Unit Commitment Strikes Again: the Convex Hull of Star-Shaped MINLPs

Antonio Frangioni\textsuperscript{1}

(with T. Bacci\textsuperscript{2}, C. Gentile\textsuperscript{2}, K. Tavlaridis-Gyparakis\textsuperscript{1})

\textsuperscript{1}Dipartimento di Informatica, Università di Pisa
\textsuperscript{2}Istituto di Analisi dei Sistemi ed Informatica “Antonio Ruberti”, C.N.R.

24\textsuperscript{th} Combinatorial Optimization Workshop
Aussois (France), January 7, 2020
1. (Perennial) Motivation: Unit Commitment Problems
2. From a DP algorithm to a new MIP formulation
3. Star-shaped MINLPs
4. More practical formulations
5. Computational results
6. Conclusions
The Electrical System

- Electrical system: the most complex machine mankind has developed

- Several sources of complexity:
  1. electricity is difficult to store \( \implies \) must be mostly produced exactly when needed
  2. electricity is difficult to route, goes where Kirchoff’s laws say
  3. growing renewables production is highly uncertain
  4. almost everything is (more or less highly) nonlinear

- All manner of (nasty) optimization problems, spanning from multi decades to sub-second

- Unit Commitment is one of the basic steps
The Unit Commitment problem

- Schedule a set of generating units over a time horizon $\mathcal{T}$ (hours/15m in day/week) to satisfy the (forecasted) demand $d_t$ at each $t \in \mathcal{T}$

- Gazzillions €€€ / $$$, enormous amount of research$^{1,2}$

- Different types of production units, different constraints:
  - Thermal (comprised nuclear): min/max production, min up/down time, ramp rates on production increase/decrease, start-up cost depending on previous downtime, others (modulation, . . . )
  - Hydro (valleys): min/max production, min/max reservoir volume, time delay to get to the downstream reservoir, others (pumping, . . . )
  - Non programmable (ROR hydro) intermittent units (solar/wind, . . . )
  - Fancy things (small-scale storage, demand response, smart grids, . . . )

- Plus the interconnection network (AC/DC, transmission/distribution) and reliability (primary/secondary reserve, $n-1$ units, . . . )

---

$^1$ van Ackooij, Danti Lopez, F., Lacalandra, Tahanan “Large-scale Unit Commitment Under Uncertainty […]” AOR, 2018

$^2$ The plan4res project: https://www.plan4res.eu/
Algorithmic approaches

- Many different pieces, many different forms of structure
- Very well-suited for decomposition methods\(^3\)
- Especially in the uncertain case\(^4\)
- Actually making this work in practice far from obvious

---


Algorithmic approaches

- Many different pieces, many different forms of structure
- Very well-suited for decomposition methods\(^3\)
- Especially in the uncertain case\(^4\)
- Actually making this work in practice far from obvious
- One would need a structured modelling system

Algorithmic approaches

- Many different pieces, many different forms of structure
- Very well-suited for decomposition methods\(^3\)
- Especially in the uncertain case\(^4\)
- Actually making this work in practice far from obvious
- One would need a structured modelling system
  . . . but this is another story
- Suffices to say, focusing on each relevant structure makes sense
- Our structure today: thermal units
- There are many others (hydro units\(^5\), . . .)

MIP Formulations of thermal units

- Standard formulations in natural variables $u^i_t \in \{0, 1\}$ and $p^i_t \in \mathbb{R}_+$: on/off state and power level of thermal unit $i \in P$ at time $t \in T$

- Standard constraints: maximum and minimum power output

$$\bar{p}^i_{\min} u^i_t \leq p^i_t \leq \bar{p}^i_{\max} u^i_t \quad t \in T \quad (1)$$

- Ramp-up/down constraints ($\Delta^i_+/\Delta^i_- = $ ramp-up/down limit, $\bar{l}^i/\bar{u}^i = $ start-up/shut-down limit)

$$p^i_t \leq p^i_{t-1} + u^i_{t-1} \Delta^i_+ + (1 - u^i_{t-1})\bar{l}^i \quad t \in T \quad (2)$$

$$p^i_{t-1} \leq p^i_t + u^i_t \Delta^i_- + (1 - u^i_t)\bar{u}^i \quad t \in T \quad (3)$$

