi})$ in terms of the convex multipliers $\theta$

$$\begin{array}{rll} \max & c\left(\sum_{\bar{u} \in U} \bar{u}\theta_{\bar{u}}\right) & \\ & A\left(\sum_{\bar{u} \in U} \bar{u}\theta_{\bar{u}}\right) & = b \\ & \sum_{\bar{u} \in U} \theta_{\bar{u}} & = 1 \quad \theta_{\bar{u}} \geq 0 \quad \bar{u} \in U \end{array}$$

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- Could this ever be a good idea? Actually, it could:
  polyhedra may have few faces and many vertices . . . or vice-versa

| | | | |
|---|---|---|---|
| $n$-cube | $\lvert u_i \rvert \leq 1 \quad \forall\, i$ | $2n$ faces | $2^n$ vertices |
| $n$-co-cube | $\sum_i \lvert u_i \rvert \leq 1$ | $2^n$ faces | $2n$ vertices |

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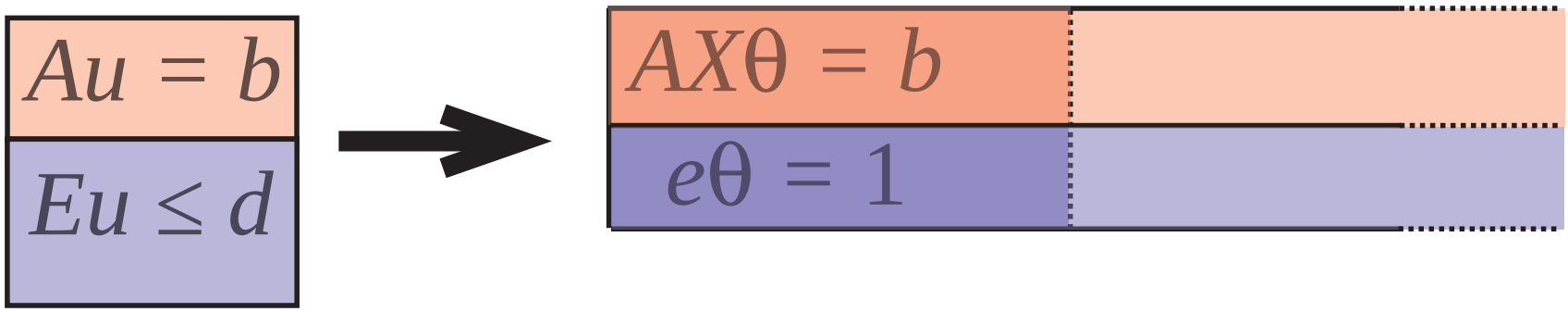

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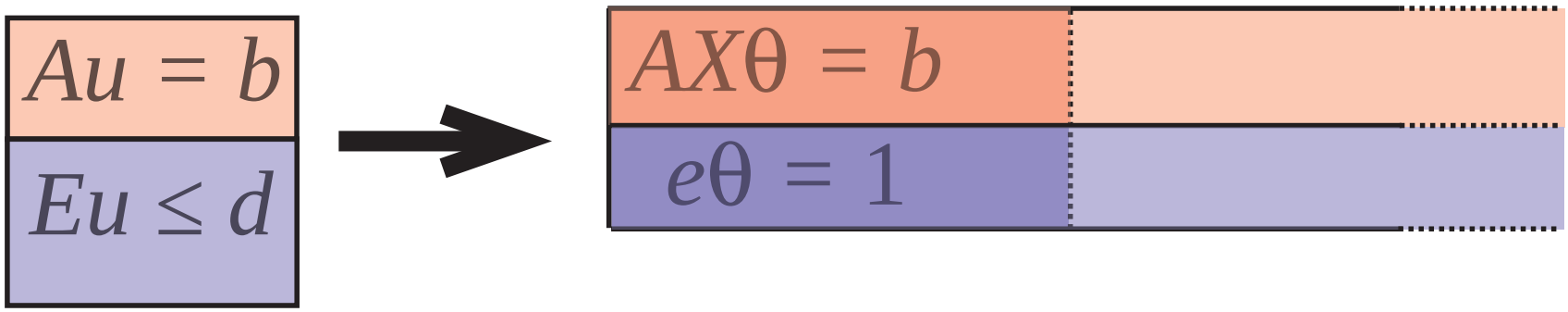


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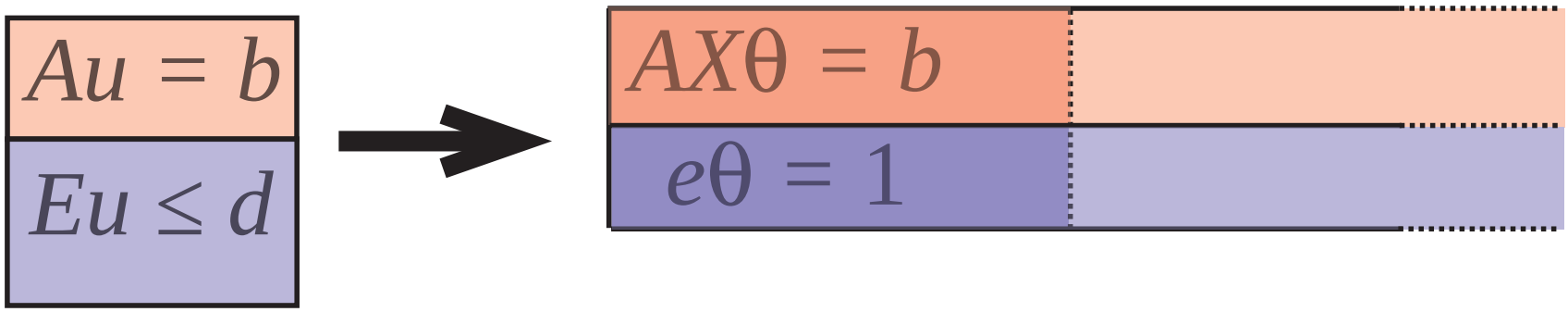



- But, if we can efficiently optimize over $U$, we can generate vertices
- $\mathcal{B} \subset U$ (small), solve restriction of $(\bar{\Pi})$ with $U \to \mathcal{B}$, i.e.,

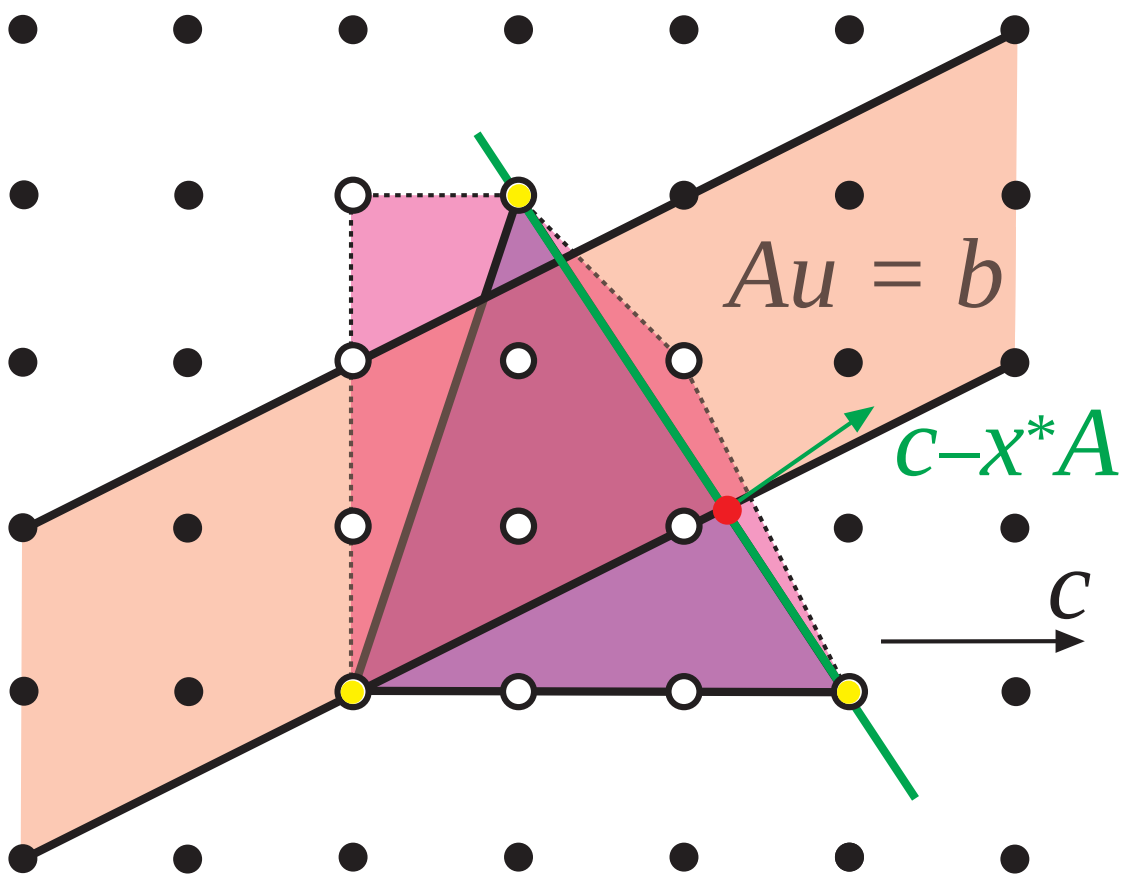
$$(\Pi_{\mathcal{B}}) \qquad \max \{ cu \ : \ Au = b \ , \ u \in conv(\mathcal{B}) \}$$

feed (partial) dual optimal solution $x^*$ (of $Au = b$) to pricing problem

$$(\Pi_x) \qquad f(x) = \max \{ (c - xA)u \ : \ u \in U \} \ + \ xb$$

a.k.a. Lagrangian relaxation/Dantzig-Wolfe decomposition

# Dantzig-Wolfe in pictures

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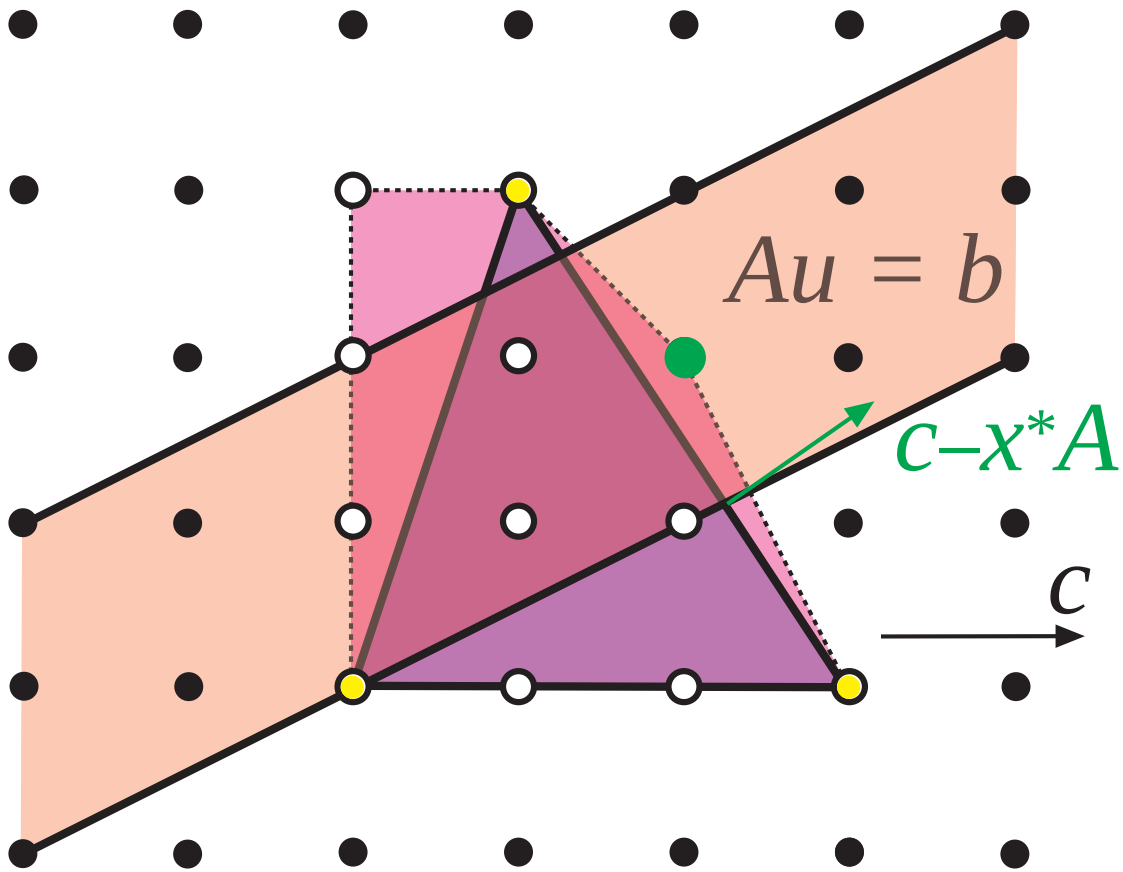


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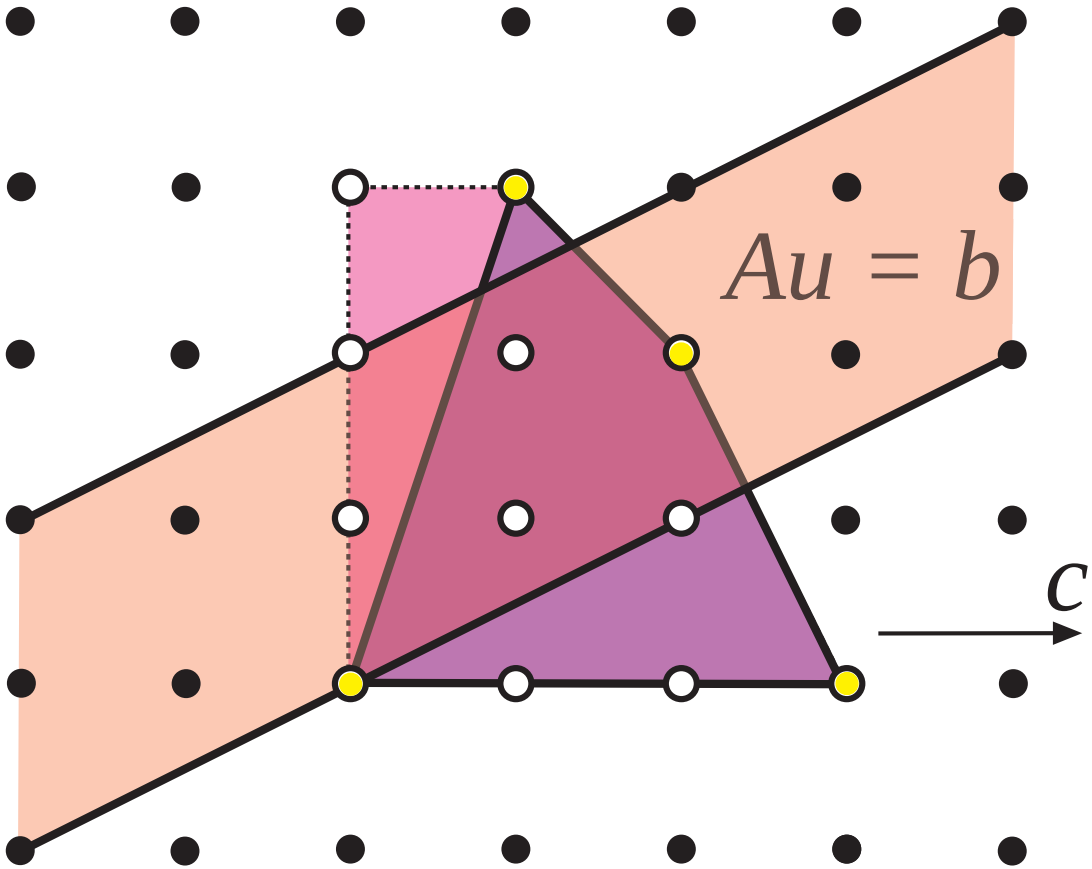