- Min up/down-time constraints ($\tau^i_+/\tau^i_- = $ min up/down-time)

$$u^i_t \leq 1 - u^i_{r-1} + u^i_r \quad t \in T, \ r \in [t - \tau^i_+, t - 1] \quad (4)$$

$$u^i_t \geq 1 - u^i_{r-1} - u^i_r \quad t \in T, \ r \in [t - \tau^i_-, t - 1] \quad (5)$$
MIP Formulations of thermal units (finish.d)

Objective function:

\[
\min \sum_{i \in P} \left( s^i(u^i) + \sum_{t \in T} \left( a_t^i(p_t^i)^2 + b_t^i p_t^i + c_t^i u_t^i \right) \right) \tag{6}
\]

- **convex nonlinear** energy cost \((a_t^i > 0)\)
- **time-dependent start-up costs** \(s^i(u^i)\): require some extra constraints and continuous variables

Global constraints: at least demand satisfaction

\[
\sum_{i \in P} p_t^i = \bar{d}_t \quad t \in T \tag{7}
\]

+ possibly several others (reserve, pollution, . . .)

- **Already a nasty MIQP**, unsolvable for few 10s of units (as-is)
- And this is the “academic” version, real-world ones are much worse
- **Especially since it needs be solved “unreasonably fast”**

---

\[^6\] Nowak, Römisch “Stochastic Lagrangian Relaxation Applied to Power Scheduling [. . .]”, Annals O.R., 2000
Improved MIP formulations (1)

- Convex hull of the min-up/down constraints (4)/(5) known\(^7\): exponential number of constraints, but separable in poly time

- Indeed, extended formulation\(^8\): start-up/shut-down \(v^i_t/w^i_t\) variables

\[
u^i_t - u^i_{t-1} = v^i_t - w^i_t \quad t \in T\]  \(8\)

- Can be extended to start-up/shut-down limits\(^9\) \((\tau_+ \geq 2 \neq \tau_+ = 1)\)

\[
\begin{align*}
p_1 &\leq \bar{p}_{\text{max}} u_t - (\bar{p}_{\text{max}} - \bar{u})w_{t+1} \\
p_t &\leq \bar{p}_{\text{max}} u_t - (\bar{p}_{\text{max}} - \bar{l})v_t - (\bar{p}_{\text{max}} - \bar{u})w_{t+1} \quad t \in [2, |T| - 1] \\
p_T &\leq \bar{p}_{\text{max}} u_t - (\bar{p}_{\text{max}} - \bar{l})v_t
\end{align*}
\]

---

\(^7\) Lee, Leung, Margot, “Min-up/Min-down polytopes”, *Disc. Opt.*, 2004

\(^8\) Rajan, Takriti, “Minimum Up/Down polytopes of the unit commitment problem with start-up costs”, IBM RC23628, 2005

Improved MIP formulations (2)

- Ramp-up and Ramp-down polytopes studied separately\textsuperscript{10}

- Ramp-up, convex hull for two-period case

\[
\begin{align*}
p_{\text{min}} u_t & \leq p_t \leq p_{\text{max}} u_t \\
0 & \leq v_{t+1} \leq u_{t+1} \\
u_{t+1} - u_t & \leq v_{t+1} \leq 1 - u_t \\
p_{\text{min}} u_{t+1} & \leq p_{t+1} \leq p_{\text{max}} u_{t+1} - (p_{\text{max}} - \bar{l}) v_{t+1} \\
p_{t+1} - p_t & \leq (p_{\text{min}} + \Delta) u_{t+1} + (\bar{l} - p_{\text{min}} - \Delta) v_{t+1} - p_{\text{min}} u_t
\end{align*}
\]

- Some valid/facet defining inequalities for the general case

- Strengthened ramp-up/down constraints under some conditions\textsuperscript{11}


\textsuperscript{11} Ostrowski, Anjos, Vannelli “Tight [...] formulations for the unit commitment problem” IEEE TPWRS, 2012
Improved MIP formulations (3)

- Convex quadratic objective function with semi-continuous variables:
  \[ \sum_{t \in T} a_t^i (p_t^i)^2 / u_t^i + b_t^i p_t^i + c_t^i u_t^i \]

- Several ways to deal with the “more nonlinearity”\(^{14,15}\)

- Start-up cost is a concave in previous shut-down period length \(\tau\):
  \[ cs(\tau) = V(1 - e^{-\lambda \tau}) + F \text{ (only required for integer } \tau) \]

- Convex hull description of the start-up cost fragment: extended formulation with temperature variables\(^{16}\)

---

\(^{12}\) F., Gentile, “Perspective cuts for a class of convex 0-1 mixed integer programs”, Math. Prog., 2006
\(^{13}\) F., Gentile, Lacalandra, “Tighter approximated MILP formulations for Unit Commitment Problems”, IEEE TPWRS, 2009
\(^{14}\) F., Gentile “A Computational Comparison of [ . . . ]: SOCP vs. Cutting Planes” ORL 2009
\(^{15}\) F., Furini, Gentile “Approximated Perspective Relaxations: a Project&Lift Approach” COAP 2016
\(^{16}\) Silbernagl, Huber, Brandenberg “[ . . . ] MIP Unit Commitment by Modeling Power Plant Temperatures”, IEEE TPWRS, 2016
Improved MIP formulations (3)

- **Convex quadratic** objective function with semi-continuous variables:

  \[ \sum_{t \in T} a_t i (p_t)^2 / u_t + b_i p_t + c_i u_t \]

- Several ways to deal with the “more nonlinearity”\(^ {14,15} \)

- Start-up cost is a concave in previous shut-down period length \( \tau \):

  \[ cs(\tau) = V(1 - e^{-\lambda \tau}) + F \] (only required for integer \( \tau \))

- Convex hull description of the start-up cost fragment: extended formulation with **temperature variables**\(^ {16} \)

- All these works deal with **partial fragments** of the (thermal) (single-)Unit Commitment problem

---

\(^{12}\) F., Gentile, “Perspective cuts for a class of convex 0-1 mixed integer programs”, *Math. Prog.*, 2006

\(^{13}\) F., Gentile, Lacalandra, “Tighter approximated MILP formulations for Unit Commitment Problems”, *IEEE TPWRS*, 2009

\(^{14}\) F., Gentile “A Computational Comparison of [ . . . ]: SOCP vs. Cutting Planes” *ORL* 2009


\(^{16}\) Silbernagl, Huber, Brandenberg “[ . . . ] MIP Unit Commitment by Modeling Power Plant Temperatures”, *IEEE TPWRS*, 2016
1. (Perennial) Motivation: Unit Commitment Problems
2. From a DP algorithm to a new MIP formulation
3. Star-shaped MINLPs
4. More practical formulations
5. Computational results
6. Conclusions
An improved DP algorithm\textsuperscript{17} based on the state-space graph $G$:

- **nodes** $(t, \uparrow)/(t, \downarrow)$: unit starts up/shuts down at time $t$
- **arc** $((h, \uparrow), (k, \downarrow))$ with $k - h + 1 \geq \tau^+$: unit on from $h$ to $k$ (endpoints included)
- **arc** $((h, \downarrow), (k, \uparrow))$ with $k - k - 2 \geq \tau^-$: unit off from $h + 1$ to $k - 1$
- An $s$-$d$ path from represents a schedule for the unit

\[ \text{“on” arcs } ((h, \uparrow), (k, \downarrow)): \text{ optimal dispatching cost } z_{hk}^* + \sum_{t=h}^{k} c_t^i \]

\[ \text{“off” arcs } ((h, \downarrow), (k, \uparrow)): \text{ start-up cost for } k - h - 2 \text{ off time periods} \]