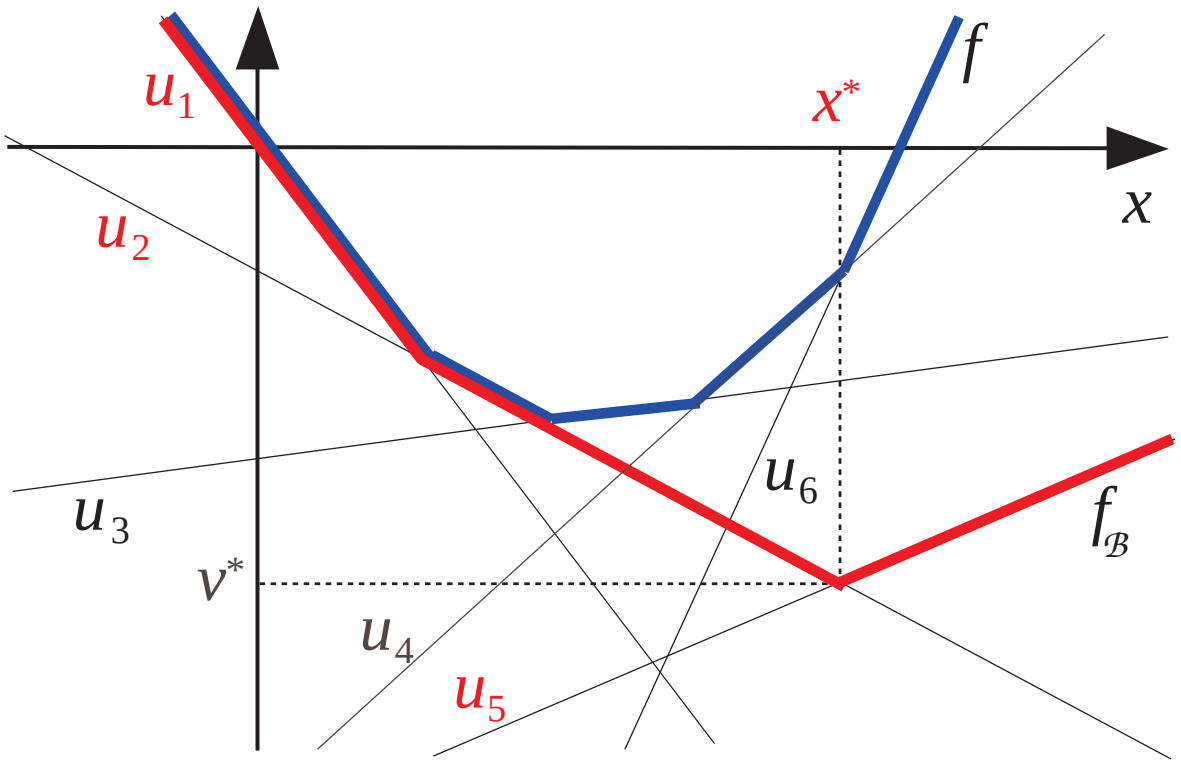

- $c - x^* A$ separates $conv(\mathcal{B}) \cap Au = b$ from all $u \in U$ better than $u^*$
- Thus, optimizing it allows finding new points (if any)
- Issue: $conv(\mathcal{B}) \cap Au = b$ must be nonempty

# Dantzig-Wolfe = Row Generation in the dual

- Dual of $(\Pi_{\mathcal{B}})$: $\min \left\{ v \ : \ v \geq (c - xA)u \ , \ u \in \mathcal{B} \right\} \quad [ + xb ]$

$$= \min \left\{ f_{\mathcal{B}}(x) = \max\{ (c - xA)u + xb \ , \ u \in \mathcal{B} \} \right\}$$

- $f_{\mathcal{B}}(x)$ = lower approximation of "true" Lagrangian function $f(x)$
  = cutting-plane model



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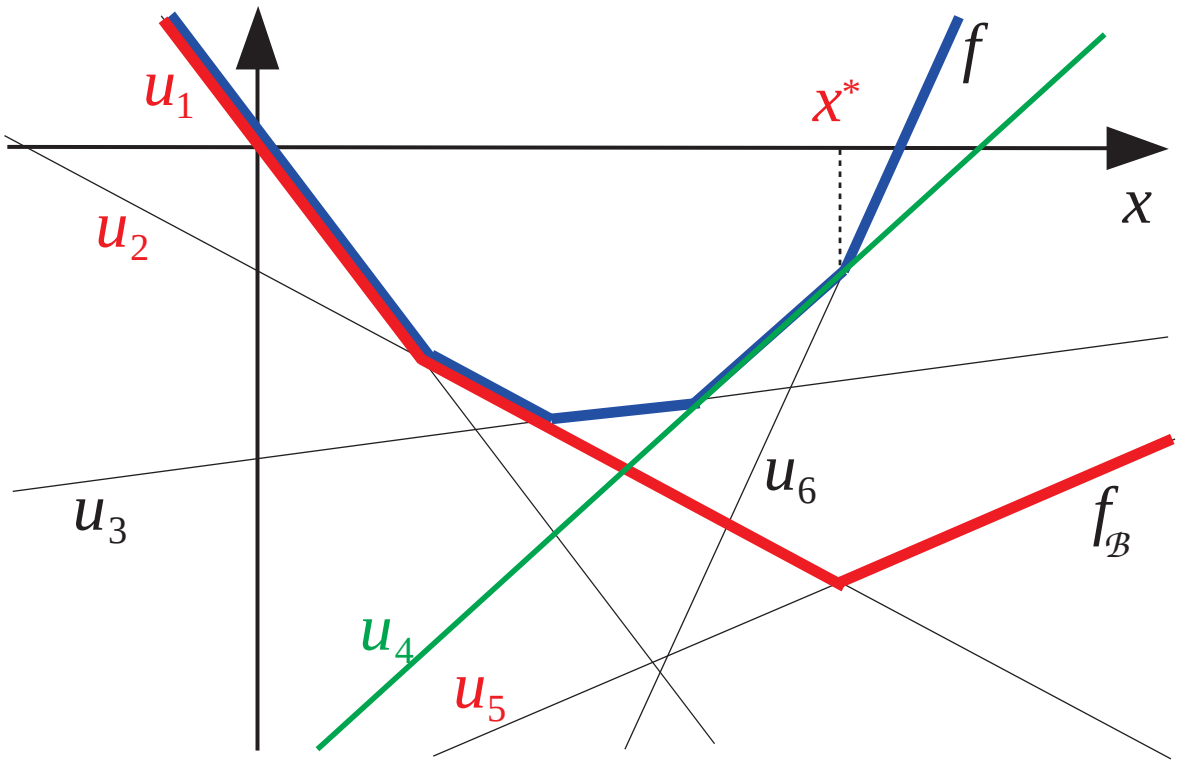

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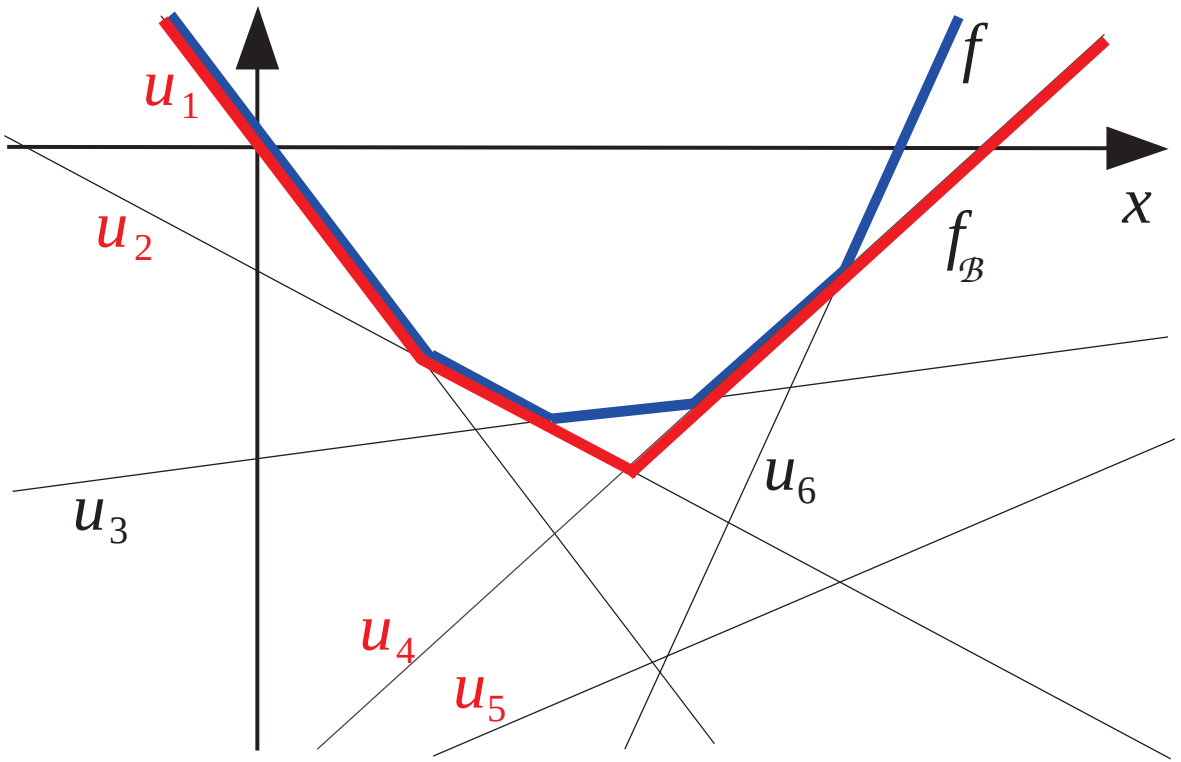

# (Not really) A Divagation: the Decomposable Case

- Decomposable $U = \mathsf{X}_{k \in K}\ U^k$ (Cartesian product), $u = [u^k]_{k \in K}$

  $\equiv \quad f(x) = xb + \sum_{k \in K} \left( f^k(\pi) = \max \left\{ (c^k - xA^k)u^k \ : \ u^k \in U^k \right\} \right)$

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- "Nonstandard" disaggregated primal master problem

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- $|K|$ times larger master problem, but better use of information

  $\Rightarrow$ faster convergence [Jones et al, 1993], for multicommodity flows

- Can we do better than that?