“on” arcs cost = Economic Dispatch

- Optimal dispatch cost $z^*_{hk}$: solving the Economic Dispatch problem $(ED_{hk})$ on $p_h, p_{h+1}, \ldots, p_k$

\[
z^*_{hk} = \min \sum_{t=h}^{k} f^t(p_t)
\]

\[p_{\min} \leq p_h \leq \bar{l}
\]

\[p_{\min} \leq p_t \leq p_{\max} \quad h + 1 \leq t \leq k - 1
\]

\[p_{\min} \leq p_k \leq \bar{u}
\]

\[p_{t+1} - p_t \leq \Delta_+ \quad t = h, \ldots, k - 1
\]

\[p_t - p_{t+1} \leq \Delta_- \quad t = h, \ldots, k - 1
\]

- Complexity:
  - acyclic graph $O(n)$ nodes, $O(n^2)$ arcs $\Rightarrow O(n^2)$ for optimal path
  - $O(n^3)$ for computing costs (specialized inner DP for $(ED_{hk})$)

$\Rightarrow O(n^3)$ overall
... to a new MIP formulation

- arc variables $y_{hk}^{on}$ on arc $((h, \uparrow), (k, \downarrow))$, $y_{hk}^{off}$ on arc $((h, \downarrow), (k, \uparrow))$

- network matrix $E$ for $G$, rhs vector $b$ for $s$-$d$ path

$$Ey = b$$ (15)

- power variables $p_t^{hk}$ for $t = h, \ldots, k$ for each “on” arc $((h, \uparrow), (k, \downarrow))$

\[
\begin{align*}
\bar{p}_{min}y_{hk}^{on} & \leq p_t^{hk} \leq \bar{y}_{on}^{hk} \\
\bar{p}_{min}y_{hk}^{on} & \leq p_t^{hk} \leq p_{max}y_{on}^{hk} & t = h + 1, \ldots, k - 1 \\
\bar{p}_{min}y_{hk}^{on} & \leq p_k^{hk} \leq \bar{u}y_{on}^{hk}
\end{align*}
\] \quad \forall (h, k) \quad (16)

\[
\begin{align*}
p_{t+1}^{hk} - p_t^{hk} & \leq y_{on}^{hk}\Delta_{+} & t = h, \ldots, k - 1 \\
p_t^{hk} - p_{t+1}^{hk} & \leq y_{on}^{hk}\Delta_{-} & t = h, \ldots, k - 1
\end{align*}
\]

(15)–(16) describes the convex hull if objective linear\(^{18}\)

- Slightly $\neq$ version (independently obtained) use DP to separate cuts\(^{19}\)

\[^{18}\text{F., Gentile “New MIP Formulations for the Single-Unit Commitment Problems with Ramping Constraints”, IASI RR, 2015}
\]
\[^{19}\text{Knueven, Ostrowski, Wang, “Generating Cuts from the Ramping Polytope for the Unit Commitment [...]”, OO 5099, 2015}
\]
About the new formulation

- $O(n^2)$ binary + $O(n^3)$ continuous variables, $O(n^3)$ constraints
- Computational usefulness dubious (but perfect for Structured DW$^{20}$)
- Convex hull proof use well-known polyhedral result
- Known for linear problems, but no reason to really require linearity
- In fact, “easy” to generalise to MI-SOCPs$^{21}$
- Useful because Perspective Reformulation is SOCP-representable:
  \[ v \geq \frac{ap^2}{u} \equiv uv \geq ap^2 \text{ (if } u \geq 0) \equiv \text{rotated SOCP constraint} \]
- And Perspective Reformulation describes the convex envelope
- General result: appropriate composition of convex hulls gives the convex hull

$^{21}$ Bacci, F., Gentile Tavlaridis-Gyparakis “New MI-SOCP Formulations for the Single-Unit Commitment […]”, IASI RR, 2019
The result: preliminaries