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- Assumption 3: easy update of rows and columns

  Given $\mathcal{B}$, $\bar{u} \in conv(U)$, $\bar{u} \notin U_{\mathcal{B}}$, it is "easy" to find $\mathcal{B}' \supset \mathcal{B}$

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- Structured master problem

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$\equiv$ structured model

$$f_{\mathcal{B}}(x) \ = \ \max\{ \ (c - xA)u + ub \ , \ u = C_{\mathcal{B}}\theta_{\mathcal{B}} \ , \ \Gamma_{\mathcal{B}}\theta_{\mathcal{B}} \leq \gamma_{\mathcal{B}} \ \}$$

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$\langle$ initialize $\mathcal{B}$ $\rangle$;
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    $\langle$ solve $(\Pi_{\mathcal{B}})$ for $u^*$, $x^*$ (duals of $Au = b$); $v^* = cu^*$ $\rangle$;
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  - . . . even if $U$ is non compact and $\mathcal{B} = \emptyset$ at start (Phase 0)
- The subproblem to be solved is identical to that of DW
- Requires ($\Rightarrow$ exploits) extra information on the structure
- Master problem with any structure, possibly much larger

# When/why is this interesting?

- Obviously generalizes DW, whose unstructured model

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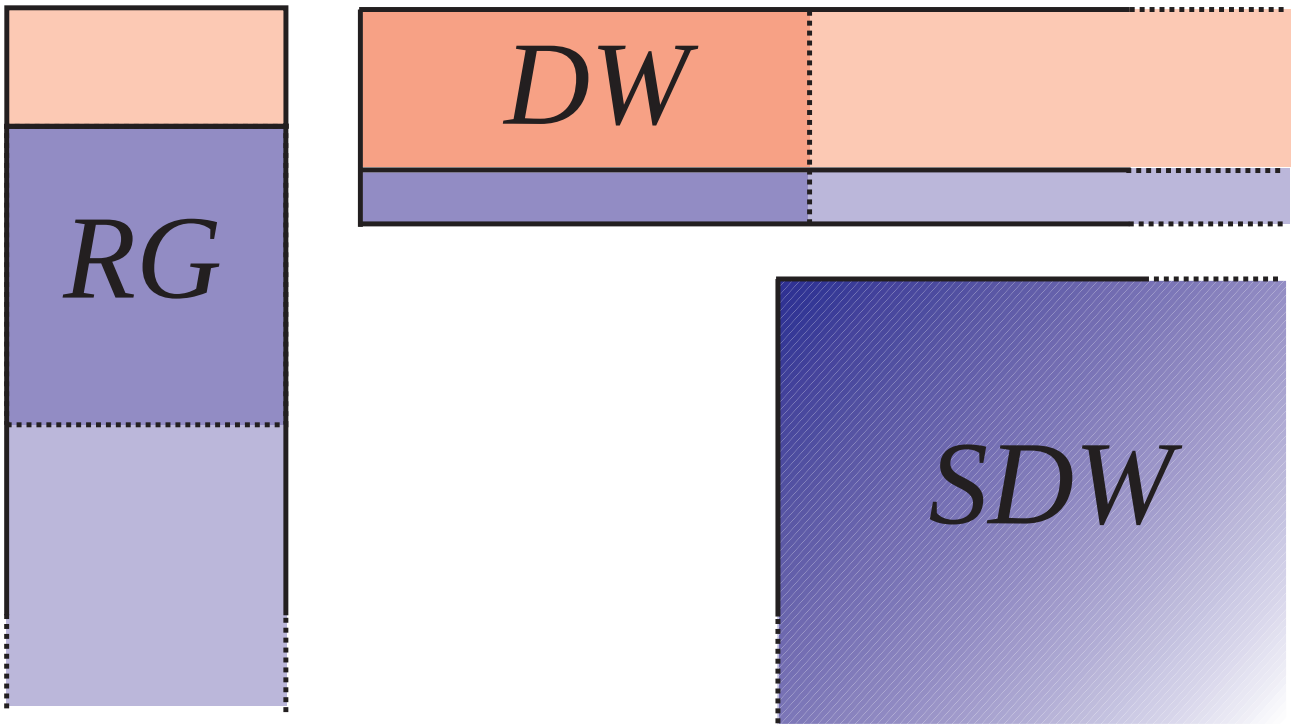
- Substantially different from both RG and DW




(say, a pseudo-polynomial number of variables and constraints)

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- Known possible applications:
  - Cutting stock [de Carvalho '02, Zak '02]
  - Time-constrained routing [Avella et al '06]
  - Multicommodity Capacitated Network Design [F., Gendron, 2009]

# Multicommodity Capacitated Network Design

- Multiple flows (commodities) $(s^k, t^k, d^k)$, $k \in K$, facility costs $f_{ij}$

$$\min \quad \sum_{k \in K} \sum_{(i,j) \in A} d^k c_{ij}^k u_{ij}^k + \sum_{(i,j) \in A} f_{ij} y_{ij}$$

$$\sum_{(i,j) \in A} u_{ij}^k - \sum_{(j,i) \in A} u_{ji}^k = \begin{cases} 1 & \text{if } i = s^k \\ -1 & \text{if } i = t^k \\ 0 & \text{otherwise} \end{cases} \qquad i \in N\,,\ k \in K$$

$$\sum_{k \in K} d^k u_{ij}^k \le a_{ij} y_{ij} \qquad (i,j) \in A$$

$$0 \le x_{ij}^k \le 1 \qquad (i,j) \in A\,,\ k \in K$$

$$y_{ij} \in \mathbb{N} \qquad (i,j) \in A$$

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- Multiple flows (commodities) $(s^k, t^k, d^k)$, $k \in K$, facility costs $f_{ij}$

$$\min \sum_{k \in K} \sum_{(i,j) \in A} d^k c_{ij}^k u_{ij}^k + \sum_{(i,j) \in A} f_{ij} y_{ij}$$

$$\sum_{(i,j) \in A} u_{ij}^k - \sum_{(j,i) \in A} u_{ji}^k = \begin{cases} 1 & \text{if } i = s^k \\ -1 & \text{if } i = t^k \\ 0 & \text{otherwise} \end{cases} \qquad i \in N\ ,\ k \in K$$

$$\sum_{k \in K} d^k u_{ij}^k \leq a_{ij} y_{ij} \qquad (i,j) \in A$$

$$0 \leq x_{ij}^k \leq 1 \qquad (i,j) \in A\ ,\ k \in K$$

$$y_{ij} \in \mathbb{N} \qquad (i,j) \in A$$

- Efficiently optimize on mutiflows + construct the graph
- $\mathcal{NP}$-hard, loads of applications, difficult in practice
- $I$ = integer formulation, $\bar{I}$ = continuous relaxation

# Row Generation vs. Dantzig-Wolfe

- Relax the flow conservation constraints, decompose by arc

$$\min \quad \sum_{(i,j)\in A} \left( \sum_{k\in K} (d^k c_{ij}^k - x_i^k + x_j^k) u_{ij}^k + f_{ij} y_{ij} \right)$$

$$U_{ij} \equiv \left\{ \begin{array}{ll} \sum_{k\in K} d^k u_{ij}^k \leq a_{ij} y_{ij} & \\ 0 \leq u_{ij}^k \leq 1 & k \in K \\ y_{ij} \in \mathbb{N} & \end{array} \right\} \qquad (i,j) \in A$$

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  $\Rightarrow$ better bound than continuous relaxation

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- Residual capacity inequalities have same separation cost

$$a_k = d^k / a_{ij} \qquad S \subseteq K \qquad a(S) = \sum_{k\in S} a_k$$

$$\sum_{k\in S} a_k (1 - u_{ij}^k) \geq (a(S) - \lfloor a(S) \rfloor)(\lceil a(S) \rceil - y_{ij})$$

and describe $conv(U_{ij})$ [Atamturk, 2002] $\Rightarrow v(DW) = v(\bar{I}+)$

# Binary formulation $B$

- Redundant upper bound constraints: $y_{ij} \leq \left\lceil \sum_{k \in K} d^k / a_{ij} \right\rceil = T_{ij}$
- Pseudo-polinomially many segments $S_{ij} = \{ 1, \ldots, T_{ij} \}$ for $y_{ij}$

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- Pseudo-polinomially many segments $S_{ij} = \{ 1, \ldots, T_{ij} \}$ for $y_{ij}$
- Reformulation in binary variables: $y_{ij} = \sum_{s \in S_{ij}} y_{ij}^s$

$$y_{ij}^s = \begin{cases} 1 & \text{if } y_{ij} = s \\ 0 & \text{otherwise} \end{cases} \qquad s \in S_{ij}$$

$$u_{ij}^{ks} = \begin{cases} u_{ij}^k & \text{if } y_{ij} = s \\ 0 & \text{otherwise} \end{cases} \qquad s \in S_{ij} \ , \ k \in K$$

$$(s-1) a_{ij} y_{ij}^s \leq \sum_{k \in K} d^k u_{ij}^{ks} \leq s a_{ij} y_{ij}^s \qquad (i,j) \in A \ , \ s \in S_{ij}$$

$$\sum_{s \in S_{ij}} y_{ij}^s \leq 1 \qquad (i,j) \in A$$

... then original variables can be removed

- Up to now, continuous relaxation bound has not improved

# Improved binary formulation $B+$

- Extended linking inequalities:

$$u_{ij}^{ks} \leq y_{ij}^{s} \qquad (i,j) \in A \ , \ k \in K \ , \ s \in S_{ij}$$

- Improved continuous relaxation bound: $v(\bar{B}+) = v(\bar{I}+) = v(DW)$
  [F., Gendron, 2009] using [Croxton et al., 2003]

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- Pseudo-polynomial number of variables and constraints

- Just perfect :-)

# Computational results for SDW

- Intel Xeon X7350@2.93GHz, 64Gb RAM, Suse Linux, `CPLEX 11.1`
- Large-scale instances ($|K| \in \{100, 200, 400\}$), very difficult
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- Stabilized DW $\equiv$ bundle much better, but only disaggreated
- Solving the root relaxation, then freezing the formulation
  + `CPLEX` polishing for one hour
- Unlike $I+$, frozen $B+$ formulations may not contain optimal solution
  $\Rightarrow$ final gap $\approx$ quality of obtained formulation
- imp = lower bound improvement (equal for all)
  gap = final gap (%), cpu = time, it = iterations

# Sample computational results ($|K| = 100$)

| Problem | | | $I+$ | | | StabDW | | StructDW | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\|A\|$ | $C$ | imp | cpu | gap | it | cpu | it | cpu | gap | it |
| 517 | 1 | 187.00 | 348 | 5.78 | 26 | 4323 | 88144 | 296 | 6.94 | 55 |
| | 4 | 138.22 | 362 | 6.42 | 25 | 3581 | 79390 | 312 | 7.48 | 44 |
| | 8 | 100.08 | 305 | 6.12 | 21 | 4054 | 88807 | 633 | 6.11 | 61 |
| | 16 | 60.49 | 249 | 6.20 | 21 | 3015 | 71651 | 1138 | 6.45 | 87 |
| 517 | 1 | 155.19 | 140 | 3.95 | 23 | 2899 | 69500 | 188 | 4.70 | 60 |
| | 4 | 122.84 | 194 | 3.87 | 26 | 2799 | 65229 | 147 | 4.15 | 39 |
| | 8 | 93.00 | 151 | 3.96 | 20 | 2824 | 66025 | 355 | 4.31 | 67 |
| | 16 | 59.68 | 116 | 4.72 | 18 | 2172 | 56184 | 551 | 4.94 | 70 |
| 669 | 1 | 114.50 | 80 | 0.50 | 26 | 330 | 11273 | 36 | 0.46 | 32 |
| | 4 | 97.32 | 78 | 0.46 | 22 | 327 | 10951 | 66 | 0.46 | 50 |
| | 8 | 79.62 | 68 | 0.46 | 19 | 323 | 11173 | 55 | 0.46 | 33 |
| | 16 | 56.19 | 58 | 0.74 | 19 | 275 | 9979 | 164 | 0.81 | 65 |

- SDW worsens as $C$ grows (tighter capacities), RG the converse

# Sample computational results ($|K| = 200$)

| Problem | | | $I+$ | | | StabDW | | StructDW | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\|A\|$ | $C$ | imp | cpu | gap | it | cpu | it | cpu | gap | it |
| 229 | 1 | 205.67 | 49081 | 28.16 | 109 | 11748 | 154821 | 525 | 10.50 | 44 |
| | 4 | 131.24 | 30899 | 25.40 | 91 | 9132 | 131674 | 807 | 13.58 | 45 |
| | 8 | 84.61 | 16502 | 21.80 | 87 | 12682 | 162766 | 1593 | 10.17 | 44 |
| | 16 | 42.78 | 2090 | 5.59 | 54 | 6541 | 97952 | 2630 | 9.20 | 73 |
| 229 | 1 | 185.17 | 18326 | 20.53 | 86 | 9261 | 132963 | 380 | 7.44 | 39 |
| | 4 | 125.39 | 15537 | 18.81 | 80 | 11791 | 147879 | 612 | 9.36 | 49 |
| | 8 | 85.31 | 9500 | 13.08 | 74 | 10702 | 146727 | 1647 | 8.87 | 68 |
| | 16 | 46.09 | 1900 | 7.19 | 52 | 7268 | 107197 | 3167 | 7.99 | 108 |
| 287 | 1 | 198.87 | 14559 | 27.86 | 66 | 8815 | 120614 | 598 | 12.54 | 53 |
| | 4 | 136.97 | 11934 | 22.52 | 62 | 8426 | 112308 | 603 | 15.07 | 37 |
| | 8 | 92.94 | 9656 | 15.28 | 64 | 10098 | 130536 | 1221 | 10.38 | 41 |
| | 16 | 53.45 | 3579 | 11.60 | 54 | 6801 | 98972 | 3515 | 9.06 | 99 |

- Same trend, but RG better only for $C = 16$

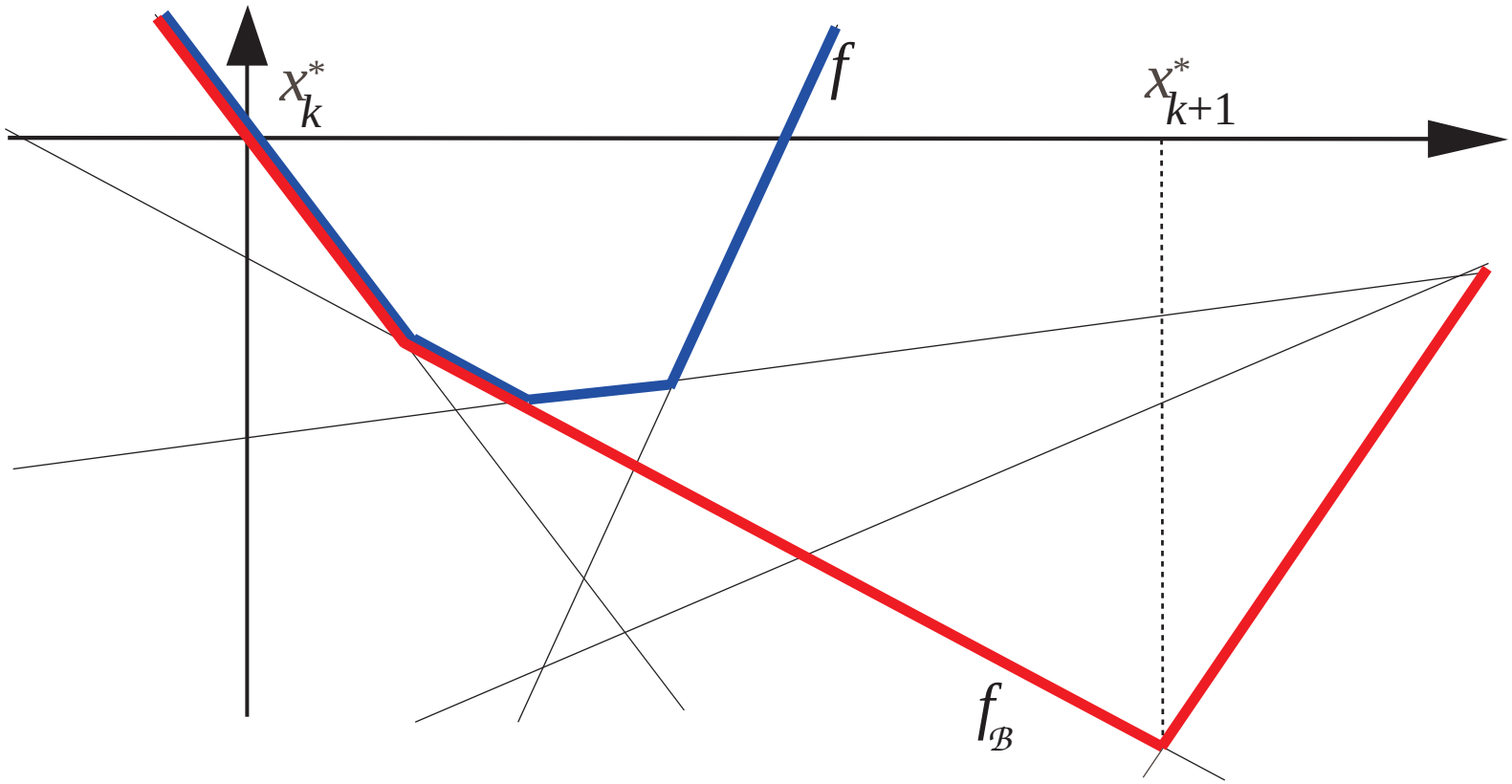
# Sample computational results ($|K| = 400$)