- Nonlinear version of “Approach no. 4”\textsuperscript{22} known since Edmonds\textsuperscript{23}
- Uses duality, hence in the nonlinear case has to be Lagrangian (was conic duality in the SOCP case)
- Closed convex \( C = \{ z \in \mathbb{R}^n : f(z) \leq 0 \} \) and its mixed-integer restriction \( S = \{ z \in C : z_k \in \mathbb{Z} \text{ for } k \in K \subseteq \{ 1, \ldots, n \} \} \)
- Arbitrary objective function \( c \in \mathbb{R}^n \), support function of \( c \):
  \[ \sigma_C(c) = \inf \{ cz : z \in C \} \]
- Arbitrary objective function \( c \in \mathbb{R}^n \), dual function of \( C \):
  \[ \sigma_C(c) \geq D(c) = \sup_{\lambda \geq 0} \{ L(\lambda; c) = \inf \{ cz + \lambda f(z) \} \} \]

\textsuperscript{22} Wolsey, “Integer Programming”, 1998
\textsuperscript{23} Edmonds, “Matroids and the greedy algorithm”, Math. Prog., 1971
Basic convex analysis: the support function does not distinguish a set from its convex hull \( \Rightarrow \) if the condition
\[
\forall \ c \in \mathbb{R}^n \ \sigma_S(c) = \inf \{ \ cz : \ z \in S \} = D(c)
\]
holds, then \( C = \text{conv}(S) \)

Dual convex hull proof: \( \forall \ c \ \text{exhibit} \ \lambda^* \ \text{s.t.} \ L(\lambda^*; c) = \sigma_S(c) \)

Depends on the description \( f \) of \( C \)

Assumption

For each (closed convex) set \( C \) represented by constraint functions \( f = [f_i]_{i=1,\ldots,m} : \mathbb{R}^n \rightarrow \mathbb{R}^m \), each \( f_i \in C^1 \) and conditions hold such that the KKT conditions are both necessary and sufficient for global optimality

Standard constraints qualification for \( f \) convex, but need not be\(^{24}\)

The result: composition operation

- Two sets $S^h \subset \mathbb{R}^{n_h} \times \mathbb{R}$ for $h = 1, 2$

- 1-sum composition:

$$S^1 \oplus S^2 = \{ (x^1, x^2, y) \in \mathbb{R}^{n_1+n_2+1} : (x^h, y) \in S^h \quad h = 1, 2 \}$$

“$S^1$ and $S^2$ only share the single variable $y$”

- 1-sum composition preserves both convexity and closedness

- The result: under mild assumptions, the 1-sum composition of convex hulls is the convex hull of the 1-sum composition
The result: statement

Lemma

For $h = 1, 2$, let $S^h \subset \mathbb{R}^{n_h} \times \mathbb{R}$. If:

i) the closed (convex) sets

$$C^h = \{ (x^h, y) \in \mathbb{R}^{n_h+1} : y \geq 0 , \ f^h(x^h, y) \leq 0 \} \quad (18)$$

describe the convex hull of $S^h$

ii) Assumption 1 holds

iii) $(x^h, y) \in S^h$ implies that $y \in \{0, 1\}$

iv) $\exists$ points $(\bar{x}^h, 0) \in S^h$ and $(\bar{x}^h, 1) \in S^h$, for $h = 1, 2$, then $C^1 \oplus C^2 = \text{conv}(S^1 \oplus S^2)$
The result: sketch of proof (1)

- Arbitrarily choose \((c^1, c^2, d) \in \mathbb{R}^{n_1+n_2+1}\)

- Define \(L = \min \{ c^1 x^1 + c^2 x^2 + dy : (x^h, y) \in S^h \quad h = 1, 2 \} \) and \(L \geq \Pi = \inf \{ c^1 x^1 + c^2 x^2 + dy : (x^h, y) \in C^h \quad h = 1, 2 \} \)