| Problem | | | StabDW | | StructDW | | |
|---|---|---|---|---|---|---|---|
| $\|A\|$ | $C$ | imp | cpu | it | cpu | gap | it |
| 519 | 1 | 100.83 | 87695 | 248746 | 9839 | 9.96 | 157 |
| | 4 | 92.54 | 88031 | 247864 | 9087 | 11.25 | 140 |
| | 8 | 82.16 | 88918 | 258266 | 11613 | 8.47 | 143 |
| | 16 | 65.53 | 85384 | 238945 | 38617 | 10.26 | 242 |
| 519 | 1 | 125.07 | 93065 | 258054 | 22246 | 14.90 | 165 |
| | 4 | 111.02 | 90573 | 250854 | 17976 | 18.22 | 131 |
| | 8 | 94.82 | 93418 | 256884 | 30460 | 18.18 | 159 |
| | 16 | 71.31 | 93567 | 265663 | 74447 | 16.50 | 176 |
| 668 | 1 | 126.02 | 98789 | 246702 | 23771 | 11.89 | 149 |
| | 4 | 115.29 | 99014 | 247620 | 28567 | 10.97 | 176 |
| | 8 | 102.03 | 104481 | 258636 | 27871 | 12.07 | 130 |
| | 16 | 80.96 | 103011 | 278905 | 58363 | 13.95 | 156 |

- SWD always better, stabilizing SDW seems promising

# Instability in (S)DW

- $x^*_{k+1}$ can be very far from $x^*_k$, where $f_{\mathcal{B}}$ is a "bad model" of $f$



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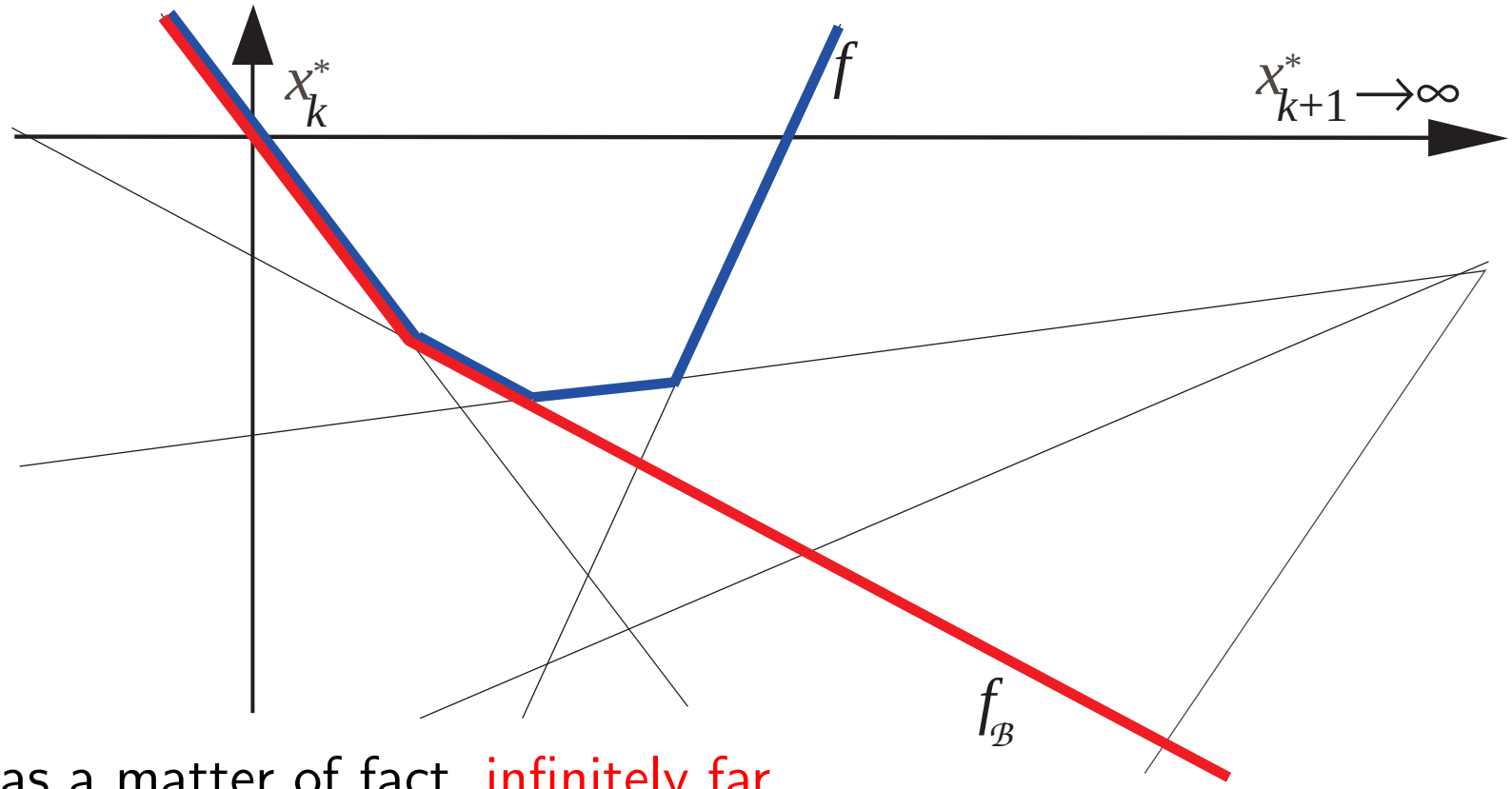


  ... as a matter of fact, infinitely far

- $(\Pi_{\mathcal{B}})$ empty $\equiv$ $(\Delta_{\mathcal{B}})$ unbounded $\Rightarrow$ Phase 0 / Phase 1 approach
- More in general: $\{x^*_k\}$ is unstable, has no locality properties
  $\Rightarrow$ convergence speed does not improve near the optimum

# Stabilizing the (Structured) Dantzig-Wolfe Approach

- Use a Proximal Point method:
  - Current point $\bar{x}$, stabilizing term $\mathcal{D}_t$, proximal parameter(s) $t$
  - Stabilized dual problem $\equiv$ Moreau–Yosida regularization

$$(\Delta_{\bar{x},\mathcal{D},t}) \quad \phi_{\mathcal{D},t}(\bar{x}) = \min \left\{ f(x) + \mathcal{D}_t(x - \bar{x}) \right\}$$

  - Proper $\mathcal{D}_t \Rightarrow \phi_{\mathcal{D},t}$ has same minima as $f$ but is smoother
- Optimal solution $x^*$: $f(x^*) < f(\bar{x})$ ($x^* = \bar{x} \Rightarrow \bar{x}$ optimum)
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- Fenchel's dual of $(\Delta_{\bar{x},\mathcal{D},t})$

$$- \min \left\{ f^*(z) - z\bar{y} + \mathcal{D}_t^*(-z) \right\}$$

$$\equiv \max \left\{ cu + \bar{x}z - \mathcal{D}_t^*(-z) \ : \ z = b - Au \ , \ u \in conv(U) \right\}$$

a generalized augmented Lagrangian of $(\Pi)$: replace $Ax = b$ with "1st-order term" $\bar{x}(b - Au)$ and "2nd-order term" $\mathcal{D}_t^*(Au - b)$

# Stabilizing Terms

- Few general properties:
  - i) $\mathcal{D}_t \geq 0$ convex, $\mathcal{D}_t(0) = 0$ | i) + ii) hold for $\mathcal{D}_t$
  - ii) $S_\delta(\mathcal{D}_t)$ compact and full-dimensional $\forall \delta > 0$ | $\iff$ hold for $\mathcal{D}_t^*$
  - iii) $\mathcal{D}_t$ differentiable in 0 $\iff$ $\mathcal{D}_t^*$ strictly convex in 0
  - iv) $\lim_{\|x\| \to \infty} D_t(x)/\|x\| = +\infty \iff \mathcal{D}_t^* < +\infty$
  - v) $\mathcal{D}_t$ ($\mathcal{D}_t^*$) de(inc)reasing in $t$, $\mathcal{D}_t \to 0$ ($\mathcal{D}_t^* \to I_{\{0\}}$) as $t \to \infty$