- Define the Lagrangian Dual of the latter

\[
\Delta = \sup_{\lambda^0 \geq 0, \lambda^1 \geq 0, \lambda^2 \geq 0} \left\{ L(\lambda^0, \lambda^1, \lambda^2) \right\},
\]

where \(L(\lambda^0, \lambda^1, \lambda^2) = \)

\[
\inf_{x^1, x^2, y \geq 0} \left\{ c^1 x^1 + c^2 x^2 + (d - \lambda^0)y + \lambda^1 f^1(x^1, y) + \lambda^2 f^2(x^2, y) \right\}
\]

- Prove that \(L = \Delta\)
The result: sketch of proof (2)

- By the assumptions the optimal solutions satisfies

\[ c^1 + \lambda^1 J_x f^1(x^1, y) = 0 \]  \hspace{1cm} (19a)
\[ c^2 + \lambda^2 J_x f^2(x^2, y) = 0 \]  \hspace{1cm} (19b)
\[ d - \lambda^0 + \lambda^1 J_y f^1(x^1, y) + \lambda^2 J_y f^2(x^2, y) = 0 \]  \hspace{1cm} (19c)
\[ \lambda^0 y = 0 \]  \hspace{1cm} (19d)
\[ \lambda^1 f^1(x^1, y) = 0 \]  \hspace{1cm} (19e)
\[ \lambda^2 f^2(x^2, y) = 0 \]  \hspace{1cm} (19f)

- For \( h = 1, 2 \) and fixed \( y \in \{0, 1\} \) define

\[ L^h_y = \min \{ c^h x^h + d y : (x^h, y) \in C^h \} . \]
The result: sketch of proof (3)

- For $h = 1, 2$ define the problems (equivalent since $C^h = \text{conv}(S^h)$)

  \[
  \sigma^h = \min \left\{ c^h x^h + (d + L^h_0 - L^h_1)y : (x^h, y) \in S^h \right\} \quad (20)
  \]

  \[
  \bar{\sigma}^h = \min \left\{ c^h x^h + (d + L^h_0 - L^h_1)y : (x^h, y) \in C^h \right\} \quad (21)
  \]

- Crucial property: $\bar{\sigma}^h = \sigma^h = L^h_0 \implies \text{both } y = 0 \text{ and } y = 1 \text{ is optimal}$

- Have dual solutions that satisfy KKT

  \[
  c^h + \lambda^h J_x f^h(x^h, y) = 0 \quad (22a)
  \]

  \[
  d + L^h_0 - L^h_1 - \lambda^0 + \lambda^h J_y f^h(x^h, y) = 0 \quad (22b)
  \]

  \[
  \lambda^0 y = 0 \quad (22c)
  \]

  \[
  \lambda^h f^h(x^h, y) = 0 \quad (22d)
  \]

  for both $y = 0$ and $y = 1$
The result: sketch of proof (4)

- Now, “easy” case: $L = L_0 \leq L_1$, i.e., $y = 0$ is optimal
- Can construct solution of (19) using these of (22) for $y = 0$
- “Complicated” case: $L = L_1 < L_0$, i.e., $y = 1$ is optimal
- Further auxiliary problem
  \[ \sigma = \min \{ (L - L_0)y : (x^1, y) \in S^1 \} = \min \{ (L - L_0)y : (x^1, y) \in C^1 \} \]
  where every $(x^1, 1) \in C^1$ is optimal, with KKT
  \[ \tilde{\lambda}^1 J_x f^1(\tilde{x}^1, 1) = 0 \]  
  \[ L - L_0 + \tilde{\lambda}^1 J_y f^1(\tilde{x}^1, 1) = 0 \]  
  \[ \tilde{\lambda}^h f^h(\tilde{x}^h, 1) = 0 \]
- Can construct solution of (19) using these of (22) for $y = 1$ and (23)
- The last step requires $f_i \in C^1$, which is used nowhere else (!?!)
Star-shaped MINLP: constructed by a set of 1-sum compositions

If each piece has the convex hull property, so does the MINLP

Our formulation is of this kind:

- network flow has the integrality property
- for generic convex $f$, the Perspective Reformulation

$$z^{hk} \geq \sum_{t \in T(h,k)} y^{hk} f(p^{hk}_t/y^{hk})$$

describes the convex hull (all $p^{hk}_t$ depend on the same $y^{hk}$)

Likely to have several other applications (Simge’s talk yesterday)
Outline

1. (Perennial) Motivation: Unit Commitment Problems
2. From a DP algorithm to a new MIP formulation
3. Star-shaped MINLPs
4. More practical formulations
5. Computational results
6. Conclusions
More practical formulations (1)

- **Idea 1:** kill the many $p_t^{hk}$ entirely

- **Obvious map** between 3-bin variables and flow ones

  $$x_{it} = \sum_{(h,k) : t \in T(h,k)} y_i^{hk}, \quad v_{it} = \sum_{k \geq t} y_i^{tk}, \quad w_{it+1} = \sum_{h \leq t} y_i^{ht}$$

- **Strengthen 3-bin formulation** using the flow variables:

  $$p_{it} - p_{it-1} \leq -l_i \sum_{h : h \leq t-1} y_i^{ht-1} + \Delta_i^+ \sum_{(h,k) : t-1 \in T(h,k-1)} y_i^{hk} + \bar{l}_i \sum_{k : k \geq t} y_i^{tk}$$

  $$p_{it-1} - p_{it} \leq -l_i \sum_{k : k \geq t} y_i^{tk} + \Delta_i^- \sum_{(h,k) : t-1 \in T(h,k-1)} y_i^{hk} + \bar{u}_i \sum_{h : h \leq t-1} y_i^{ht-1}$$

  $$l_i \sum_{(h,k) : t \in T(h,k)} y_i^{hk} \leq p_{it} \leq u_i \sum_{(h,k) : t \in T(h,k)} y_i^{hk}$$

  $$p_{it} \leq \bar{l}_i \sum_{k : k \geq t} y_i^{tk} + \bar{u}_i \sum_{h : h \leq t} y_i^{ht} + \sum_{(h,k) : h < t < k} \psi_{it} y_i^{hk}$$

  (some changes needed when $\tau_i^+ = 1$ and at the beginning of time)
More practical formulations (2)

- Idea 2: aggregate the many $p_{it}^{hk}$ somehow
  \[ p_{it} = \sum_{h:h \leq t} p_{it}^h \]
  (only starting time, not ending one)

- Modified formulation
  \[
  p_{it}^h - p_{it-1}^h \leq -l_i y_i^{ht-1} + \Delta^+ \sum_{k:k \geq t} y_i^{hk} \\
  p_{it-1}^h - p_{it}^h \leq \bar{u}_i y_i^{ht-1} + \Delta^+ \sum_{k:k \geq t} y_i^{hk} \\
  p_{i1}^0 \leq (\Delta^+ + p_0) \sum_{k:1 \leq k} y_i^{0k} \\
  - p_{i1}^0 \leq (\Delta^- - p_0) \sum_{k:1 \leq k} y_i^{0k} \\
  l_i \sum_{k:k \geq t} y_i^{hk} \leq p_{it}^h \leq u_i \sum_{k:k \geq t} y_i^{hk} \\
  p_{ih}^h \leq \bar{l}_i \sum_{k:k \geq h} y_i^{hk} + \min\{\bar{l}_i, \bar{u}_i\} y_i^{hh} \\
  p_{it}^h \leq \bar{u}_i y_i^{ht} + \sum_{k:k > t} \psi_{it}^{hk} y_i^{hk}
  \]
1. (Perennial) Motivation: Unit Commitment Problems

2. From a DP algorithm to a new MIP formulation

3. Star-shaped MINLPs

4. More practical formulations

5. Computational results

6. Conclusions
Computational results: root note bound

<table>
<thead>
<tr>
<th>n</th>
<th>3-bin</th>
<th>DP</th>
<th>(p_t)</th>
<th>(p^h_t)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>time</td>
<td>time</td>
<td>time</td>
<td>time</td>
</tr>
<tr>
<td>10</td>
<td>0.12</td>
<td>16.40</td>
<td>0.59</td>
<td>2.83</td>
</tr>
<tr>
<td></td>
<td>1.52</td>
<td>0.67</td>
<td>0.93</td>
<td>0.92</td>
</tr>
<tr>
<td>20</td>
<td>0.28</td>
<td>59.07</td>
<td>1.46</td>
<td>8.53</td>
</tr>
<tr>
<td></td>
<td>1.43</td>
<td>0.51</td>
<td>0.78</td>
<td>0.76</td>
</tr>
<tr>
<td>50</td>
<td>0.96</td>
<td>300.53</td>
<td>4.32</td>
<td>22.42</td>
</tr>
<tr>
<td></td>
<td>0.87</td>
<td>0.08</td>
<td>0.30</td>
<td>0.29</td>
</tr>
</tbody>
</table>

- Artificial (but allegedly realistic) instances
- Obvious trade-off between root bound and LP cost
- Picture significantly murkier after Cplex cuts added
### Computational results: overall B&C

<table>
<thead>
<tr>
<th>n</th>
<th>3-bin</th>
<th>DP</th>
<th>(p_t)</th>
<th>(p_t^h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>time</td>
<td>opt</td>
<td>nodes</td>
<td>gap</td>
</tr>
<tr>
<td></td>
<td>28</td>
<td>5</td>
<td>275</td>
<td>0.01</td>
</tr>
<tr>
<td>20</td>
<td>7036</td>
<td>2</td>
<td>3561</td>
<td>0.08</td>
</tr>
<tr>
<td></td>
<td>10000</td>
<td>0</td>
<td>1619</td>
<td>0.12</td>
</tr>
<tr>
<td>50</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>5</td>
<td>163</td>
<td>0.09</td>
</tr>
<tr>
<td></td>
<td>6002</td>
<td>2</td>
<td>1980</td>
<td>0.11</td>
</tr>
<tr>
<td></td>
<td>6052</td>
<td>2</td>
<td>1042</td>
<td>0.14</td>
</tr>
</tbody>
</table>

- Above stop gap \(1e-4\), below stop gap \(1e-3\) (even less in practice)
- \(p_t\) formulation promising: maybe smaller exact formulation?
- Structured DW may make DP/\(p_t^h\) competitive
Outline

1. (Perennial) Motivation: Unit Commitment Problems
2. From a DP algorithm to a new MIP formulation
3. Star-shaped MINLPs
4. More practical formulations
5. Computational results
6. Conclusions
Conclusions

- Unit Commitment problem = an endless source of inspiration
- “Challenging problems require good methodologies, challenging problems motivate methodological advances”: very true for me
- 1st complete (correct and correctly proven) convex hull formulation for (single)-UC with ramping and nonlinear costs
- Uses Perspective Reformulation, of course :-)
- Technical lemma fully expected but still possibly useful
- $f \in C^1$?? I don’t really think so
- Possibly several other more star-shaped MINLPs
- “Large” formulations possibly useful, trade-offs to be navigated (did I mention Structured DW already?)
Conclusions

Just a small step in a long chain of problems... but this is another story.

A. Frangioni (DI — UniPi)
UC & star-shaped MINLP
Aussois 2020
Conclusions

- Just a small step in a long chain of problems
Conclusions²

- Just a small step in a long chain of problems
Conclusions

- Just a small step in a long chain of problems

... but this is another story
Acknowledgements

Copyright © PLAN4RES Partners 2020, all rights reserved.

This document may not be copied, reproduced, or modified in whole or in part for any purpose without written permission from the PLAN4RES Consortium. In addition, an acknowledgement of the authors of the document and all applicable portions of the copyright notice must be clearly referenced.

This document may change without notice.

The content of this document only reflects the author’s views. The European Commission / Innovation and Networks Executive Agency is not responsible for any use that may be made of the information it contains.

This project has received funding from the European Union’s Horizon 2020 research and innovation programme under grant agreement No 773897