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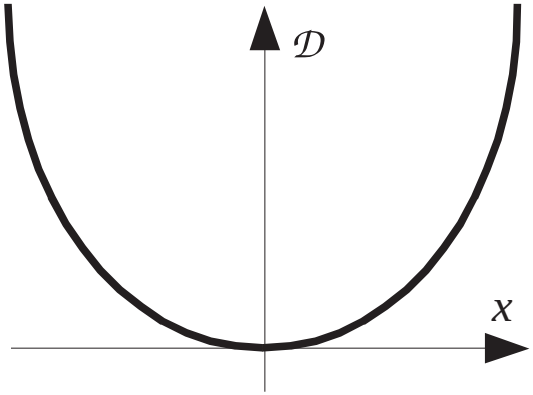
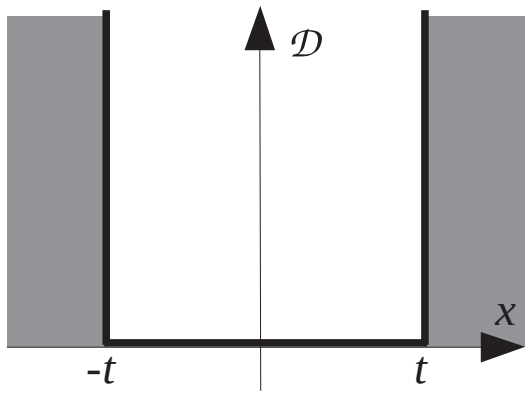
  iii) can be relaxed somewhat, albeit at a cost
- Simple and robust choice: $\|\cdot\|_2^2$ [Lemaréchal et al, 2006]
- Reasonable choices: piecewise-linear functions $\Rightarrow (\Delta_{\bar{x},\mathcal{D},t})$ is a LP
  - 1-piece = boxstep [Marsten et al, 1975]
  - 2-pieces [Kim et al, 1995]
  - 3-pieces [Du Merle et al, 1999]
  - 5-pieces [Ben Hamor et al, 2009]

# In practice

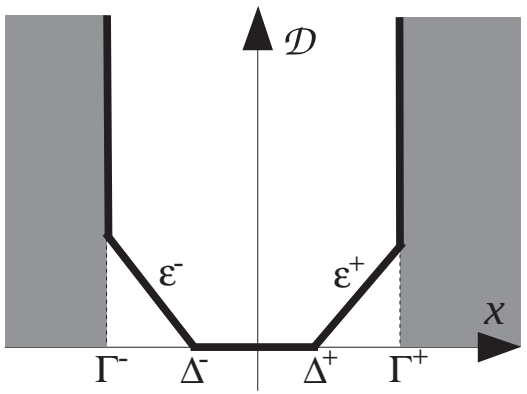
a penalty

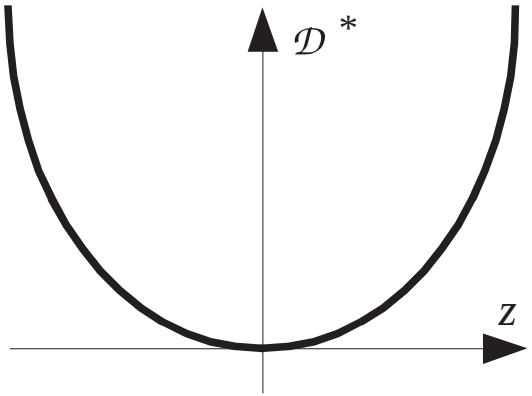


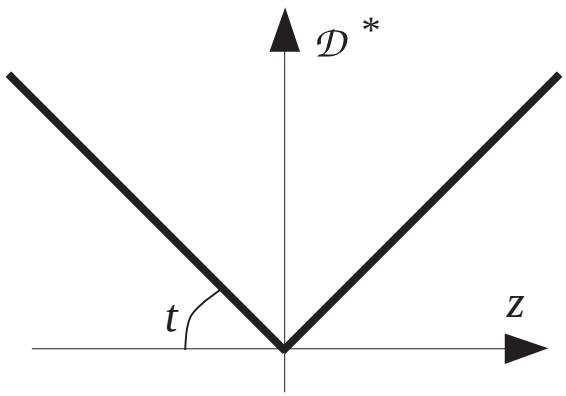

a trust region

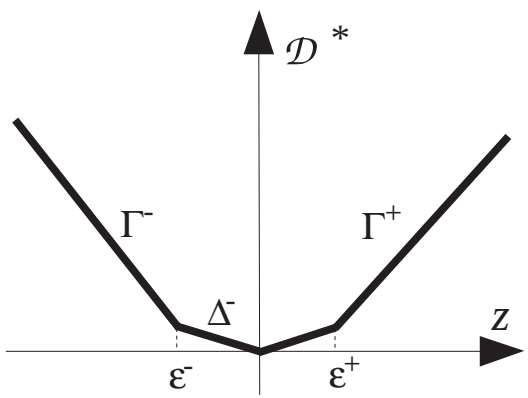



or both




$$\mathcal{D}_t = \frac{1}{2t}\|\cdot\|_2^2$$
$$\mathcal{D}_t^* = \frac{1}{2}t\|\cdot\|_2^2$$

$$\mathcal{D}_t = I_{B_\infty(t)}$$
$$\mathcal{D}_t^* = t\|\cdot\|_1$$

$$\mathcal{D}_{\Gamma^\pm,\Delta^\pm,\varepsilon^\pm} = \cdots$$
$$\mathcal{D}^*_{\Gamma^\pm,\Delta^\pm,\varepsilon^\pm} = \cdots$$

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  obviously not, think of the disaggregated case [F., 2002]

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- Even simpler from the primal viewpoint:

$$\max \left\{ cu + \bar{x}z - \mathcal{D}^*(-z) \ : \ z = b - Au \ , \ u = C_{\mathcal{B}}\theta_{\mathcal{B}} \ , \ \Gamma_{\mathcal{B}}\theta_{\mathcal{B}} \leq \gamma_{\mathcal{B}} \right\}$$

- With proper choice of $\mathcal{D}_t$, still a Linear Program; e.g.

$$\begin{aligned} \max \quad & \ldots - (\Delta^- + \Gamma^-)z_2^- - \Delta^- z_1^- - \Delta^+ z_1^+ - (\Delta^+ + \Gamma^+)z_2^+ \\ & z_2^- + z_1^- - z_1^+ - z_2^+ = b - Au \ , \ \ldots \\ & z_2^+ \geq 0 \ , \ \varepsilon^+ \geq z_1^+ \geq 0 \ , \ \varepsilon^- \geq z_1^- \geq 0 \ , \ z_2^- \geq 0 \end{aligned}$$

- Dual optimal variables of "$z = b - Au$" still give $x^*$

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$$\max \left\{ cu + \bar{x}z - \mathcal{D}^*(-z) \ : \ z = b - Au \ , \ u = C_{\mathcal{B}}\theta_{\mathcal{B}} \ , \ \Gamma_{\mathcal{B}}\theta_{\mathcal{B}} \leq \gamma_{\mathcal{B}} \right\}$$

- With proper choice of $\mathcal{D}_t$, still a Linear Program; e.g.

$$\begin{aligned} \max \quad & \ldots - (\Delta^- + \Gamma^-)z_2^- - \Delta^- z_1^- - \Delta^+ z_1^+ - (\Delta^+ + \Gamma^+)z_2^+ \\ & z_2^- + z_1^- - z_1^+ - z_2^+ = b - Au \ , \ \ldots \\ & z_2^+ \geq 0 \ , \ \varepsilon^+ \geq z_1^+ \geq 0 \ , \ \varepsilon^- \geq z_1^- \geq 0 \ , \ z_2^- \geq 0 \end{aligned}$$

- Dual optimal variables of "$z = b - Au$" still give $x^*$
- Convergence theory basically the same as in [F., 2002]

  even somewhat simpler because $\mathcal{B}$ is inherently finite
- NS/SS decision, handling of $t$, handling of $\mathcal{B}$

# Aggregation & the Structured Dantzig-Wolfe Algorithm

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$$\max \begin{cases} cu + (1-\rho)cu^* + \bar{x}z - \mathcal{D}_t^*(-z) \\ u = C_{\mathcal{B}}\theta_{\mathcal{B}} \ , \ \Gamma_{\mathcal{B}}\theta_{\mathcal{B}} \leq \rho\gamma_{\mathcal{B}} \\ z = Au + (1-\rho)Au^* - b \ , \quad \rho \in [0,1] \end{cases}$$

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- "Knob": $\rho = 0 \ \Rightarrow \ \gamma_{\mathcal{B}} = 0 \ \Rightarrow \ u = u^*$, $\rho = 1 \ \Rightarrow \ u \in U_{\mathcal{B}}$
- Possible use: avoid Phase 0 when $\mathcal{D}_t$ "not steep"
  given $u^* \in conv(U)$ (e.g. $u^* \in U$) such that $Au^* = b$

# Computational results

- Same machine/instances as before
- Comparing SDW with S$^2$DW
- No removal/aggregation for $\mathcal{B}$, fixed $t$ (class-specific tuning)
- Different stabilizing terms: $\mathcal{D}_t = \frac{1}{2t}\|\cdot\|_2^2$ vs $\mathcal{D}_t = I_{B_\infty(t)}$
  (QP vs LP, Lemaréchal vs Marsten)
- Different warm-start: "standard" MCF initialization (used for all) vs MCF + subgradient warm-start (few iterations, class-specific tuning)
- gap = final gap (%), cpu = time, it = iterations, ss = serious steps

# Sample computational results ($|K| = 100$)

| | StructDW | | | $S^2DW_2$ | | | | $S^2DW_\infty$ | | | | $S^2DW_\infty$–$ws^2$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C$ | cpu | gap | it | cpu | gap | it | ss | cpu | gap | it | ss | cpu | gap | it | ss |
| 1 | 296 | 6.94 | 55 | 16380 | 6.57 | 51 | 15 | 223 | 2.97 | 66 | 58 | 357 | 1.52 | 91 | 84 |
| 4 | 312 | 7.48 | 44 | 17091 | 5.87 | 47 | 12 | 298 | 2.72 | 70 | 54 | 270 | 1.48 | 69 | 60 |
| 8 | 633 | 6.11 | 61 | 22176 | 7.16 | 37 | 14 | 280 | 2.70 | 64 | 34 | 277 | 1.44 | 65 | 47 |
| 16 | 1138 | 6.45 | 87 | 27033 | 6.08 | 43 | 18 | 190 | 2.78 | 60 | 21 | 119 | 1.52 | 40 | 18 |
| 1 | 188 | 4.70 | 60 | 5802 | 4.01 | 42 | 13 | 205 | 2.56 | 71 | 57 | 222 | 1.43 | 85 | 71 |
| 4 | 147 | 4.15 | 39 | 6453 | 4.32 | 39 | 15 | 215 | 2.43 | 79 | 40 | 91 | 1.39 | 41 | 36 |
| 8 | 354 | 4.31 | 67 | 5752 | 4.40 | 31 | 12 | 167 | 2.38 | 62 | 25 | 124 | 1.42 | 50 | 21 |
| 16 | 551 | 4.94 | 70 | 10154 | 5.07 | 40 | 14 | 163 | 2.76 | 61 | 20 | 113 | 1.53 | 50 | 19 |
| 1 | 36 | 0.46 | 32 | 2405 | 0.46 | 47 | 15 | 84 | 0.41 | 76 | 48 | 78 | 0.33 | 72 | 66 |
| 4 | 66 | 0.46 | 50 | 1964 | 0.46 | 45 | 14 | 67 | 0.41 | 74 | 24 | 81 | 0.33 | 73 | 56 |
| 8 | 55 | 0.46 | 33 | 1974 | 0.46 | 44 | 15 | 50 | 0.41 | 57 | 18 | 40 | 0.33 | 49 | 20 |
| 16 | 164 | 0.81 | 65 | 1408 | 0.80 | 38 | 17 | 47 | 0.61 | 52 | 16 | 44 | 0.40 | 52 | 22 |

- $S^2DW_2$ converges faster but slow, $ws^2$ best in gap and often time

# Sample computational results ($|K| = 200$)

| | StructDW | | | $S^2DW_2$ | | | | $S^2DW_\infty$ | | | | $S^2DW_\infty$–$ws^2$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C$ | cpu | gap | it | cpu | gap | it | ss | cpu | gap | it | ss | cpu | gap | it | ss |
| 1 | 525 | 10.50 | 44 | 1.8e4 | 12.11 | 32 | 17 | 860 | 4.16 | 76 | 73 | 907 | 1.32 | 129 | 119 |
| 4 | 807 | 13.58 | 45 | 2.7e4 | 10.20 | 29 | 15 | 1091 | 2.79 | 89 | 87 | 1460 | 1.23 | 126 | 118 |
| 8 | 1593 | 10.17 | 44 | 8.3e4 | 10.12 | 40 | 17 | 1027 | 3.03 | 78 | 61 | 1237 | 1.20 | 99 | 77 |
| 16 | 2630 | 9.20 | 73 | 1.1e5 | 9.21 | 54 | 16 | 399 | 2.12 | 65 | 31 | 804 | 1.02 | 114 | 73 |
| 1 | 380 | 7.44 | 39 | 1.0e4 | **** | 29 | 14 | 557 | 2.61 | 80 | 71 | 592 | 1.30 | 101 | 95 |
| 4 | 612 | 9.36 | 49 | 1.3e4 | 10.33 | 25 | 15 | 755 | 2.87 | 80 | 68 | 930 | 1.22 | 98 | 95 |
| 8 | 1647 | 8.87 | 68 | 3.3e4 | 10.61 | 30 | 14 | 468 | 2.75 | 50 | 43 | 761 | 1.33 | 83 | 66 |
| 16 | 3167 | 7.99 | 108 | 7.0e4 | 8.32 | 47 | 17 | 476 | 2.22 | 67 | 30 | 357 | 1.10 | 53 | 39 |
| 1 | 598 | 12.54 | 53 | 2.1e4 | 16.31 | 39 | 15 | 1019 | 3.92 | 98 | 93 | 1327 | 1.65 | 149 | 143 |
| 4 | 603 | 15.07 | 37 | 1.8e4 | 13.78 | 27 | 15 | 1001 | 3.72 | 90 | 79 | 891 | 1.60 | 98 | 94 |
| 8 | 1221 | 10.38 | 41 | 5.2e4 | 11.81 | 29 | 14 | 909 | 3.68 | 73 | 50 | 1040 | 1.63 | 102 | 96 |
| 16 | 3515 | 9.06 | 99 | 1.3e5 | 10.11 | 54 | 17 | 513 | 2.93 | 59 | 25 | 555 | 1.26 | 62 | 45 |

- $S^2DW_2$ exceedingly slow, $ws^2$ best in gap, not always time

# Sample computational results ($|K| = 400$)

| | StructDW | | | $S^2DW_\infty$ | | | | $S^2DW_\infty$–$ws^2$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $C$ | cpu | gap | it | cpu | gap | it | ss | cpu | gap | it | ss |
| 1 | 9839 | 9.96 | 157 | 2473 | 2.23 | 76 | 55 | 1857 | 2.31 | 53 | 38 |
| 4 | 9087 | 11.25 | 140 | 2140 | 2.33 | 68 | 54 | 2487 | 2.36 | 66 | 44 |
| 8 | 11613 | 8.47 | 143 | 2338 | 2.45 | 66 | 45 | 1813 | 2.30 | 52 | 30 |
| 16 | 38617 | 10.26 | 242 | 3403 | 2.66 | 77 | 39 | 2570 | 2.26 | 58 | 23 |
| 1 | 22246 | 14.90 | 165 | 4811 | 3.31 | 87 | 76 | 4668 | 3.06 | 66 | 55 |
| 4 | 17976 | 18.22 | 131 | 4324 | 2.57 | 77 | 64 | 4373 | 3.19 | 66 | 45 |
| 8 | 30460 | 18.18 | 159 | 5224 | 3.14 | 85 | 60 | 4209 | 2.86 | 57 | 36 |
| 16 | 74447 | 16.50 | 176 | 5532 | 3.14 | 67 | 46 | 5191 | 3.02 | 64 | 23 |
| 1 | 23771 | 11.89 | 149 | 9215 | 2.96 | 97 | 78 | 6815 | 3.01 | 69 | 56 |
| 4 | 28567 | 10.97 | 176 | 6766 | 2.99 | 79 | 63 | 6506 | 3.07 | 69 | 45 |
| 8 | 27871 | 12.07 | 130 | 7560 | 2.67 | 87 | 56 | 5765 | 2.78 | 61 | 37 |
| 16 | 58363 | 13.95 | 156 | 8626 | 3.14 | 83 | 45 | 3764 | 2.95 | 41 | 18 |

- SDW always slower, $ws^2$ most often faster, $S^2DW$ gaps much better

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  just a bundle method with a different model
- Implementation non straightforward but possible
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- Stabilizing SDW possible, little theoretical issues:
  just a bundle method with a different model
- Implementation non straightforward but possible
- Computational results quite promising
- To do: implement generic version (`FiOracle` class)
- To do: application to other interesting problems
- To do: something better than `CPLEX` to solve the quadratic version

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